

MEDIAN AND INJECTIVE METRIC SPACES

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ABSTRACT. We describe a construction which associates to any median metric space a pseudometric satisfying the binary intersection property for closed balls. Under certain conditions, this implies that the resulting space is, in fact, an injective metric space, bilipschitz equivalent to the original metric. In the course of doing this, we derive a few other facts about median metrics, and the geometry of CAT(0) cube complexes. One motivation for the study of such metrics is that they arise as asymptotic cones of certain naturally occurring spaces.

1. INTRODUCTION

In this paper, we discuss some connections between median metric spaces and injective metric spaces.

Median metric spaces have been studied for some time, see for example, [Ve, ChaDH, Bo3] and references therein. The definition is quite simple, and is given in Section 2 here. The geometry of such a space is closely linked to that of a cube complex. In particular, any finite median metric space is canonically the vertex set of a “CAT(0)” cube complex, where we put an l^1 metric on each of the cells (see the discussions in [BaC, Mi1] and Section 7 here). Moreover, any finite subset of a median metric space lies inside another finite subset which is median in the restricted metric. Here, we interpret the term “CAT(0)” to refer to the combinatorial characterisation of such a complex. (Except in dimension 1, the median metric is different from the usual CAT(0) metric.) Of particular interest here is the “finite-rank” case, where there is a bound on the dimensions of the cube complexes which arise in this way.

Another interesting class of metric spaces are the injective metric spaces. Formally, these are the injective objects in the category of metric spaces where the morphisms are taken to be 1-lipschitz maps. This is expressed more concretely in Section 2. These have also been studied for some time. Some early references are [AP, I]. A recent account, with some nice applications in geometric group theory, can be found in [L]. Injective metric spaces are always complete contractible geodesic spaces. Part of the general motivation for studying them arises from the fact that they have nice fixed-point properties.

In this paper (Section 3), we give a simple construction which canonically associates to any median metric space, (M, ρ) , a pseudometric, σ_M , on M . If one

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places additional conditions on (M, ρ) , one can say more about the resulting space, (M, σ_M) . In particular, if we assume that (M, ρ) is complete, connected and of finite rank, then (M, σ_M) is a bilipschitz equivalent injective metric space (Theorem 7.8). In this case, another construction of an injective metric has been given independently in [Mi2]. One can verify that the two constructions give rise to the same metric in this case. We remark that in [Bo3] it is shown that, under the same hypotheses, (M, ρ) also admits a canonical bilipschitz equivalent CAT(0) metric. One immediate consequence of either result is that (M, ρ) is contractible. Our argument also gives a simple proof that median intervals in (M, ρ) are compact (Corollary 5.2).

A simple example to illustrate this is when $M = \mathbb{R}^n$, with ρ the l^1 metric. This is median, and σ_M is the l^∞ metric. (The CAT(0) described in [Bo3] is then the euclidean, l^2 metric.)

Another example comes from CAT(0) cube complexes. If we put the euclidean (l^2) metric on each cube, we get the usual CAT(0) metric. If, instead, we put the l^1 metric on each cube, the resulting path metric is a median metric. If we use the l^∞ metric we get an injective metric space. (This is shown for finite CAT(0) cube complexes in [MaT, Va].) Our construction again gets us from the l^1 metric to the l^∞ metric (see Section 7), and so gives another proof of injectivity. See also [Mil] for further discussion.

One can also apply the same construction to the space of l^1 sequences, $l^1(\mathbb{N})$, with ρ the l^1 metric. This is again median. In this case, σ_M , is the l^∞ metric on the same set. This is not complete. However its completion is the space of sequences tending to 0, with the l^∞ metric. As a closed subspace of $l^\infty(\mathbb{N})$, this is also an injective metric space. (For more discussion of these examples, see Section 8.)

We remark that a combinatorial analogue of our construction can be found in [BaC], see Proposition 3.2 thereof.

One of the main motivations for studying median metric spaces of this type is that they arise, up to bilipschitz equivalence, as asymptotic cones of various naturally occurring spaces. Examples include the mapping class groups of surfaces [BeDS], the Teichmüller space in either the Teichmüller or Weil-Petersson metric, CAT(0) cube complexes, various relatively hyperbolic groups etc. In particular, any asymptotic cone of such a space is a complete, connected metric space, which has a canonical structure as a topological median algebra of finite rank; indeed it is bilipschitz equivalent to a median metric space inducing the same median. The existence of such a structure has consequences for the large-scale geometry of the space. For example, it can be used to prove various rank and rigidity results, see [Bo1, Bo4] and references therein. The contractibility of the asymptotic cone also gives polynomial bounds on higher isoperimetric functions [Ri].

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2. BASIC DEFINITIONS AND FACTS

In this section we assemble various basic facts about median metric spaces, median algebras, and injective metric spaces. For further background, see for example, [BaH, Ve, Ro, BaC, L, Bo3] and references therein.

Let (M, ρ) be a metric space. We will write $B_\rho(a; r)$ for the closed r -ball centred on $a \in M$. Given $a, b \in M$, write $[a, b] = [a, b]_\rho = \{x \in M \mid \rho(a, b) = \rho(a, x) + \rho(x, b)\}$.

Definition. We say that (M, ρ) is a *median metric space* if, for all $a, b, c \in M$, $[a, b] \cap [b, c] \cap [c, a]$ consists of a single point of M .

We refer to this point as the *median* of a, b, c , and denote it by $\mu(a, b, c)$. This gives a ternary operation $\mu : M^3 \rightarrow M$, and with this structure, (M, μ) is a median algebra. One checks that $[a, b] = \{x \in M \mid \mu(a, b, x) = x\}$, i.e. the *median interval* from a to b . In fact, many definitions and constructions depend only on the median structure.

A subset, A , of M (or any median algebra) is a *subalgebra* if it is closed under μ . It is *convex* if $[a, b] \subseteq A$ for all $a, b \in A$. An *n -cube* in M is a subalgebra median isomorphic to $\{0, 1\}^n$ (i.e. the direct product of n two-point median algebras). The (possibly infinite) *rank* of M is the maximal n such that M contains an n -cube. A *homomorphism* between median algebras is a map respecting the respective median structures.

We recall the following basic facts about median algebras and metrics. Any finite subset of a median algebra is contained in a finite subalgebra. Any finite median algebra can be canonically identified with the vertex set of a (combinatorial) CAT(0) cube complex. (See Section 7 for more discussion of cube complexes.)

The median operation on any median metric space is continuous; in fact, 1-lipschitz with respect to the l^1 metric on the product, M^3 . The completion of any median metric space is also a median metric space. Any subalgebra of a median metric space is a median metric space in the restricted metric.

Definition. Let \mathcal{F} be a set of subsets of (any set) M . We say that \mathcal{F} has the (*finite*) *binary intersection property* if given any (finite) subset $\mathcal{E} \subseteq \mathcal{F}$ of pairwise intersecting sets (i.e. $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{E}$), then $\bigcap \mathcal{E} \neq \emptyset$. (We adopt the convention that $\bigcap \emptyset = M$.)

We note that the family of convex subsets of any median algebra has the finite binary intersection property. This is called the ‘‘Helly property’’. We will only make essential use of it when $|\mathcal{E}| = 3$. (Here, the argument is simple: if $\mathcal{E} = \{A, B, C\}$, choose $a \in B \cap C$, $b \in C \cap A$ and $c \in A \cap B$, then $\mu(a, b, c) \in A \cap B \cap C$.) From this, one can easily proceed by induction to the general case.

Next we recall the notion of an injective metric space:

Definition. A metric space, (M, σ) is *injective* if, given any other metric space X , any subset $Y \subseteq X$, and any 1-lipschitz map of Y to M , there is a 1-lipschitz extension of this to a map of X to M .

Any injective metric space is complete, geodesic, and contractible. (We remark that if $a, b, c \in M$, then $[a, b]_\sigma \cap [b, c]_\sigma \cap [c, a]_\sigma$ is nonempty, but may consist of more than one point.)

Injectivity can be equivalently formulated in terms of hyperconvexity. For future reference, note that the following definition makes sense for any pseudometric space, (M, σ) .

Definition. We say that (M, σ) is *hyperconvex* if, given any family, $((x_i, r_i))_{i \in \mathcal{I}}$, of points in $X \times [0, \infty)$ with $\sigma(x_i, x_j) \leq r_i + r_j$ for all i, j in the indexing set, \mathcal{I} , then $\bigcap_{i \in \mathcal{I}} B_\sigma(x_i; r_i) \neq \emptyset$.

It was shown in [AP] that a metric space is injective if and only if it is hyperconvex. (For another proof, see [L].)

Let $\mathcal{B} = \{B_\sigma(x; r) \mid x \in M, r \geq 0\}$ be the set of all closed metric balls in (M, σ) . Clearly hyperconvexity implies that \mathcal{B} has the binary intersection property. The converse holds if, given any $x, y \in M$ and $t \leq \sigma(x, y)$, there is some $z \in M$ with $\sigma(x, z) = t$ and $\sigma(y, z) = \sigma(x, y) - t$. Clearly, this is implied if (M, σ) is a geodesic space. (If M is complete, this is, in fact, equivalent to saying that (M, σ) is geodesic.)

3. CONSTRUCTION OF THE NEW METRIC

In this section, we describe how to associate a canonical pseudometric to a median metric.

Definition. A *contraction* between two median metric spaces is a 1-lipschitz median homomorphism.

Clearly, this is closed under composition. Also, if (M, ρ) is a median metric space, and $a, b \in M$, then the map $[x \mapsto \mu(a, b, x)] : M \rightarrow [a, b]$ is a contraction of M onto $[a, b]$.

Given $a, b \in M$, let $\sigma(a, b) = \sigma_M(a, b)$ be the supremum of those r for which there is a contraction, $\phi : M \rightarrow [0, r]$, of M to the real interval $[0, r] \subseteq \mathbb{R}$, with $\phi a = 0$ and $\phi b = r$.

We will refer to such a ϕ as a “contraction (onto $[0, r]$) for a, b ”.

Note that, in this definition, we could equivalently replace M by $[a, b]$ (by precomposing any contraction of $[a, b]$ with the map $[x \mapsto \mu(a, b, x)]$). Clearly $\sigma(a, b) = \sigma(b, a) \leq \rho(a, b)$.

Lemma 3.1. σ is a pseudometric.

Proof. Suppose $a, b \in M$ and $r \leq \rho(a, b)$. Let $\phi : M \rightarrow [0, r]$ be a contraction for a, b , and let $s = \phi c$. Postcomposing with the contractions of $[0, r]$ to $[0, s]$ and to $[s, r]$ (and translating the latter) we see that $\sigma(a, c) + \sigma(c, b) \geq r$. This proves the triangle inequality. \square

Lemma 3.2. *Suppose $a, b, c, d \in M$ with $b \in [c, d]$. Then $\sigma(a, b) \leq \max\{\sigma(a, c), \sigma(a, d)\}$.*

Proof. Let $r < \sigma(a, b)$, and let $\phi : M \rightarrow [0, r]$ be a contraction for a, b . Now $r = \phi(b) \in \phi([c, d]) = [\phi c, \phi d]$, so either $\phi c = r$ or $\phi d = r$ (or both). Thus, $r \leq \max\{\sigma(a, c), \sigma(a, d)\}$. \square

In other words, if $a \in M$ and $r \geq 0$, then $B_\sigma(a; r)$ is (median) convex. From the Helly property mentioned above we get:

Corollary 3.3. *The set of closed balls in (M, σ) has the finite binary intersection property.*

For future reference, we make the elementary observation that if $R \subseteq M$ is any subalgebra, then $\sigma_M \leq \sigma_R$.

4. COMPLETE SPACES

In this section, we show that, under some additional assumptions, we can deduce that (M, σ) has the binary intersection property for closed balls. First we recall the notion of a gate in a median algebra.

Let (M, ρ) be a median metric space, and $C \subseteq M$ be convex. If $a \in M$, we say that $b \in C$ is a *gate* for a in C if $[a, b] \cap C = \{b\}$. An equivalent formulation is to say that $b \in C$ and that $b \in [a, c]$ for all $c \in C$. Note that if $d \in [a, b]$, then $[d, b] \subseteq [a, b]$, and so b is also a gate for d . If a gate exists, then it is unique. The above only requires the median structure. In terms of the metric, a gate is simply a point, b , of C which uniquely minimises $\rho(a, b)$.

Definition. We say that a median metric space (M, ρ) has *compact intervals* if $[a, b]$ is compact for all $a, b \in M$.

We will see (Corollary 5.2) that this holds for any complete finite-rank median metric space.

Note that if M has compact intervals then gates always exist if C is closed and non-empty.

Lemma 4.1. *Let (M, ρ) be a complete median metric space with compact intervals. Then the set of closed convex bounded subsets of M has the binary intersection property.*

Proof. Given that all sets are well orderable, this is equivalent to the following claim. Let $(C_\beta)_{\beta < \alpha}$ be a family of closed convex bounded subsets, indexed by an ordinal α , and such that $C_\beta \cap C_\gamma \neq \emptyset$ for all $\beta, \gamma < \alpha$. Then $\bigcap_{\beta < \alpha} C_\beta \neq \emptyset$. We will prove this claim by transfinite induction. We suppose that the statement

holds for all families of subsets satisfying the hypotheses which are indexed by any strictly smaller ordinal.

So let $(C_\beta)_{\beta < \alpha}$ satisfy the above hypotheses.

Suppose first, that $\alpha = \delta + 1$ is a successor ordinal. For $\beta < \delta$, let $E_\beta = C_\beta \cap C_\delta$. If $\beta, \gamma < \delta$, then $E_\beta \cap E_\gamma = C_\beta \cap C_\gamma \cap C_\delta \neq \emptyset$ (by the Helly property for triples). Thus, $(E_\beta)_{\beta < \delta}$ satisfies the hypotheses, so by the inductive hypothesis, $\bigcap_{\beta < \alpha} C_\beta = \bigcap_{\beta < \delta} E_\beta \neq \emptyset$, as required.

Now suppose that α is a limit ordinal. Given $\beta < \alpha$, let $D_\beta = \bigcap_{\gamma < \beta} C_\gamma$. This is closed convex and bounded, and by the inductive hypothesis it is non-empty. Now $(D_\beta)_{\beta < \alpha}$ is a non-increasing sequence with $\bigcap_{\beta < \alpha} D_\beta = \bigcap_{\beta < \alpha} C_\beta$. We want to show that this is also non-empty.

Fix any $x_0 \in D_0$. For $\beta < \alpha$, let $x_\beta \in D_\beta$ be the gate for x_0 in D_β . If $\delta < \gamma < \alpha$, then $x_\gamma \in D_\delta$, so $x_\delta \in [x_0, x_\gamma]$. It follows that x_γ is also the gate for x_δ in D_γ . Suppose that $\delta < \gamma < \beta < \alpha$. Then $x_\beta \in D_\gamma$, so $x_\gamma \in [x_\delta, x_\beta]$. Thus, $\rho(x_\delta, x_\beta) = \rho(x_\delta, x_\gamma) + \rho(x_\gamma, x_\beta)$. In other words, $(x_\beta)_{\beta < \alpha}$ is geodesic.

The map $[\beta \mapsto \rho(x_0, x_\beta)]$ is non-decreasing and bounded. Let $r = \sup\{\rho(x_0, x_\beta) \mid \beta < \alpha\}$. We can find a sequence, $(\beta_i)_{i \in \mathbb{N}}$, indexed by \mathbb{N} , so that $\rho(x_0, x_{\beta_i}) \rightarrow r$. Write $x_i = x_{\beta_i}$. Thus, $(x_i)_{i \in \mathbb{N}}$ is a bounded geodesic sequence, hence Cauchy. Since M is complete, $x_i \rightarrow y$ for some $y \in M$. We claim that $y \in \bigcap_{\beta < \alpha} D_\beta$.

For suppose not. Then $y \notin D_\beta$ for some $\beta < \alpha$. Since D_β is closed, there is some $i \in \mathbb{N}$ such that $[x_i, y] \cap D_\beta = \emptyset$. Since $x_i \in D_{\beta_i}$, we must have $\beta > \beta_i$. But then $\rho(x_0, x_\beta) > \rho(x_0, x_i)$, and so $\rho(y, x_\beta) < \rho(y, x_i)$. It follows (by continuity of the median) that $x_\beta \in [x_i, y]$, contradicting the fact that $[x_i, y] \cap D_\beta = \emptyset$. \square

From the fact that closed balls in (M, σ_M) are median convex, we deduce immediately:

Corollary 4.2. *Let (M, ρ) be a complete median metric space with compact intervals. Then the collection of closed balls in (M, σ_M) has the binary intersection property.*

5. FINITE MEDIAN ALGEBRAS

We assume basic facts about median algebras. See [BaH, Ve, Ro, Bo1, Bo2] and references therein.

Let Π be a finite median algebra of rank ν . This can be canonically identified with the vertex set of a finite CAT(0) cube complex, Δ (see [Che]). For the moment, we view this as a combinatorial statement. We will discuss the geometry of cube complexes further in Section 7. One can show that the rank of Π is equal to the dimension of Δ .

A *wall* of Π is an unordered partition of Π into two disjoint non-empty convex subsets. It can be identified with a hyperplane in Δ . (See [Ro] for more discussion.) We write $\mathcal{W}(\Pi)$ for the set of walls of Π . If $W, W' \in \mathcal{W}(\Pi)$, we say that

W, W' cross, if the corresponding hyperplanes in Δ intersect. The rank, ν , of Π is equal to the maximal cardinality of a set of pairwise crossing walls.

If $a, b \in \Pi$, we write $\mathcal{W}(a, b) \subseteq \mathcal{W}(\Pi)$ for the set of walls that separate a and b . This has a natural partial order given by nesting. Two walls are comparable in this order if and only if they do not cross.

A subset of $\mathcal{W}(a, b)$ is a *chain* if no two of its elements cross — in other words the induced order is a total order. We write $\mathcal{C}(a, b)$ for the set of all chains in $\mathcal{W}(a, b)$. An antichain in $\mathcal{W}(a, b)$ (i.e. a set of pairwise crossing elements) has cardinality at most ν . It follows by Dilworth's Lemma (cf. [BrCGNW]) that $\mathcal{W}(a, b)$ is a union of (at most) ν chains, which we can take to be disjoint. This, in turn implies that there is a median monomorphism of Π into the real cube $[0, 1]^\nu$.

Now suppose that (Π, ρ) is a finite median metric space. We have a map, $\lambda : \mathcal{W}(\Pi) \rightarrow (0, \infty)$, where $\lambda(W)$ is the *width* of the wall W , i.e. it equals $\rho(x, y)$ where $\{x, y\}$ is any 1-face of Π crossing W . Conversely, the metric ρ can be recovered from the median structure, μ , together with the map λ . In fact, given $\mathcal{S} \subseteq \mathcal{W}(\Pi)$, write $\Lambda(\mathcal{S}) = \sum_{W \in \mathcal{S}} \lambda(W)$. One checks that $\rho(a, b) = \Lambda(\mathcal{W}(a, b))$.

A simple application of Dilworth's Lemma, as mentioned above, now shows that if $a, b \in \Pi$, then $[a, b]$ isometrically embeds into an l^1 product of real intervals $\prod_{i=1}^\nu [0, r_i]$ with $\rho(a, b) = \sum_{i=1}^\nu r_i$. An immediate consequence is that if $A \subseteq [a, b]$ is ϵ -separated (i.e. $\rho(x, y) \geq \epsilon > 0$ for all distinct $x, y \in A$), then $|A|$ is bounded above in terms of $\rho(a, b)$, ϵ and ν .

We will say more about finite metric spaces in Section 6. For the moment, we draw some immediate consequences for more general median metric spaces.

Let (M, ρ) be a median metric space of rank $\nu < \infty$.

Suppose $a, b \in M$ and that $A \subseteq [a, b]$ is ϵ -separated. Now $A \cup \{a, b\}$ is contained in a finite subalgebra of $[a, b] \subseteq M$. Applying the above, we see once again, that $|A|$ is bounded above in terms of $\rho(a, b)$, ϵ and ν . We immediately deduce:

Lemma 5.1. *Any interval in a finite-rank median metric space is totally bounded.*

It follows that any interval is separable. In fact, given $n \in \mathbb{N}$, let $B_n \subseteq [a, b]$ be a maximal $(1/n)$ -separated set. Let $A_n = \bigcup_{i=1}^n B_i$ and let $\Pi_n \subseteq [a, b]$ be a finite subalgebra containing $A_n \cup \{a, b\}$. Thus $\Pi_n \subseteq \Pi_{n+1}$ for all n , and so $\Pi = \bigcup_n \Pi_n$ is a countable dense subalgebra of $[a, b]$.

Given that complete and totally bounded implies compact, another immediate consequence is:

Corollary 5.2. *Any interval in a complete finite-rank median metric space is compact.*

(In fact, this is shown under more general hypotheses in [Bo2], though by a more involved argument.)

Note that, in this case, it follows by Corollary 4.2 that (M, σ_M) has the binary intersection property for closed balls.

6. BILIPSCHITZ EQUIVALENCE

In this section, we will show that for a finite-rank median metric space, (M, ρ) , the pseudometric σ_M is bilipschitz equivalent to ρ , hence a metric. (Recall that in general, $\sigma_M \leq \rho$.) We begin with some further discussion of the finite case.

Let (Π, ρ) be a finite median metric space, and let $\lambda : \mathcal{W}(\Pi) \rightarrow (0, \infty)$ be the associated map.

Lemma 6.1. *If $a, b \in \Pi$, then $\sigma_\Pi(a, b) = \max\{\Lambda(\mathcal{S}) \mid \mathcal{S} \in \mathcal{C}(a, b)\}$.*

Proof. To simplify notation, we can assume that $\Pi = [a, b]$ so $\mathcal{W}(\Pi) = \mathcal{W}(a, b)$. Write $\mathcal{C} = \mathcal{C}(a, b)$.

Suppose that $\phi : \Pi \rightarrow [0, r]$ is a median homomorphism with $\phi a = 0$ and $\phi b = r$. Then $\phi\Pi \subseteq [0, r]$ is a rank-1 median algebra. The walls, $\mathcal{W}(\phi\Pi)$, of $\phi\Pi$ are in bijective correspondence with the connected components of $[0, r] \setminus \phi\Pi$. Now any wall $W \in \mathcal{W}(\phi\Pi)$ gives rise to a wall, $W' \in \mathcal{W}(\Pi)$ (by taking the preimage of the partition of $\phi\Pi$ in Π). Note that if $\{x, y\}$ is a 1-face of Π crossing W' , then $\phi x, \phi y$ are the endpoints of the corresponding component of $[0, r] \setminus \phi\Pi$. Thus, if ϕ is 1-lipschitz, then this component has length at most $\lambda(W') = \rho(x, y)$. Now the set of all walls in $\mathcal{W}(\Pi)$ arising in this way from $\mathcal{W}(\phi\Pi)$ forms a chain $\mathcal{S} \in \mathcal{C}$. Thus, summing over all the components of $[0, r] \setminus \phi\Pi$, we see that $r \leq \Lambda(\mathcal{S})$. This shows that $\sigma_\Pi(a, b) \leq \max\{\Lambda(\mathcal{S}) \mid \mathcal{S} \in \mathcal{C}\}$.

Conversely, if $\mathcal{S} \in \mathcal{C}$, then \mathcal{S} gives rise to a median homomorphism of Π to a finite totally ordered set, which we can embed in a compact real interval, say $[0, r]$, with $\phi a = 0$ and $\phi b = r$. In fact, we can realise this so that the interval corresponding to ϕW for $W \in \mathcal{S}$ has length exactly $\lambda(W)$. It is now easily checked that ϕ is 1-lipschitz. Note that $r = \Lambda(\mathcal{S})$ in this case, so we get $\sigma_\Pi(a, b) \geq \max\{\Lambda(\mathcal{S}) \mid \mathcal{S} \in \mathcal{C}\}$. \square

Note that, in this case, the supremum defining σ_Π is attained. The construction of ϕ in the second part of the proof will be elaborated upon in Section 7.

Note that Π has rank $\nu < \infty$, then there is some $\mathcal{S} \in \mathcal{C}(a, b)$ with $\rho(a, b) \leq \nu\Lambda(\mathcal{S})$. (This is an immediate consequence of Dilworth's Lemma, or can be seen by a more direct argument.) Thus, $\rho(a, b) \leq \nu\sigma_\Pi(a, b)$. In fact, from this, we get:

Lemma 6.2. *Let (M, ρ) be a median metric space of rank ν . Then $\rho \leq \nu\sigma_M$.*

Proof. Let $a, b \in M$. Let $\Pi_n \subseteq [a, b]$ be an increasing sequence of finite subalgebras, each containing $\{a, b\}$, with $\Pi = \bigcup_n \Pi_n$ dense in $[a, b]$ (as in Section 5). Let $r < \rho(a, b)/\nu$. Given any n , we have $\sigma_{\Pi_n}(a, b)/\nu > r$, by the above, so we can find a retraction, $\phi_n : \Pi_n \rightarrow [0, r]$ with $\phi_n a = 0$ and $\phi_n b = r$. Passing to a diagonal subsequence, we can suppose that $(\phi_i x)_{i=1}^n$ converges for all $x \in \Pi_n$ for all n . Given $x \in \Pi$, set $\phi x = \lim_{i \rightarrow \infty} \phi_i x$. One sees easily that $\phi : \Pi \rightarrow [0, r]$ is a contraction. Since $\Pi \subseteq [a, b]$ is dense, this extends continuously to a map $\phi : [a, b] \rightarrow [0, r]$, which is also a contraction. Since this holds for all such r , we get $\sigma_M(a, b) \geq \rho(a, b)/\nu$ as required. \square

Thus, $\sigma_M \leq \rho \leq \nu\sigma_M$, so in the finite-rank case, σ_M and ρ are bilipschitz equivalent.

For use in Section 7 (see Lemma 7.7) note that the sequence $\sigma_{\Pi_n}(a, b)$ is non-increasing in n , and that $\sigma_M(a, b) \leq \sigma_{\Pi_n}(a, b)$. If $s = \lim_{n \rightarrow \infty} \sigma_{\Pi_n}(a, b)$, we can take ϕ_n to be a contraction of Π_n to $[0, s]$. This shows that $\sigma_{\Pi_n}(a, b) \rightarrow \sigma_M(a, b)$.

7. CUBE COMPLEXES

We describe the l^p metric on the cube complex associated to a finite median metric space. We will apply this to the case of a finite-rank median metric space. (The term ‘‘cube’’ in this context does not imply that it is regular — it may have differing side-lengths.)

Further discussion of the l^p metric can be found in [BaC]. It is shown that the l^1 metric on a CAT(0) cube complex is injective. (Another argument will be given below.)

We have already noted that a finite median algebra, Π , is canonically the vertex set of a finite cube complex, $\Delta = \Delta(\Pi)$. One way to see this is to note that Π naturally embeds into the cube $\{0, 1\}^{\mathcal{W}(\Pi)}$, where the coordinates correspond to the walls, $\mathcal{W}(\Pi)$, of Π . This in turn embeds into the real cube, $[0, 1]^{\mathcal{W}(\Pi)}$. These embeddings are median monomorphisms. We can construct Δ as the full subcomplex of $[0, 1]^{\mathcal{W}(\Pi)}$ with vertex set Π . Note that Δ is also a subalgebra of $[0, 1]^{\mathcal{W}(\Pi)}$. Taking the induced euclidean metric from the regular unit cube would give us the usual CAT(0) metric on Δ (hence the terminology of ‘‘CAT(0) complex’’).

We can generalise this as follows. Suppose that (Π, ρ) is a finite median metric space. We have median embeddings, $\Pi \subseteq \prod_{W \in \mathcal{W}(\Pi)} \{0, \lambda(W)\} \subseteq \prod_{W \in \mathcal{W}(\Pi)} [0, \lambda(W)]$, which are isometric. Given $p \in [0, \infty]$, we put the l^p metric on this product, and write τ_p for the induced path metric on Δ . Since Δ is compact, this is a geodesic metric.

If $p = 1$, this is a median metric extending the metric ρ . (We will just write $\rho = \tau_1$ in this case.) If $p = 2$, this is CAT(0). If $p = \infty$, it is an injective metric (see [MaT, Va]). We will give another proof of this last fact by showing that $\tau_\infty = \sigma_\Delta$.

We begin with some preliminary observations. Suppose that $\Pi' \subseteq \Delta = \Delta(\Pi)$ is a median subalgebra containing Π . We can identify $\Delta' = \Delta(\Pi')$ with Δ as sets, and indeed as median algebras. In fact, Δ' is obtained by subdividing the cells of Δ into smaller cubes. It is easily checked that for any p , the metrics τ_p obtained from Δ and Δ' agree. Now, any finite subset, $A \subseteq \Delta$, is contained in the vertex set of some such subdivision (by taking a finite subalgebra, $\Pi' \subseteq \Delta$ containing $\Pi \cup A$). In particular, if $a, b \in \Delta$, we see that we can realise the median interval $[a, b]$ intrinsically as a cube complex of the above form. (Thus, for many purposes, one can assume that $\Delta = [a, b]$.)

Before investigating τ_∞ , we make the following simple observation.

Lemma 7.1. σ_Π equals σ_Δ restricted to Π .

Proof. Since Π is a subalgebra, it is immediate that $\sigma_\Delta \leq \sigma_\Pi$. For the reverse inequality, let $a, b \in \Pi$ and $r = \sigma_\Pi(a, b)$. Thus, $r = \Lambda(\mathcal{S})$ for some $\mathcal{S} \in \mathcal{C}(a, b)$, as given by Lemma 6.1. We can suppose that $\Pi = [a, b]$. Now we have a contraction $\phi : \Pi \rightarrow [0, r]$ as described in the proof of Lemma 6.1. We can extend this to a contraction $\phi : \Delta \rightarrow [0, r]$ by taking the affine extension on each cell. This shows that $r \leq \sigma_\Delta(a, b)$. \square

We now make some general observations about τ_p .

Let $P \subseteq \Delta$ be a cell of this complex. This is convex, and we have a map $\theta = \theta_P : \Delta \rightarrow P$, so that $[x, \theta(x)] \cap P = \{\theta(x)\}$ (i.e. $\theta(x)$ is a gate for x in P , as discussed in Section 6). In terms of the embedding of Δ into $\prod_{W \in \mathcal{W}(\Pi)} [0, \lambda(W)]$, this is just projection to a face. From this, it is easily seen that τ_p restricted to P is just the l^p metric, and that θ is 1-lipschitz with respect to this metric. Note that, if a, b are opposite corners of P , then $P = [a, b]$, and $\theta(x) = \mu(a, b, x)$. Furthermore, if $c, d \in P$, then the map $[x \mapsto \mu(c, d, x)]$ is 1-lipschitz for τ_p , since it is the composition of θ with projection of P to $[c, d]$. In fact:

Lemma 7.2. *If $a, b \in \Delta$, then the map $[x \mapsto \mu(a, b, x)] : \Delta \rightarrow [a, b]$ is 1-lipschitz with respect to τ_p .*

Proof. Let $c, d \in \Delta$. It is easily seen that there is a geodesic, α , from c to d of the form $\alpha = \alpha_1 \cup \dots \cup \alpha_n$, where each α_i lies in some cell of Δ . Let $c = x_0, x_1, \dots, x_n = d$ be the breakpoints, and let $y_i = \mu(a, b, x_i)$. Thus, $y_{i-1} = \mu(y_{i-1}, y_i, x_{i-1})$ and $y_i = \mu(y_{i-1}, y_i, x_i)$ (by basic properties of projection, given that $[y_{i-1}, y_i] \subseteq [a, b]$). It is also easily checked that, since x_{i-1} and x_i lie in a cell of Δ , so do y_{i-1} and y_i . Now, from the earlier observation, we have that $\tau_p(y_{i-1}, y_i) \leq \tau_p(x_{i-1}, x_i)$. Summing over all i , we get that $\tau_p(\mu(a, b, c), \mu(a, b, d)) = \tau_p(y_0, y_n) \leq \tau_p(x_0, x_n) = \tau_p(c, d)$. \square

We say that a sequence, $(x_i)_{i=0}^n$, in Δ is *monotone* if $x_j \in [x_i, x_k]$ whenever $i \leq j \leq k$. Clearly, this is equivalent to being geodesic in the metric, ρ . In fact, it is also equivalent to saying that $x_{i+1} \in [x_i, x_n]$ for all i .

Lemma 7.3. *If $a, b \in \Delta$, then there is a monotone sequence, $a = x_0, x_1, \dots, x_n = b$, with $\tau_p(a, b) = \sum_{i=1}^n \tau_p(x_{i-1}, x_i)$ and with x_{i-1}, x_i lying in a cell of Δ for all i .*

Proof. Start with any sequence x_0, x_1, \dots, x_n , as in the proof of Lemma 7.2. Now, Lemma 7.2 tells us that its projection to $[a, b]$ under the map $[x \mapsto \mu(a, b, x)]$ has the same property, so we can suppose that $x_1 \in [a, b]$. We can now similarly project the sequence starting at x_1 to $[x_1, b]$, and thereby suppose that $x_2 \in [x_1, b]$. Continuing inductively, we can suppose that $x_{i+1} \in [x_i, b]$ for all i , and so $(x_i)_i$ is monotone, as required. \square

Given any cell, P of Δ , we write $\mathcal{W}(P) \subseteq \mathcal{W}(\Pi)$ for the set of walls of Π which cross P (so we can identify $\mathcal{W}(P)$ with the intrinsically defined set $\mathcal{W}(P \cap \Pi)$).

Thus, the elements of $\mathcal{W}(P)$ pairwise cross. We say that P is *trivial* if it is just a point (i.e. $\mathcal{W}(P) = \emptyset$). If P is non-trivial, we write $\lambda(P) = \max\{\lambda(W) \mid W \in \mathcal{W}(P)\}$, and $\mathcal{W}^0(P) = \{W \in \mathcal{W}(P) \mid \lambda(W) = \lambda(P)\}$. We say that P is *regular* if it is non-trivial and $\mathcal{W}^0(P) = \mathcal{W}(P)$.

We now fix a, b , and let $(x_i)_i$ be as given by Lemma 7.3. After subdivision of Δ , we can suppose that $x_i \in \Pi$ for all i . It follows that $P_i = [x_{i-1}, x_i]$ is a cell of Δ . Write $\lambda_i = \lambda(P_i)$, $\mathcal{W}_i = \mathcal{W}(P_i)$ and $\mathcal{W}_i^0 = \mathcal{W}^0(P_i)$. Now $\tau_\infty(x_{i-1}, x_i) = \lambda_i$ and so $\tau_\infty(a, b) = \sum_{i=1}^n \lambda_i$.

Since $(x_i)_i$ is monotone, it is easily seen that $\mathcal{W}(a, b) = \bigsqcup_{i=1}^n \mathcal{W}_i$. If $\mathcal{S} \in \mathcal{C}(a, b)$, we see that $|\mathcal{S} \cap \mathcal{W}_i| \leq 1$ for all i , and so $\Lambda(\mathcal{S}) = \sum_{W \in \mathcal{S}} \lambda(W) \leq \sum_{i=1}^n \lambda_i = \tau_\infty(a, b)$. By Lemma 6.1 we deduce that $\sigma_\Pi(a, b) \leq \tau_\infty(a, b)$.

For the reverse inequality, we will need:

Lemma 7.4. *There is some $\mathcal{S} \in \mathcal{C}(a, b)$ with $\mathcal{S} \cap \mathcal{W}_i^0 \neq \emptyset$ for all i .*

In other words, we can find $W_i \in \mathcal{W}_i^0$ such that W_1, \dots, W_n is a chain in $\mathcal{W}(a, b)$.

For the proof, we will use the following construction. Suppose that P is a regular cell of Δ . If $x \in P \cap \Pi$ is a vertex, let $y \in P \cap \Pi$ be the opposite vertex (so $P = [x, y]$). Given $t \in (0, \lambda(P))$, write $q = q(P, x, t)$ for the unique point with $\tau_\infty(x, q) = t$ and $\tau_\infty(y, q) = \lambda(P) - t$. (In other words, it lies on the euclidean diagonal between these two corners.)

We will prove Lemma 7.4 by induction on the length, n , of the sequence $(x_i)_i$. In fact, the inductive hypothesis will be stronger. We show that there is a regular face, F , of P_1 , containing $a = x_0$, with the following two properties. First, for each $W \in \mathcal{W}(F) \subseteq \mathcal{W}_1$, there is a chain, $\mathcal{S} \in \mathcal{C}(a, b)$, with $W \in \mathcal{S}$ and with $\mathcal{S} \cap \mathcal{W}_i^0 \neq \emptyset$ for all i . Second, there is some $t \in (0, \lambda(F))$ such that $\tau_\infty(q(F, a, t), b) = \tau_\infty(a, b) - t$. Note that the first condition implies that $W \in \mathcal{W}_i^0$, so in fact $\mathcal{W}(F) \subseteq \mathcal{W}_i^0$, and so $\lambda(F) = \lambda(P_1) = \lambda_1$. The second condition is equivalent to asserting that $q(F, a, t)$ lies on some τ_∞ -geodesic from a to b (though not necessarily with all breakpoints at vertices of Π).

In what follows, we refer to faces, R, R' , of a cell P as *complementary* if $\mathcal{W}(P) = \mathcal{W}(R) \sqcup \mathcal{W}(R')$. This implies that $|R \cap R'| = 1$. In fact, each vertex of R gives us exactly one complementary face, and these faces are all parallel.

Now, suppose the inductive hypothesis holds for any strictly smaller n . Applying this to the sequence x_1, x_2, \dots, x_n , gives us a regular face $G \subseteq P_2$ satisfying the conclusion of the inductive hypothesis (with G playing the role of F).

Now consider the (possibly empty) set, $\mathcal{W}' \subseteq \mathcal{W}_1^0$ of walls in \mathcal{W}_1^0 which cross every element of $\mathcal{W}(G)$. These determine a vertex or regular face, H , of P_1 , containing x_1 , and with $\mathcal{W}(H) = \mathcal{W}'$. Note that G, H are complementary faces of a cell K , of Δ , with $\mathcal{W}(K) = \mathcal{W}(G) \sqcup \mathcal{W}(H)$. (Possibly, $H = \{x_1\}$ and $K = G$.) Let $F \subseteq P_1$, be the face of P_1 containing a , with $\mathcal{W}(F) = \mathcal{W}_1^0 \setminus \mathcal{W}(H)$. Thus, F is either trivial, or regular with $\lambda(F) = \lambda_1$. We claim that F satisfies the conclusion of the inductive hypothesis (and so must be non-trivial).

First note that if $W \in \mathcal{W}(F)$, then $W \notin \mathcal{W}(H)$, so there is some $W_2 \in \mathcal{W}(G) \subseteq \mathcal{W}_2^0$ such that W does not cross W_2 . By the inductive hypothesis, we can find $W_i \in \mathcal{W}_i^0$, for $i \geq 2$ so that W_2, \dots, W_n is a chain in $\mathcal{W}(x_1, b)$. Setting $W_1 = W$ now gives us our chain $\mathcal{S} = \{W_1, W_2, \dots, W_n\}$ in $\mathcal{W}(a, b)$ as required for the first part.

For the second part, note that by the inductive hypothesis, we can find some $t \in (0, \lambda_2)$ such that $\tau_\infty(q(G, x_1, t), b) = \tau_\infty(x_1, b) - t$. Since t can be taken arbitrarily small, we can also suppose that $\lambda(W) + t < \lambda_1$ for all $W \in \mathcal{W}_1 \setminus \mathcal{W}_1^0$. Set $z = q(G, x_1, t)$. We define points, $q \in F$ and $y \in H$ as follows. If F is trivial (i.e. $F = \{a\}$) set $q = a$, otherwise, set $q = q(F, a, t)$. If H is trivial, set $y = x_1$, otherwise, set $y = q(H, x_1, t)$. Note that F and H cannot both be trivial. In fact, since $\mathcal{W}(P_1) = \mathcal{W}_1 = \mathcal{W}(F) \sqcup \mathcal{W}(H)$, and since a, x_1 are opposite corners of P_1 , we see that $\tau_\infty(a, x_1) = \lambda_1 - t$. Moreover, since y and z both lie in the cell K , and are distant 0 or t from the corner, x_1 , we see that $\tau_\infty(y, z) = t$. It follows that $\tau_\infty(q, b) \leq \tau_\infty(q, y) + \tau_\infty(y, z) + \tau_\infty(z, b) = (\lambda_1 - t) + t + (\tau_\infty(x_1, b) - t) = \tau_\infty(a, x_1) + \tau_\infty(x_1, b) - t = \tau_\infty(a, b) - t$. It follows that $q \neq a$. Therefore, in fact, F must be non-trivial, and $q = q(F, a, t)$. Since $\tau_\infty(a, q) = t$, we then have $\tau_\infty(q, b) = \tau_\infty(a, b) - t$. We have therefore verified the second part of the inductive hypothesis.

We have proven Lemma 7.4.

Note that we have $\tau_\infty(a, b) = \sum_{i=1}^n \lambda_i = \Lambda(\mathcal{S})$ for this $\mathcal{S} \in \mathcal{C}(a, b)$.

Lemma 7.5. $\sigma_\Delta = \tau_\infty$.

Proof. We have already seen that $\sigma_\Delta \leq \tau_\infty$. For the reverse inequality, let $a, b \in \Delta$. After subdividing, we can suppose that $a, b \in \Pi$. Let $\mathcal{S} \in \mathcal{C}(a, b)$ be as given by Lemma 7.4. By Lemmas 6.1 and 7.1 we have, $\sigma_\Delta(a, b) \geq \Lambda(\mathcal{S})$, so $\tau_\infty(a, b) \leq \sigma_\Delta(a, b)$ as required. \square

In particular, this shows that σ_Δ is a geodesic metric. Since it also has the binary intersection property for closed balls, it follows that (Δ, σ_Δ) is an injective metric space.

In fact, we can prove a more general statement. First note that Lemma 7.5 implies a result about finite median metrics.

Let (Π, ρ) be a finite median metric space. Write $\omega(\Pi) = \max\{\lambda(W) \mid W \in \mathcal{W}(\Pi)\}$.

Lemma 7.6. *If $a, b \in \Pi$ and $t \leq \sigma_\Pi(a, b)$, then there is some $c \in [a, b]$ with $\sigma_\Pi(a, c) \leq t$ and $\sigma_\Pi(b, c) \leq \sigma_\Pi(a, b) - t + \omega(\Pi)$.*

Proof. Let Δ be the associated cube complex. Recall that by Lemma 7.1, σ_Π is just σ_Δ restricted to Π . Also, by Lemma 7.5, σ_Δ is a geodesic metric. Therefore, there is some $d \in \Delta$ with $\sigma_\Delta(a, d) = t$ and $\sigma_\Delta(b, d) = \sigma_\Delta(a, b) - t$. Moreover, we can take d to lie in the interval $[a, b]$ in Δ . Let $c \in \Pi$ be the vertex of the cell of Δ containing d and with $c \in [a, d]$. Then $\sigma(a, c) \leq t$ and $\sigma_\Delta(c, d) \leq \omega(\Pi)$, and so $\sigma_\Delta(b, c) \leq \sigma_\Delta(a, b) - t + \omega(\Pi)$ as required. \square

Now let (M, ρ) be a complete connected, finite-rank median metric space. (Such a space is called a “proper” median metric space in [Bo2].)

Lemma 7.7. *(M, σ_M) is a geodesic space.*

Proof. It is enough to show that, given $a, b \in M$ and $t \leq \sigma_M(a, b)$, there is some $c \in [a, b]$ with $\sigma_M(a, c) = t$ and $\sigma_M(b, c) = \sigma_M(a, b) - t$. To see this, recall that $[a, b]$ compact (by Corollary 5.2) and connected (since it is the image of M under the map $[x \mapsto \mu(a, b, x)]$). Let $(\Pi_n)_n$ be an increasing sequence of finite subalgebras containing a, b with $\Pi = \bigcup_n \Pi_n$ dense in $[a, b]$ (cf. Corollary 5.2). It is easily checked that $\omega(\Pi_n) \rightarrow 0$. Moreover, as remarked after Lemma 6.2, we have $\sigma_{\Pi_n}(a, b) \rightarrow \sigma_M(a, b)$. Lemma 7.6 now gives us some $c_n \in \Pi_n$ with $\sigma_M(a, c_n) \leq \sigma_{\Pi_n}(a, c_n) \leq t$ and $\sigma_M(b, c_n) \leq \sigma_{\Pi_n}(b, c_n) \leq \sigma_{\Pi_n}(b, c_n) - t + \omega(\Pi_n)$. Passing to a subsequence, we have $c_n \rightarrow c \in [a, b]$ with the required properties. \square

By Lemma 4.1, (M, σ_M) is hyperconvex, hence injective. We have shown:

Theorem 7.8. *If (M, ρ) is a complete connected finite-rank median metric space, then (M, σ_M) is injective.*

As noted in the introduction, an equivalent construction of σ_M in this case is given in [Mi2], and is shown to be injective.

8. FUNCTION SPACES

We explain what these constructions give for certain l^1 spaces. We begin with a brief discussion of direct products.

If M, M' are median algebras, then so is their direct product, $M \times M'$, with the median defined coordinatewise. Clearly, the coordinate projections are median epimorphisms.

Lemma 8.1. *Any median homomorphism from $M \times M'$ to any rank-1 median algebra factors through projection to one of the factors.*

Proof. We make use of the fact that this is easily seen to be the case where both factors are two-point median algebras, so that their product is a square.

Let T be any rank-1 median algebra, and $\phi : M \times M' \rightarrow T$ any homomorphism. If $x, y \in M$ and $z, w \in M'$, write $(x, z) \sim (y, w)$ to mean $\phi((x, z)) = \phi((y, w))$. Note that, by the above observation, given any $x, y \in M$ and $z, w \in M'$, at least one of $(x, z) \sim (y, z)$ or $(x, z) \sim (x, w)$ must hold. We can assume that there exist such x, y, z, w with $(x, z) \not\sim (y, w)$, and so, after swapping M and M' , we can assume that $(x, z) \not\sim (y, z)$. Now, given any $u \in M$, after swapping x and y , we can suppose that $(u, z) \not\sim (x, z)$. It now follows that for any $v \in M'$, we have $(u, v) \sim (u, z)$. This shows that $\phi((u, v))$ depends only on u . In other words, ϕ factors through projection to M . \square

Note that the above extends to finite products, thereby justifying a statement made earlier about projections of a finite product of real intervals to a real interval.

We now consider function spaces.

Let X be any set, and write \mathbb{R}^X be the set of functions from X to \mathbb{R} . Given $f, g \in \mathbb{R}^X$, define $f \wedge g$ and $f \vee g$ by $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$. Given $f, g, h \in \mathbb{R}^X$, write $\mu(f, g, h) = (f \wedge g) \vee (g \wedge h) \vee (h \wedge f) = (f \vee g) \wedge (g \vee h) \wedge (h \vee f)$. Then (\mathbb{R}^X, μ) is a median algebra. Note that if $X = A \sqcup B$, then \mathbb{R}^X is canonically isomorphic to $\mathbb{R}^A \times \mathbb{R}^B$.

Suppose that m is a measure on X , with associated σ -algebra, \mathcal{M} , of measurable sets. Let $L(X)$ be the set of L^1 -functions with respect to this measure. Thus, $f \in L(X)$, if $\|f\|_1 = \int_X |f| dm$ is defined and finite. Write $\rho(f, g) = \|f - g\|_1$ and define the equivalence relation, \sim , on $L(X)$ by $f \sim g$ if $\|f - g\|_1 = 0$. Let $\mathcal{L}(X) = L(X)/\sim$. Thus, $\mathcal{L}(X)$ is the L^1 space, and ρ descends to a metric, also denoted ρ , on $\mathcal{L}(X)$.

Now $L(X)$ is a median subalgebra of \mathbb{R}^X . Moreover, if $f, g, h, h' \in L(X)$ with $h \sim h'$, then $\mu(f, g, h) \sim \mu(f, g, h')$, so we also get a median, μ , defined on $\mathcal{L}(X)$. Now, $(\mathcal{L}(X), \rho)$ is a median metric space with associated median μ . Note that if $A \in \mathcal{M}$, then $\mathcal{L}(X) \cong \mathcal{L}(A) \times \mathcal{L}(X \setminus A)$.

Suppose that $\phi : \mathcal{L}(X) \rightarrow \mathbb{R}$ is a non-trivial median homomorphism. If $A \in \mathcal{M}$, then by Lemma 8.1, ϕ factors through projection to exactly one of the factors, $\mathcal{L}(A)$ or $\mathcal{L}(X \setminus A)$. Let $\mathcal{A} \subseteq \mathcal{M}$ be the set of those $A \in \mathcal{M}$ for which ϕ factors through projection to $\mathcal{L}(A)$. Note that if $A \in \mathcal{M}$, then precisely one of A or $X \setminus A$ lies in \mathcal{A} . Also, if $A, B \in \mathcal{M}$ with $A \in \mathcal{A}$ and $A \subseteq B$, then $B \in \mathcal{A}$. In other words, \mathcal{A} , satisfies the axioms of an ultrafilter, with domain restricted to \mathcal{M} .

Now consider the case where X is countable, \mathcal{M} is the whole power set of X , and $m(A) = |A|$. (Then \mathcal{A} is an ultrafilter in the usual sense.) In this case, $\mathcal{L}(X) \cong L(X)$. Let $\phi : \mathcal{L}(X) \rightarrow \mathbb{R}$ be a contraction. We claim that \mathcal{A} is a principal ultrafilter, i.e. $\mathcal{A} = \mathcal{A}(x) = \{A \subseteq X \mid x \in A\}$ for some $x \in X$.

To see this, write $X = \bigcup_{i=0}^{\infty} X_n$, where $(X_n)_n$ is an increasing union of finite subsets of X . Suppose that \mathcal{A} is non-principal. Then $X \setminus X_n \in \mathcal{A}$ for all n . Given $f \in \mathcal{L}(X)$, define $f_n \in \mathcal{L}(X)$ by $f_n(x) = 0$ if $x \in X_n$ and $f_n(x) = f(x)$ if $x \notin X_n$. Then $f_n \rightarrow 0$ in $(\mathcal{L}(X), \rho)$, so $\phi f_n \rightarrow 0$. But since $f_n|_{(X \setminus X_n)} = f|_{(X \setminus X_n)}$ and $X \setminus X_n \in \mathcal{A}$, we have $\phi f_n = \phi f$. Thus, $\phi f = 0$. We get the contradiction that ϕ is trivial.

We have shown:

Lemma 8.2. *If X is countable, with uniform measure, then any contraction of $\mathcal{L}(X)$ into \mathbb{R} factors through evaluation at some point of X .*

Now write $\|f\|_{\infty} = \max\{|f(x)| \mid x \in X\}$, and set $\tau_{\infty}(f, g) = \|f - g\|_{\infty}$ for the l^{∞} metric. Let σ be the pseudometric arising from the median metric space $(\mathcal{L}(X), \rho)$ by the above construction. We claim:

Lemma 8.3. $\sigma = \tau_{\infty}$.

Proof. Suppose $f, g \in \mathcal{L}(X)$. Then there is some $x \in X$ with $|f(x) - g(x)| = \tau_\infty(f, g)$. Let $\phi : \mathcal{L}(X) \rightarrow \mathbb{R}$ be evaluation at x . This is a contraction, and $|\phi f - \phi g| = |f(x) - g(x)| = \tau_\infty(f, g)$. It follows that $\sigma(f, g) \geq \tau_\infty(f, g)$.

For the reverse inequality, let $\phi : \mathcal{L}(X) \rightarrow \mathbb{R}$ be any contraction. By Lemma 8.2, there is some $x \in X$ such that ϕ factors through evaluation at x . It follows $|\phi f - \phi g| \leq |f(x) - g(x)| \leq \tau_\infty(f, g)$, so we get $\sigma(f, g) = \tau_\infty(f, g)$. \square

Note that the supremum defining the metric σ is attained in this case.

In summary, we have shown that if we start with the complete space median metric space, $l^1(\mathbb{N})$, of l^1 sequences with the l^1 metric, then our construction gives us the space of l^1 sequences in the l^∞ metric. This is a non-closed subspace of the complete injective metric space $l^\infty(\mathbb{N})$. Note that this subspace is a geodesic metric space, and it satisfies the finite binary interersection property for closed balls (by Corollary 3.3). However, the hypotheses of Corollary 4.2 fail in this case (since $l^1(\mathbb{N})$ does not have compact intervals), and so does its conclusion (as it must: since otherwise this space would be hyperconvex, hence injective by [AP], hence complete, hence a closed subspace of $l^\infty(\mathbb{N})$, which it is not). We note however that the completion of this space, that is, its closure in $l^\infty(\mathbb{N})$, is an injective metric space (namely the space of sequences converging to 0, equipped with with l^∞ metric).

As another example, consider the case where $X = [0, 1]^\nu \subseteq \mathbb{R}^\nu$, with the ν -dimensional Lebesgue measure. In this case, the ultrafilter determines a limit point, $a \in X$, with the property that every neighbourhood of a lies in \mathcal{A} . If we take a neighbourhood base, $(U_n)_n$, of a in X , and set $X_n = X \setminus U_n$, then the earlier argument shows that ϕ is trivial. In other words, any contraction on $\mathcal{L}(X)$ is trivial in this case.

Similarly, if $X = \mathbb{R}^\nu$, we also get a limit point, though we need to allow $a = \infty$. In the case when $a = \infty$, we take $(X_n)_n$ to be any compact exhaustion of \mathbb{R}^ν . In this case, we deduce that ϕ is trivial. We see:

Lemma 8.4. *Every contraction on $\mathcal{L}([0, 1]^\nu)$ or on $\mathcal{L}(\mathbb{R}^\nu)$ is trivial.*

Thus, in these cases, σ_M is identically 0.

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