

# THE ENDING LAMINATION THEOREM

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ABSTRACT. We give a proof of the Ending Lamination Theorem, due (in the indecomposable case) to Brock, Canary and Minsky. This was originally conjectured by Thurston. It states that a complete topologically finite hyperbolic 3-manifold is determined, up to isometry, by its topology together with a set of invariants associated to its ends. Moreover, the manifold is bilipschitz equivalent to a model manifold constructed directly from this data. We give a summary of background material and some related results.

## 0. PREFACE

This paper is based on a combination of two earlier manuscripts both originally produced in 2005. Many of the ideas for the first preprint were worked out while visiting the Max Planck Institute in Bonn. The remainder and most of the write-up of this was carried out when I was visiting the Tokyo Institute of Technology at the invitation of Sadayoshi Kojima. Most of the second preprint was written while visiting the Centre Bernoulli, E.P.F. Lausanne, as part of the programme organised by Christophe Bavard, Peter Buser and Ruth Kellerhals. I thank all three institutions for their generous hospitality. I also thank Dave Gabai for his many helpful comments on the former preprint; in particular, for the argument that appears at the end of Section 2.3, and his permission to include it here. The final drafts of these preprints were prepared at the University of Southampton. The preprints were substantially reworked and combined, and new material added at the University of Warwick. I thank Al Marden for his comments on the first few sections of the current manuscript. I also thank Makoto Sakuma and Eriko Hironaka for their interest and comments.

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## 1. GENERAL BACKGROUND

## 1.1. INTRODUCTION.

In this paper, we give an account of the Ending Lamination Theorem. This was proven in the “indecomposable” case by Minsky, Brock and Canary [Mi4, BrocCM], who also announced a proof in the general case. It gives a reasonably complete understanding of the geometry of complete hyperbolic 3-manifolds with finitely generated fundamental group. An informal statement is given as Theorem 1.2.1 here, and a more formal statement as Theorem 1.5.4.

The Ending Lamination Conjecture originates in the seminal work of Thurston in the late 1970s [Th1]. As well as a plethora of new results, this threw up many new questions (see Section 6 of [Th2]). Undoubtedly the most important of these was the Geometrisation Conjecture which asserts that any compact 3-manifold can canonically be cut into “geometric” pieces — each admitting a metric locally modelled on one of eight geometries (Question 1 of [Th2]). It is hyperbolic geometry that provides the richest source of these examples, so that it might be said, in a certain sense, that “most” 3-manifolds are hyperbolic. Three other key conjectures related to non-compact hyperbolic 3-manifolds, namely the Tame-ness Conjecture, the Ending Lamination Conjecture and the Density Conjecture (respectively, Questions 5, 11 and 6 of [Th2]). Major progress towards these conjectures was subsequently made by many authors. Then, in the space of three years, between 2002 and 2004, proofs of the first three conjectures were produced: respectively by Perelman [Pe], by Agol and Calegari and Gabai [Ag, CalG], and by Minsky, Brock and Canary [Mi4, BrocCM]. This paved the way to a proof of the Density Conjecture, finally completed in [Oh, NS]. As well as its intrinsic interest, the Ending Lamination Theorem, and the technology involved in its proof, has many applications beyond 3-manifolds. For example, it has consequences for the geometry of the mapping class groups and Teichmüller space.

In this paper, we restrict to the orientable case: by default any manifold will be assumed orientable. Indeed, most of the literature restricts to this case. Corresponding results in the non-orientable case can often be derived by passing to the orientable cover.

In 2 dimensions, it is well known that a closed surface of genus at least 2 admits a rich “Teichmüller space” of hyperbolic structures: in other words complete metrics locally modelled on the hyperbolic plane. In contrast, in higher dimensions, a closed manifold admits at most one hyperbolic structure. This is the Mostow Rigidity Theorem [Most]. If we allow non-compact manifolds, then in general, we might expect to get many such structures. In dimensions greater than 3, there are lots of interesting examples; though there is no well developed general theory,

and it would seem that hyperbolic structures do not arise in such a natural way. The focus has therefore been on the 3-dimensional situation.

To outline this, let  $M$  be a complete hyperbolic 3-manifold. We can write  $M = \mathbb{H}^3/\Gamma$ , where  $\mathbb{H}^3$  is hyperbolic 3-space, and  $\Gamma \cong \pi_1(M)$  acts isometrically and freely properly discontinuously on  $\mathbb{H}^3$ . In general, one could construct lots of exotic examples, so it is natural to restrict to the case where  $\Gamma$  is finitely generated. In this case the Tameness Theorem tells us that  $M$  is homeomorphic to the interior of a compact manifold with (possibly empty) boundary. This is equivalent to saying that each end,  $e$ , of  $M$  has a neighbourhood homeomorphic to  $\Sigma \times [0, \infty)$ , where  $\Sigma = \Sigma(e)$  is a closed surface. In general the situation may be complicated by the existence of cusps, where a given curve in  $M$  can be made arbitrarily short by homotoping it out one of the ends. To simplify the discussion, we will assume in this introduction that  $M$  has no cusps. In this case, each  $\Sigma(e)$  has genus at least 2. A consequence of (the proof of) the Tameness Theorem is that we can associate to  $e$  an “end invariant”,  $a(e)$ . In fact, there are two cases.

Maybe  $e$  is “geometrically finite”. Here the geometry is relatively simple. The end expands exponentially, and can be naturally compactified by adjoining a Riemann surface. This surface determines an element of the Teichmüller space of  $\Sigma(e)$ , which is defined to be the end invariant in this case.

Otherwise  $e$  is “degenerate”. In this case we can find surfaces, homotopic to  $\Sigma(e) \times \{0\}$ , which go out the end, and have bounded area in the induced metric. The induced structure on these converge, in a certain sense, to a well defined “ending lamination”, which is defined to be the end invariant in this case. This latter can be loosely thought of as an ideal point of the Teichmüller space of  $\Sigma(e)$  (though that is not entirely accurate).

In the case where there are no cusps, the Ending Lamination Theorem tells us that  $M$  is determined up to isometry by its topology and its family of end invariants. (Figure 1.1.) There is a reasonably good understanding of the structure of end invariants, and so this gives a fairly satisfactory answer to the question of what such 3-manifolds look like. (Though of course, other questions remain open.) In particular, if  $M$  is closed, then there are no end invariants. Therefore, the Ending Lamination Theorem implies the Mostow Rigidity Theorem.

An important case is where  $M$  is homeomorphic to  $\Sigma \times \mathbb{R}$ . (In this case, Tameness is due to Bonahon [Bon].) One can very loosely think of the geometry of  $M$  determining a coarse path in the Teichmüller space of  $\Sigma$ , parameterised by the  $\mathbb{R}$ -coordinate. If both ends are degenerate, then this path is bi-infinite, and we can think of it as limiting on the respective ending laminations. Again, this is only very approximately true (though it can be made more precise in certain cases, see [Mil] for example). In this way, a connection can be made with the geometry of Teichmüller space. Indeed, understanding this particular case, is in some sense, the key to understanding the Ending Lamination Theorem in general.

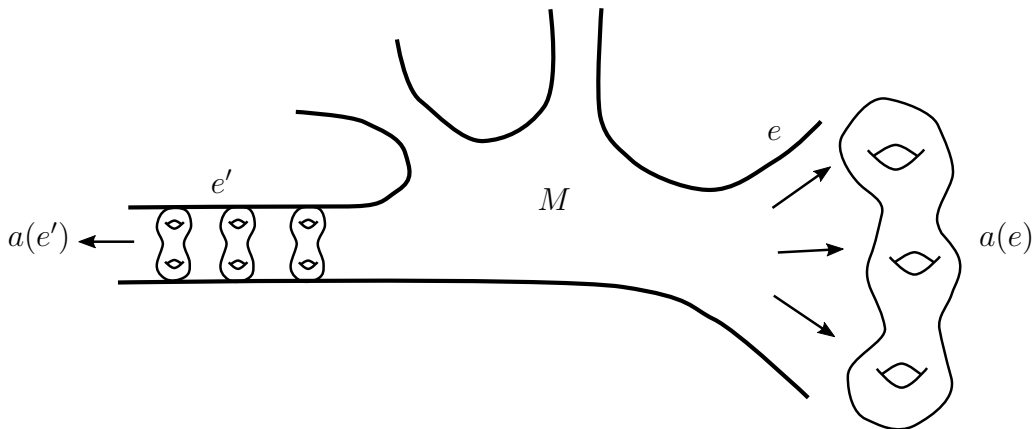


FIGURE 1.1. Ends of a manifold,  $M$ , without cusps:  $e$  geometrically finite,  $e'$  degenerate.

The proof of the Ending Lamination Theorem here, as in [Mi4, BrocCM], works by constructing a model space. This is a riemannian manifold,  $P$ , homeomorphic to  $M$ , whose construction depends only on the data given by the (degenerate) end invariants. One then constructs a bilipschitz map from  $P$  to  $M$ . If  $M'$  is a hyperbolic 3-manifold homeomorphic to  $M$  and has the same end invariants, we therefore get a bilipschitz map from  $M$  to  $M'$ , via  $P$ . Some standard results from the theory of quasiconformal maps allows us to promote this to an isometry. The construction of  $P$  is of intrinsic interest, since it makes much use of the geometry of curve complexes, and so one obtains new information about these spaces: both along the way and in retrospect.

The fact that the map from  $P$  to  $M$  is lipschitz, though highly non-trivial, is significantly simpler than showing that it has a lipschitz inverse. In fact, for the proof of the Ending Lamination Theorem, we only need a weaker statement, namely that the map lifts to a quasi-isometry of universal covers. This is what we prove first. The promotion to a bilipschitz map is given in Chapter 4, which is not directly relevant to the rest of the paper.

As we have indicated, the discussion is somewhat simpler in the indecomposable case (see Section 1.4 for a definition). It was this case that was dealt with in [BrocCM], and we will discuss that case first.

The outline of this paper is as follows. In the Section 1.2, we give a first informal statement of the Ending Lamination Theorem. In the remainder of Chapter 1, we describe the relevant background, outline the main results, and describe the ingredients of the proof. In Sections 2.2 to 2.14, we proceed to a proof of the Ending Lamination Theorem in the indecomposable case. In Chapter 3, we describe the modifications necessary to deal with the general case. In Chapter 4, we explain how the map from the model space can be promoted to be bilipschitz.

In Section 5.1, we give an account of the Uniform Injectivity Theorem applicable to our situation.

## 1.2. AN INFORMAL SUMMARY OF THE ENDING LAMINATION THEOREM.

As noted in Section 1.1, the Ending Lamination Theorem is a major component of the classification of finitely generated Kleinian groups, or equivalently complete hyperbolic 3-manifolds with finitely generated fundamental group. It can be viewed as the “uniqueness” part of the classification. It shows that such manifolds are determined by their “end invariants”. The other main components of the classification are the Tameness Theorem [Bon, Ag, CalG], and the existence of manifolds with prescribed end invariants (see [Oh] and [NS]). (The latter is also key to the Density Theorem which asserts that all such Kleinian groups are limits of geometrically finite groups.)

Since the original papers of Minsky, Brock and Canary [Mi4, BrocCM], a number of other approaches to the Ending Lamination Theorem have been proposed, for example, by Brock, Bromberg, Evans and Souto (unpublished), and by Soma [So2].

We proceed with an informal statement of the Ending Lamination Theorem. For a more precise statement, see Theorem 1.5.4. We will assume some background knowledge of hyperbolic geometry, which will be discussed further in Sections 1.3 and 1.4.

Let  $M$  be a complete hyperbolic 3-manifold with  $\pi_1(M)$  finitely generated. The “thin part” of  $M$  is the (open) subset where the injectivity radius is less than some sufficiently small fixed “Margulis” constant. The unbounded components of the thin part form a (possibly empty) finite set of “cusps” of  $M$ . These cusps are of two types, depending on whether the boundary is a torus or a bi-infinite cylinder. These will be referred to respectively as  $(\mathbb{Z} \oplus \mathbb{Z})$ -*cusps* and  $\mathbb{Z}$ -*cusps* (based on their fundamental groups). Removing the interiors of all these cusps, we obtain the *non-cuspidal* part,  $\Psi = \Psi(M)$  of  $M$ . The Tameness Theorem tells us that  $\Psi$  is topologically finite. This means that there is a compact manifold,  $\bar{\Psi}$ , with boundary  $\partial\bar{\Psi}$ , and a closed subsurface  $\partial_I\bar{\Psi} \subseteq \partial\bar{\Psi}$ , such that  $\Psi(M)$  is homeomorphic to  $\bar{\Psi} \setminus \partial_I\bar{\Psi}$ . We will usually abbreviate  $\partial_I\bar{\Psi}$  to  $\partial_I\Psi$ . Fixing some such (proper homotopy class of) homeomorphism, we can identify  $\partial\Psi(M)$  with  $\partial_V\Psi = \partial\bar{\Psi} \setminus \partial_I\bar{\Psi}$ , which we refer to as the *vertical boundary* of  $\Psi$ . Each torus component of  $\partial\bar{\Psi}$  bounds a cusp of  $M$ , and does not meet  $\partial_I\Psi$ . All other components of  $\partial\bar{\Psi}$  have genus at least 2. (Figure 1.2.)

Note that the ends of  $\Psi$  are in bijective correspondence with the components of  $\partial_I\Psi$ . Each end  $e$  has a neighbourhood homeomorphic to  $\Sigma \times [0, \infty)$ , where  $\Sigma = \Sigma(e)$  is such a component. Note that this meets  $\partial_V\Psi$  in  $\partial\Sigma \times [0, \infty)$ . Associated to each such end we have a geometric “end invariant”, which is either a

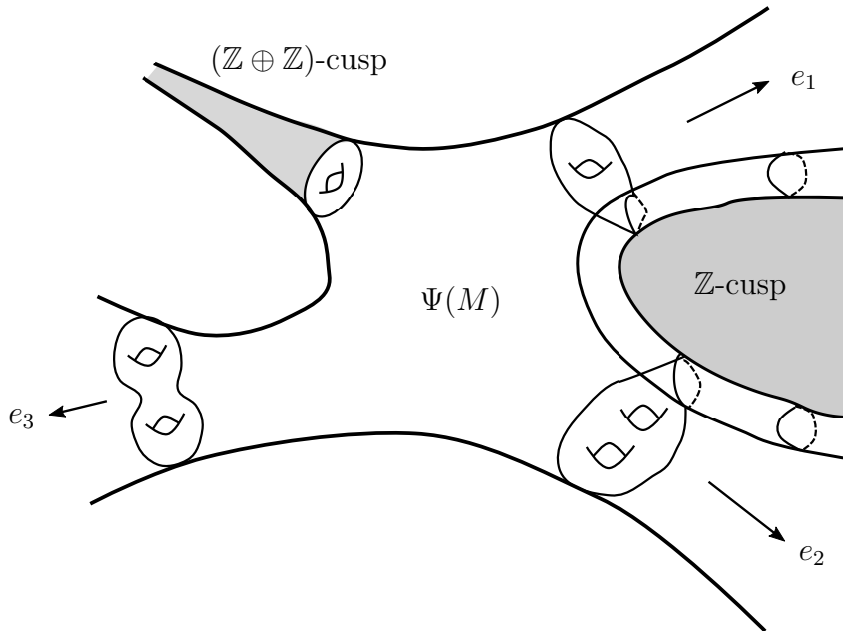


FIGURE 1.2. Here the non-cuspidal part,  $\Psi(M)$ , had three ends:  $e_1$ ,  $e_2$  and  $e_3$ . The manifold  $M$  itself also has three ends: the  $(\mathbb{Z} \oplus \mathbb{Z})$ -cusp,  $e_3$ , and a genus-3 end containing  $e_1$ ,  $e_2$  and the  $\mathbb{Z}$ -cusp.

Riemann surface (for a “geometrically finite” end) or a geodesic lamination (for a “degenerate” end). The Ending Lamination Theorem asserts:

**Theorem 1.2.1.**  *$M$  is determined up to isometry by the topology of its non-cuspidal part,  $\Psi(M)$ , together with its end invariants.*

Implicit in this is a preferred proper homotopy class of homeomorphism of  $\Psi$  with a given topological model. This gives rise to “markings” of the end invariants which we assume to be part of the data. The proper homotopy class of the isometry of the end will respect these markings. (In the “indecomposable case” proper homotopy and homotopy turn out to be essentially equivalent for the purposes of our discussion.)

As we mentioned in Section 1.1, one can get a simpler picture by considering the case where  $M$  has no cusps, so that  $\Psi(M) = M$ . In this case,  $\bar{\Psi}$  is a compactification of  $M$ , obtained by adjoining a surface (of genus at least 2) to each end of  $M$ . These surfaces are just the components of  $\partial\Psi$ , and each has an end invariant associated to it.

The above will be discussed in more detail in Section 1.5. A more formal statement of the Ending Lamination Theorem is given as Theorem 1.5.4.

The statement and proof are somewhat simpler in the indecomposable case. This means that  $\partial_I\Psi$  is incompressible. In other words, if a curve in  $\partial_I\Psi$  bounds



a disc in  $M$ , then it bounds a disc in  $\partial_I \Psi$ . We will say more about this in Section 1.4.

Particular cases of the Ending Lamination Theorem were known before the work of Minsky et al. If  $M$  is closed, so that  $\Psi(M) = M$  and there are no end invariants, then  $M$  is determined by its topology. This is Mostow rigidity [Most] in dimension 3. The same applies in the more general situation, where  $M$  has finite volume, so that each boundary component of  $\Psi(M)$  is a torus corresponding to a cusp of  $M$ . Again,  $M$  is determined by its topology. This was shown in [Mar1] and [Pr]. More generally still, if  $M$  is geometrically finite (i.e. all its ends are geometrically finite) then the Ending Lamination Theorem is shown in [Mar1]. (This used arguments similar to those of Section 2.14 here.) Indeed there was a complete classification in this case, following from the deformation theory of Ahlfors, Bers, Marden, Maskit etc. (see [Mar1]).

As noted in Section 1.1, the proof of the Ending Lamination Theorem proceeds by comparing the geometry of the hyperbolic 3-manifold with that of a riemannian “model space”, constructed using only its topological structure and set of end invariants. This model captures the large scale geometry of the hyperbolic manifold. A key step in relating the model to the hyperbolic manifold is the A-priori Bounds Theorem of Minsky [Mi4] (in the indecomposable case) which bounds the length of certain closed geodesics in the hyperbolic manifold.

Our account of the Ending Lamination Theorem broadly follows the strategy of the original, though the logic is somewhat different. Notably, we take the A-priori Bounds Theorem as a starting point, rather than as a result embedded in the proof. An independent argument for this is given in [Bow3]. (See also [So1] for some simplifications of this argument.) We will also need a version for the decomposable case, which we discuss in Section 3.5. The model spaces we use are essentially the same as those in [Mi4], though we give a combinatorial description that bypasses much of the theory of hierarchies as developed in [MasM2]. (In particular, our account makes no explicit use of “subsurface projections” as developed in [MasM2], though these are central to the more general theory.)

We shall first prove the theorem in the indecomposable case. Some additional ingredients will be needed for the general case, mainly to give a proper description of the end invariants, and to “isolate” an end of  $\Psi$  from the “core” of the manifold. Apart from that, it will only call for reinterpreting certain constructions.

We will present the main part of the argument in the specific context of a doubly degenerate manifold, namely, where  $\Psi(M)$  is a topological product,  $\Psi(M) \cong \Sigma \times \mathbb{R}$ , for a compact surface,  $\Sigma$ , and both ends are degenerate. We do this for several reasons.

Firstly it greatly simplifies the exposition. Most of the main ideas can be seen in this context. What remains for the general indecomposable case is largely a matter of describing how the various bits fit together in a more complicated situation.

Secondly, these ideas have further applications to Teichmüller theory and the geometry of the curve complex, etc. As far as these are concerned, one only really needs to consider such product manifolds. In the case of a doubly degenerate group, we get a somewhat cleaner, and stronger statement. In particular, one can show that the quasi-isometry (or bilipschitz) constants are uniform, in that they depend only on the topology of the base surface. (A similar uniformity in this case is obtained in [BroCM].) This is lost (at least without more a lot work) in the general indecomposable case.

A third, though relatively minor, reason is that one is obliged to give some special consideration to the doubly degenerate case, since there we have to check that each end of the model gets sent to “right” end of  $M$  — a fact that is automatic from the topology in all other situations. (Indeed, precisely this issue caused Bonahon a certain amount of strife in [Bon].)

Our proof of the Ending Lamination Theorem proceeds by showing that there is a Lipschitz map from the model space to the hyperbolic 3-manifold, which lifts to a quasi-isometry of universal covers. This is sufficient to apply the relevant results from complex analysis. In Chapter 4 we explain how the map can be promoted to a bilipschitz map (as was achieved in [Mi4, BroCM] in the indecomposable case). For the purposes of proving the main result, this is not needed. However, the existence of a bilipschitz map is useful in other contexts.

### 1.3. BASIC NOTIONS.

In this section we briefly summarise some of the background material used at various points of the paper. Elaborations of these notions will be given as appropriate when they are used.

We first consider group actions on hyperbolic  $n$ -space,  $\mathbb{H}^n$ . For general discussion, see [Bea, Ra], or [Mar2] more specifically for  $n = 3$ .

Suppose that  $\Gamma$  a group acting freely properly discontinuously and isometrically on  $\mathbb{H}^n$ . The quotient,  $M = \mathbb{H}^n/\Gamma$  is a complete hyperbolic  $n$ -manifold, with  $\Gamma \cong \pi_1(M)$ . Indeed every complete hyperbolic  $n$ -manifold arises in this way. The elements of  $\Gamma$  can be classified as *loxodromic* or *parabolic*. In the former case, its conjugacy class gives rise to a closed geodesic in  $M$ , which is the quotient of the bi-infinite geodesic axis in  $\mathbb{H}^n$ . In the latter case, the conjugacy class corresponds to a cusp of  $M$ : in particular, it can be represented by arbitrarily short closed curves in  $M$ .

Recall that  $\mathbb{H}^n$  has a natural compactification,  $\mathbb{H}^n \cup \partial\mathbb{H}^n$ , to a topological ball, by adjoining the ideal sphere,  $\partial\mathbb{H}^n$ . The action of  $\Gamma$  on  $\mathbb{H}^n$  extends to an action by homeomorphism on  $\mathbb{H}^n \cup \partial\mathbb{H}^n$ . This gives rise to a partition of  $\partial\mathbb{H}^n$  into the limit set,  $L(\Gamma)$ , and discontinuity domain,  $D(\Gamma)$ . This is the set of accumulation points of some (hence any)  $\Gamma$ -orbit in  $\mathbb{H}^n$ . One can show that  $\Gamma$  acts properly discontinuously on  $\mathbb{H}^n \cup D(\Gamma)$ . (The limit set and discontinuity

domain are commonly denoted “ $\Lambda$ ” and “ $\Omega$ ” respectively, but we will be using these symbols for other purposes.)

Write  $\text{hull}(L(\Gamma))$  for the intersection with  $\mathbb{H}^n$  of the convex hull of  $L(\Gamma)$  in  $\mathbb{H}^n \cup \partial\mathbb{H}^n$ . The quotient,  $\text{core}(M) = \text{hull}(L(\Gamma))/\Gamma$ , is the **convex core** of  $M$ . This is a closed submanifold of  $M$ , whose inclusion into  $M$  is a homotopy equivalence. Except in very special cases (when  $L(\Gamma)$  lies in a lower-dimensional round sphere in  $\partial\mathbb{H}^n$ ), this has codimension 0.

Given  $x \in M$ , write  $\text{sys}(M, x)$  for the length of the shortest essential curve in  $M$  containing  $x$ . This is the **systole** of  $M$  at  $x$ . It is exactly twice the injectivity radius at  $x$ . We define the **systole** of  $M$  by  $\text{sys}(M) = \inf\{\text{sys}(M, x) \mid x \in M\}$ .

Given  $\eta > 0$ , let  $\Theta_\eta(M) = \{x \in M \mid \text{sys}(M, x) \leq \eta\}$ . Now,  $\Theta_\eta(M)$  is closed in  $M$ , and is referred to as the  $\eta$ -**thick part** of  $M$ . The closure of  $M \setminus \Theta_\eta(M)$  is referred to as the  $\eta$ -**thin part**. If  $\eta$  is chosen less than some fixed “Margulis constant”, depending only on  $n$ , then the thin part is a disjoint union of relatively simple “Margulis tubes” and “Margulis cusps”. The former are always compact. We refer to  $M$  with the interiors of the Margulis cusps removed as the  $\eta$ -**noncuspidal part** of  $M$ .

If  $n = 2$ , then each Margulis tube is a topological annulus, with “core curve” a closed geodesic. A Margulis cusp is properly embedded copy of  $S^1 \times [0, \infty)$ , isometric to the quotient of a horodisc in  $\mathbb{H}^2$ .

If  $n = 3$ , then each Margulis tube is a solid torus, again with “core curve” a closed geodesic. Their intrinsic geometry is discussed in Section 2.6. A Margulis cusp has one of two forms. It may be a  $\mathbb{Z}$ -**cusps**, that is, a properly embedded copy of  $S^1 \times \mathbb{R} \times [0, \infty)$ ; or it may be a  $(\mathbb{Z} \oplus \mathbb{Z})$ -**cusps**, that is, a properly embedded copy of  $S^1 \times S^1 \times [0, \infty)$ . Their boundaries are intrinsically a bi-infinite euclidean cylinder and a euclidean torus, respectively.

Given a compact orientable surface,  $\Sigma$ , we write  $\xi(\Sigma) = 3g + p - 3$  for the **complexity** of  $\Sigma$ , where  $g$  is the genus, and  $p$  is the number of boundary components. If  $\xi(\Sigma) > 0$ , we write  $\mathbb{T}(\Sigma)$  for the Teichmüller space of  $\Sigma$ . This is the space of marked complete finite-area hyperbolic structures on  $\text{int } \Sigma = \Sigma \setminus \partial\Sigma$ . Equivalently it is the space of finite-type conformal structures on  $\text{int}(\Sigma)$ . (Here, “finite-type” means compact with finitely many punctures.) Note that the mapping class group,  $\text{Map}(\Sigma)$ , acts on  $\mathbb{T}(\Sigma)$  with quotient the moduli space of unmarked structures. It turns out that  $\mathbb{T}(\Sigma)$  is homeomorphic to  $\mathbb{R}^{2\xi(\Sigma)}$ , and has a natural compactification as a  $(2\xi(\Sigma))$ -ball, obtained by adjoining the Thurston boundary,  $\partial\mathbb{T}(\Sigma)$ . This boundary can be identified with the space of projective laminations on  $\Sigma$ . This will be discussed further in Section 1.5. One general reference to Teichmüller theory is [IT].

Note that  $\partial\mathbb{H}^3$  has a natural conformal structure isomorphic to the Riemann sphere. Suppose that  $U \subseteq \partial\mathbb{H}^3$  is an open set with  $|\partial\mathbb{H}^3 \setminus U| \geq 3$ . The Uniformisation Theorem tells us that  $U$  admits a natural complete hyperbolic metric, referred to as the **Poincaré metric**. If  $U$  is simply connected (equivalently if  $\mathbb{H}^3 \setminus U$

is connected) then  $U$  is isometric to  $\mathbb{H}^2$ . This metric will be used to deal with “geometrically finite” ends, and is discussed further in Sections 2.14 and 3.8. To estimate the Poincaré metric, we will make use of the Koebe Quarter Theorem. This says that if  $f : D \rightarrow \mathbb{C}$  is an injective conformal map of the unit disc to the complex plane with  $|f'(0)| \geq 1$ , then  $f(D)$  contains a disc of radius  $1/4$  centred on  $f(0)$ . (Here the norm and radius are measured in the euclidean metric on  $\mathbb{C}$ .) We will also use some basic facts about quasiconformal mappings (see [LehV]). In particular, we note that a quasiconformal map which is conformal almost everywhere (with respect to Lebesgue measure) is conformal.

Throughout this paper we will be dealing with “geodesic spaces”. Let  $(X, d)$  be a metric space. We will write  $N(x, r) = \{y \in X \mid d(x, y) \leq r\}$ . A map  $\gamma : I \rightarrow X$ , from an interval,  $I \subseteq \mathbb{R}$ , is **geodesic** if  $d(\gamma(t), \gamma(u)) = |t - u|$  for all  $t, u \in I$ . We will sometimes abuse terminology, and refer to  $\gamma(I) \subseteq X$  as a “geodesic”. We say that  $X$  is a **geodesic space** if any pair of points of  $X$  are connected by a geodesic. This applies to  $\mathbb{H}^n$ , and indeed to any complete riemannian manifold with the induced path-metric (by the Hopf-Rinow Theorem).

*Remark.* Note that here “geodesic” is being used in a different sense to the riemannian notion. The latter is essentially equivalent to being locally geodesic in the above sense. When we earlier referred to “closed geodesics” in a hyperbolic manifold, we meant in the riemannian sense. Throughout this paper, we will use the term “geodesic” always in the stronger metric sense, except when talking about a “closed geodesic” in a riemannian manifold (or singular polyhedral space). The meaning should be clear from context.

Let  $(X, d)$  and  $(X', d')$  be geodesic spaces. The following is a key notion:

**Definition.** A map  $\phi : X \rightarrow X'$  is a **quasi-isometric embedding** (or just “quasi-isometric”) if there exist constants,  $k_1 > 0, k_2, k_3, k_4 \geq 0$  such that for all  $x \in X$  we have

$$k_1 d(x, y) - k_2 \leq d'(\phi x, \phi y) \leq k_3 d(x, y) + k_4.$$

If, in addition, there is some  $k_5 \geq 0$  with  $X' = N(\phi(X), k_5)$ , we say that  $\phi$  is a **quasi-isometry**

We refer to the constants  $k_i$  collectively as the “constants” or “parameters” of quasi-isometry. We say that  $X$  and  $X'$  are **quasi-isometric** if there exists a quasi-isometry between them. One can check that this is an equivalence relation, so it makes sense to talk about quasi-isometry classes.

Note that we do not assume a quasi-isometric embedding to be continuous. However, in certain cases, it can be approximated up to bounded distance by a continuous map. This is the case, for example, if the domain is an interval,  $I \subseteq \mathbb{R}$ . Indeed, in this case, a quasi-isometric map  $\beta : I \rightarrow X$ , can be assumed, up to bounded distance to satisfy  $\text{length}(\beta|[t, u]) \leq h d(\beta(t), \beta(u)) + h'$ , for all  $t, u \in I$ , for constants  $h, h' \geq 0$  (depending only on the original constants of

quasi-isometry). We refer to a rectifiable path satisfying the above condition as a *quasigeodesic*. As with geodesics, we sometimes abuse notation by identifying a path with its image as subset of  $X$  (even if the path is not injective).

Recall that a geodesic space,  $X$ , is **(Gromov) hyperbolic** if there is a constant,  $k \geq 0$ , such that every geodesic triangle in  $X$  has a “ $k$ -centre”: that is a point distant at most  $k$  from each of its sides. To any geodesic space,  $X$ , we can canonically associate an ideal boundary,  $\partial X$ . This can be thought of as the set of parallel classes of quasi-geodesic rays in  $X$ . (In fact, for the hyperbolic spaces we deal with in this paper, one could equivalently use geodesic rays, though this is not the case in general.) Hyperbolicity is invariant under quasi-isometry. Moreover, any quasi-isometry,  $\phi : X \rightarrow X'$ , between hyperbolic spaces induces a homeomorphism from  $\partial X$  to  $\partial X'$ . The main cases of interest in this paper are  $\mathbb{H}^n$  (where  $\partial \mathbb{H}^n$  is the usual ideal boundary), as well as the curve graphs associated to surfaces (as discussed in Section 1.5).

A key fact about hyperbolic spaces is the following stability result for quasi-geodesics. If  $\alpha$  is a geodesic, and  $\beta$  is a quasigeodesic in a  $k$ -hyperbolic space  $X$  with the same endpoints, then the Hausdorff distance between  $\alpha$  and  $\beta$  is bounded above by a constant depending only on  $k$  and the constants,  $h, h'$  of the quasigeodesic,  $\beta$ . A refinement of this statement is given as Proposition 1.6.10 here.

Another class of metrics are used to deal with compressible ends. These are the locally  $\text{CAT}(k)$  spaces. Such a space is a geodesic metric space satisfying a certain comparison axiom locally, which is intended to capture the notion of the space having “curvature at most  $k$ ”. All the  $\text{CAT}(k)$  spaces used here will be proper (that is, complete and locally compact). The main interest is when  $k = -1$ . An example is  $\mathbb{H}^n$ . Given a free properly discontinuous group action on such a space, one can classify its non-trivial elements as loxodromic or parabolic, similarly as for  $\mathbb{H}^n$ . We discuss this further in Section 3.3.

Further background to hyperbolic and  $\text{CAT}(k)$  spaces, quasi-isometries etc. can be found in [GhH, BridH].

Finally, we will also, of course, be making much use of basic 3-manifold topology. Some of the relevant background will be discussed in the next section, and other aspects will be explained when they are used. Some references to the topological theory can be found in [Hem1, Ja], and discussion of hyperbolic 3-manifolds in [Ka, Mar2]. In particular, we will be making use of standard results such as Dehn’s Lemma and the Sphere Theorem.

We should make the general observation that in dimensions at most 3, the topological, PL and smooth categories are essentially equivalent (see, for example [Moi]). As a consequence, one can freely pass between them. We will typically assume that maps defined by any topological construction are “tame”, in the sense that they are smooth or PL, and in general position. We will not delve into

the technical issues behind this. More formal justifications can be found in the standard references.

In Section 4.4, we will make use of the fact that bounded geometry riemannian manifolds are uniformly triangulable [BoiDG, Bow10], though this is not needed for anything we discuss before then.

#### 1.4. BACKGROUND TO 3-MANIFOLDS.

We give a summary of the main ideas behind the Ending Lamination Theorem and the classification of finitely generated kleinian groups. We include some historical background, though our account is not strictly chronological. Much of the discussion of this section is not logically essential to understanding the statement or proof as presented in this paper. The bits that are will be reviewed again later. In particular, a more formal discussion of end invariants will be given in Section 1.5. In what follows, a *Kleinian group* will be a group acting properly discontinuously on  $\mathbb{H}^3$ .

Before the late 1970s, much of the theory of 3-manifolds and of Kleinian groups had developed separately. Prior to this, most major results of 3-manifold theory were based on combinatorial or topological techniques. General accounts of the topological theory of 3-manifolds can be found in [Hem1] and [Ja]. Meanwhile, the theory of Kleinian groups tended to use analytical machinery, focusing on the action of the group on the Riemann sphere. An account of the state of the art with regard to Kleinian groups around this time can be found in [BerK]. In the background, though never fully exploited, was hyperbolic geometry arising from the fact that a kleinian group acts properly discontinuously on hyperbolic 3-space, so that the quotient (in the torsion-free case) is a hyperbolic 3-manifold. It was gradually recognised, through work of Riley, Jørgensen and others, that hyperbolic structures were natural and commonplace in the world of 3-manifolds (see for example [Ri, Jo]). Marden was one of the first to bring 3-manifold theory properly into play in the theory of Kleinian groups, notably through the seminal work of [Mar1]. Then, in the late 1970s, the subject was revolutionised through work of Thurston [Th1], as mentioned in Section 1.1.

A key topological result is the Scott Core Theorem [Sc1, Sc2] and a generalisation thereof to the relative case due to McCullough [Mc]. The latter states:

**Theorem 1.4.1.** *Suppose  $\Psi$  is a 3-manifold with (possibly empty) boundary,  $\partial\Psi$  and  $F \subseteq \partial\Psi$  be any compact subset. Then there is a compact connected submanifold  $\Psi_0 \subseteq \Psi$ , with  $\Psi_0 \cap \partial\Psi = F$ , such that the induced map from  $\pi_1(\Psi_0)$  to  $\pi_1(\Psi)$  is an isomorphism.*

We refer to  $\Psi_0$  as a “compact core”.

Suppose now that  $M$  is a complete hyperbolic 3-manifold with  $\Gamma = \pi_1(M)$  finitely generated. Then  $M = \mathbb{H}^3/\Gamma$ , where  $\Gamma \equiv \pi_1(M)$  acts properly discontinuously on hyperbolic 3-space. The action extends to the ideal sphere  $\partial\mathbb{H}^3$ , which

decomposes as the limit set,  $L(\Gamma)$ , and the discontinuity domain,  $D(\Gamma)$ . We write  $R(M)$  for the (possibly empty) quotient surface  $D(\Gamma)/\Gamma$ . Note that  $\Gamma$  acts by Möbius transformations on  $\partial\mathbb{H}^3$ , and hence by conformal isometries on  $D(\Gamma)$ . It follows that  $R(M)$  has a natural complex structure. Ahlfors's Finiteness Theorem [Ah] says:

**Theorem 1.4.2.**  *$R(M)$  is a finite disjoint union of Riemann surfaces of finite type.*

By “finite type” we mean compact with finitely many punctures (possibly none).

Let  $\Psi = \Psi(M)$  be the non-cuspidal part of  $M$ . One can show that  $\partial\Psi$  has finitely many components, each an annulus or torus. We can thus find a compact core  $\Psi_0 \subseteq \Psi$  which includes each torus component of  $\partial\Psi$  and which meets each annular component of  $\partial\Psi$  in a compact annular core. (This is a consequence of Theorem 1.4.1, together with a simple Euler characteristic argument.) The ends of  $\Psi$  are in bijective correspondence with the components of  $\partial\Psi_0 \setminus \partial\Psi$ . Such an end is **geometrically finite** if it has a neighbourhood which meets no closed geodesic in  $M$ . One can show that the geometry of such an end is relatively simple. In particular, it can be constructed from a finite-sided polyhedron by carrying out side identifications. (Though that is not necessarily the most natural way of viewing it.) The geometrically finite ends are in bijective correspondence with the components of  $R(M)$ . Indeed any component,  $S$ , of  $R(M)$  admits a natural homotopy equivalence to the corresponding component of  $\partial\Psi_0 \setminus \partial\Psi$ . This determines a marking of  $S$ , and hence identifies  $S$  as a point in the Teichmüller of this surface (at least in the indecomposable case).

**Definition.** We say that  $M$  is **geometrically finite** if each end of  $\Psi(M)$  is geometrically finite.

This can be reformulated in a number of equivalent ways. For example, it is equivalent to saying that  $M$  has a finite-sided fundamental polyhedron.

The deformation theory of geometrically finite manifolds has been well understood for some time, due the work of Ahlfors, Bers, Marden, Maskit and others (see [Mar1], and references therein). In particular, we have:

**Theorem 1.4.3.** *If  $M$  is geometrically finite, then it is determined up to isometry by the topological type of  $\Psi_0$ , and the collection of end invariants,  $R(M)$  (viewed as elements of Teichmüller space).*

The above two theorems, and their proofs, are essentially analytic in nature. Gradually the theory of 3-manifolds was brought into play, notably through work of Marden. In particular, in [Mar1], Marden asked if every complete hyperbolic 3-manifold,  $M$ , with finitely generated fundamental group is topologically finite, that is, homeomorphic to the interior of a compact 3-manifold with boundary.

This question was given a geometric reinterpretation by Thurston. He defined the notion of a “simply degenerate” end of  $\Psi$ . In these terms,  $M$  is said to be **tame**

if each of its ends is either geometrically finite or simply degenerate. Using work of [BrinT] he showed that any tame manifold is topologically finite. He asked if any complete hyperbolic 3-manifold with finitely generated fundamental group is tame — the “Tameness Conjecture”. He also defined an end invariant associated to each simply degenerate end. Such an invariant is a geodesic lamination [Th1, CasB]. (To first approximation, a geodesic lamination can be thought of as an ideal point of Teichmüller space, though this is not completely accurate: see Section 1.5.) Thurston asked whether every tame hyperbolic 3-manifold was determined by its topology and its end invariants — the “Ending Lamination Conjecture”.

It turns out to be simpler to study the case of “indecomposable” manifolds. This can be defined algebraically in terms of  $\Gamma = \pi_1(M)$  and the collection of parabolic subgroups arising from its action on  $\mathbb{H}^3$ .

**Definition.** We say that  $M$  is *indecomposable* if, for every decomposition of  $\Gamma$  as a free product,  $\Gamma \cong A * B$ , there is a parabolic subgroup of  $\Gamma$  (acting on  $\mathbb{H}^3$ ) that cannot be conjugated into either  $A$  or  $B$ .

(Clearly if  $\Gamma$  has no non-trivial free product decomposition at all, then  $M$  is indecomposable.)

In [Bon], Bonahon proved Marden’s conjecture for indecomposable manifolds. Using this work, Canary [Cana] proved a converse to Thurston’s result, thereby showing that tameness was equivalent to topological finiteness. After this, Marden’s conjecture became largely synonymous with the Tameness Conjecture.

The Tameness Conjecture was finally proven independently by Agol [Ag] and Calegari and Gabai [CalG]. (See also [So1, Bow6] for other accounts.) We therefore have: We therefore have:

**Theorem 1.4.4.** *Let  $M$  be a complete hyperbolic 3-manifold with  $\pi_1(M)$  finitely generated. Then the non-cuspidal part,  $\Psi(M)$ , is topologically finite.*

It follows that  $M$  is also topologically finite.

In fact,  $\Psi$  is homeomorphic to the relative interior of the compact core  $\Psi_0$ . We can embed  $\Psi$  into a compact manifold,  $\bar{\Psi} \cong \Psi_0$  in such a way that  $\bar{\Psi} \cap \partial\Psi$  gets identified with  $\Psi_0 \cap \partial\Psi$ .

Thus, in retrospect, in describing the end invariants of  $\Psi$ , we can replace  $\Psi_0$  by  $\bar{\Psi}$ , and subsequently forget about  $\Psi_0$ . In fact, in our discussion, we will have no formal use of the Scott Core Theorem, though, of course, it remains an essential ingredient in the proof of tameness. As before, we write  $\partial_V \bar{\Psi} = \Psi \cap \partial\Psi$ . We note that, via the Dehn Lemma of Papakyriakopoulos [Hem1], there is an equivalent topological formulation of indecomposability. This says that there is no disc in  $\bar{\Psi}$  whose boundary is an essential curve in  $\partial_I \bar{\Psi}$ . (This was the way we described it in Section 1.1.)

Prior to the proof of tameness in the general case, Minsky had already made significant progress towards the Ending Lamination Conjecture. He proved it in the special case of manifolds of bounded geometry (where the injectivity radius



is bounded below) [Mi1] and for punctured torus groups [Mi3]. The general indecomposable case was finally dealt with in [Mi4, BrocCM]. They also announced the result for the decomposable case. While it was generally accepted that the decomposable case should indeed be amenable to an elaboration of these arguments, no formal proof based on that work materialised.

The overall strategy of the above proofs are similar. Based on the topological data and end invariants, one constructs a “model space”, which is a riemannian manifold,  $P$ . One then constructs a bilipschitz map from  $P$  to  $M$ . Given another hyperbolic manifold,  $M'$ , diffeomorphic to  $M$ , and with the same end invariants, one obtains, via  $P$ , a bilipschitz map from  $M$  to  $M'$ . This gives rise to a bilipschitz map between the universal covers, each isometric to  $\mathbb{H}^3$ , that is equivariant with respect to the respective actions of  $\pi_1(M)$  and  $\pi_1(M')$ . The earlier deformation theory (as with the geometrically finite case) can now be brought into play to show that these actions are in fact conjugate by isometry of  $\mathbb{H}^3$ . It follows that  $M$  and  $M'$  are isometric.

In fact, we only really need an equivariant quasi-isometry of  $\mathbb{H}^3$  to make this work. This is how we approach it here. The promotion to a bilipschitz map will be treated as an afterthought in Chapter 4.

In fact, most of the work is involved in understanding the geometry of the degenerate ends of  $\Psi$ . We can thus effectively reduce (at least in the indecomposable case) to the case where  $\Psi$  is diffeomorphic to  $\Sigma \times \mathbb{R}$ , where  $\Sigma$  is a compact surface. This is in turn closely related to understanding the large scale geometry of the Teichmüller space of  $\Sigma$ . Indeed, in the bounded geometry case [Mi1], Minsky originally constructed a model out of a Teichmüller geodesic. (This is related to the singular sol geometry manifold used in [CannT]). Another viewpoint on this is discussed in [Mosh] and [Bow7], where a characterisation of geodesics in the thick part of Teichmüller space is described. (We will not be explicitly using that material in this paper.)

The general case [Mi4, BrocCM] uses a different construction of the model, based on Harvey’s curve complex [Har]. This relies on work of Masur and Minsky [MasM1, MasM2] (some of which will be discussed in Section 2.2). This work has many other applications.

The present work uses essentially the same model as Minsky, though described in a somewhat different way. (A further variation of this, with some simplifications, is described in [So2].) We take as starting point the A-priori Bounds Theorem of Minsky [Mi4], reproven by a more direct argument in [Bow3]. For the decomposable case we will need a generalisation of this statement, which we describe in Section 3.5.

A different proof, in the general case, based on surgery arguments to reduce to the geometrically finite case, has been proposed by Brock, Bromberg, Evans and Souto. Their idea also starts with an A-priori Bounds Theorem, though it avoids the use of models. However no written account of this has appeared.

The Ending Lamination Theorem does not complete the classification of finitely generated Kleinian groups. For this, one needs to describe the set of ending laminations that can arise for any given manifold. (There are no restrictions on the end invariants of geometrically finite ends.) For example, in the case where  $\Psi$  is a topological product,  $\Sigma \times \mathbb{R}$ , the only restriction is that we cannot have both end invariants equal to the same lamination. If  $\Psi$  is a handlebody (with  $\partial\Psi = \emptyset$ ) then the set permissible ending laminations is the ‘‘Masur domain’’ [Mas, Ot1] (see the discussion at the end of Section 1.6). The general case can be described by a generalisation of Masur’s construction. To construct a manifold with specified admissible end invariants, the general strategy is to take a sequence of geometrically finite manifolds whose end invariants tend to the prescribed ones, and prove that everything converges in an appropriate sense. This represents the culmination of work of many people (see for example [KleS, KiLO]). A general account of this can be found in [Oh, NS].

As mentioned earlier, the Ending Lamination Conjecture, and the methods involved in its proof, have many other applications. One particularly notable consequence is the Density Conjecture [BrocB, NS]:

**Theorem 1.4.5.** *Any finitely generated kleinian group is an algebraic limit of geometrically finite kleinian groups.*

This basically means that if  $M = \mathbb{H}^3/\Gamma$  is a hyperbolic 3-manifold, then there is a sequence,  $M_n = \mathbb{H}^3/\Gamma_n$ , of geometrically finite manifolds, and isomorphisms,  $\theta_n : \Gamma \rightarrow \Gamma_n$ , such that  $\theta_n(g)$  converges to  $g$  for all  $g \in \Gamma$ . (One can assume that each  $\theta_n$  sends parabolics to parabolics.)

The models can also be used to give a description of geometric limits of finitely generated kleinian groups. For product manifolds, this is described in [OhS].

Some applications of this technology outside the domain of Kleinian groups can be found in [Bow5] and [Ta].

## 1.5. END INVARIANTS.

In this section, we describe more carefully how end invariants are defined, and give a more precise statement of the Ending Lamination Theorem as Theorem 1.5.4.

The main case of interest will be that of ‘‘degenerate’’ ends, where the end invariant is a ‘‘lamination’’. Traditionally, this is viewed as a partial foliation of a surface. However, we will work with an equivalent formulation in terms of the boundary of the curve graph. The connection between these formulations will be discussed at the end of this section.

For basic properties of end invariant, we refer to [Th1] and [Bon], from which the statements here can be readily derived, at least in the incompressible case. We

will also give an account in Chapter 3 here, which is aimed generalising various results to the compressible case: it applies equally well to incompressible ends.

We begin by recalling particular spaces associated to a surface, namely the Teichmüller space and the curve graph.

Let  $\Sigma$  be a compact orientable surface. We recall the following from Section 1.3.

**Definition.** The *complexity*,  $\xi(\Sigma)$ , of  $\Sigma$  is defined to be  $\xi(\Sigma) = 3g + p - 3$ , where  $g$  is the genus and  $p$  is the number of boundary components.

One readily checks that  $\xi(\Sigma)$  equals the number of curves in  $\Sigma$  needed to cut it into a collection of 3-holed spheres.

The main significance of this quantity here is that it controls the topological type of  $\Sigma$ , and will be used in various induction arguments. We will always assume here that  $\xi(\Sigma) \geq 0$ . If  $\xi(\Sigma) = 0$ , then  $\Sigma$  is a three-holed sphere, abbreviated to “3HS”. If  $\xi(\Sigma) = 1$ , then  $\Sigma$  is either a one-holed torus or a four-holed sphere, abbreviated respectively to “1HT” and “4HS”.

We write  $\mathbb{T}(\Sigma)$  for the Teichmüller space of  $\Sigma$ . Here, we think of this as the space of marked finite-type conformal structures on  $\text{int}(\Sigma)$ .

We will define the end invariant of a simply degenerate end in terms of the curve graph — the 1-skeleton of the curve complex introduced by Harvey [Har]. Let  $\mathbf{C}(\Sigma)$  be the set of simple non-trivial non-peripheral closed curves in  $\Sigma$ , defined up to homotopy. We shall frequently refer to elements of  $\mathbf{C}(\Sigma)$  simply as “curves”. The curve graph,  $\mathcal{G} = \mathcal{G}(\Sigma)$  has vertex set  $\mathbf{C}(\Sigma)$ , and two curves are deemed to be adjacent if they have minimal possible intersection number in  $\Sigma$ . If  $\xi(\Sigma) > 1$  (the “non-exceptional cases”) the minimal possible intersection number is 0. In other words, two curves are deemed adjacent if they can be realised disjointly in  $\Sigma$ . For the exceptional cases of the 1HT and 4HS, the minimal possible intersection number is 1 and 2 respectively. In each of these cases, the curve graph,  $\mathcal{G}(\Sigma)$  is isomorphic to the Farey graph. In the case of the 3HS, the curve graph is empty.

It is easily seen that the Farey graph is hyperbolic in the sense of Gromov. In fact, a remarkable theorem of Masur and Minsky [MasM1] tells us that all curve graphs have this property:

**Theorem 1.5.1.** [MasM1] *If  $\Sigma$  is a compact surface with  $\xi(\Sigma) \geq 1$ , then  $\mathcal{G}(\Sigma)$  is Gromov hyperbolic.*

In fact, the curve graphs have since been shown to be uniformly hyperbolic; that is, the hyperbolicity constant can be taken to be independent of  $\xi(\Sigma)$  [Ao, Bow8, CIRS, HenPW]. However, this is not directly relevant to this paper, since we will only be dealing with finitely many surface types (namely subsurfaces of a given surface) at any given time.

One can associate to such a Gromov hyperbolic space its Gromov boundary,  $\partial\mathcal{G}(\Sigma)$ . It was shown in [Kla] that  $\partial\mathcal{G}(\Sigma)$  can be naturally identified with the space of “arational laminations”. The theory of laminations was introduced by Thurston

[Th1]. The general theory is discussed, for example, in the book [CasB]. The result of [Kla] provides the link with end invariants as they are more traditionally defined (as in [Th1, Bon] etc.). A different approach to this result can be found in [Ham]. In this paper, we will define an end invariant directly as an element of  $\partial\mathcal{G}(\Sigma)$ . Therefore we won't formally need the equivalence for our statement or proof of the Ending Lamination Theorem. Nevertheless, we will give some further background to this towards the end of this section.

We now move on to 3-manifolds. We need to clarify how we understand the ‘‘marking’’ of an end invariant. Suppose that  $\Psi$  is a 3-manifold with a topologically finite end,  $e$ . This means that there is a compact surface,  $\Sigma$ , and a proper injective map  $\theta : \Sigma \rightarrow \Psi$  so that  $\theta(\Sigma \times [0, \infty))$  is a neighbourhood of the end (and hence a homeomorphism to its range). If  $\theta' : \Sigma' \times [0, \infty) \rightarrow \Psi$  is another such map, then there is a canonically defined homotopy equivalence from  $\Sigma$  to  $\Sigma'$ . To describe this, take any  $t \geq 0$  large enough so that  $\theta(\Sigma \times \{t\}) \subseteq \theta'(\Sigma' \times [0, \infty))$ , and postcompose  $(\theta')^{-1} \circ \theta|_{(\Sigma \times \{t\})}$  with projection to  $\Sigma'$ . This has degree 1, and we get a similar map  $\Sigma' \rightarrow \Sigma$ . It follows that these are homotopy inverses. In particular, the map  $\Sigma \rightarrow \Sigma'$  is a homotopy equivalence of pairs,  $(\Sigma, \partial\Sigma)$  and  $(\Sigma', \partial\Sigma')$ . It follows in particular, that  $\Sigma$  and  $\Sigma'$  are homeomorphic. This gives us a basis for using  $e$  (thought of formally as a directed set of subsets of  $\Psi$ ) as a topological model for marking structures associated to  $\Sigma$ . In particular, we can define the Teichmüller space,  $\mathbb{T}(e)$  associated to  $e$  by canonically identifying it with  $\mathbb{T}(\Sigma)$  via  $\theta$ . Similarly, we define the curve graph,  $\mathcal{G}(e)$ , by identifying it with  $\mathcal{G}(\Sigma)$ . Note that any element of  $\mathbf{C}(e)$  can be realised as a curve in any neighbourhood of  $e$ . We refer to  $\Sigma = \Sigma(e)$  as the **base surface** of the end. (This implicitly implies a choice of map  $\theta$ . If we use a different map, then the identification changes by an element of the mapping class group of  $\Sigma$ . This induces automorphisms of  $\mathbb{T}(\Sigma)$  and of  $\mathcal{G}(\Sigma)$ . However,  $\mathbb{T}(e)$  and  $\mathcal{G}(e)$  remain unaltered.)

Note that if we embed  $\Psi$  in a compact manifold  $\bar{\Psi}$ , with  $\partial_V \bar{\Psi} = \bar{\Psi} \cap \partial\Psi$  and  $\partial_I \bar{\Psi} = \partial\Psi \setminus \partial_V \bar{\Psi}$ , as described in Section 1.2, then we can identify  $\Sigma(e)$  (at least up to homotopy) with a component of  $\partial_I \bar{\Psi}$ . We can therefore also regard this component of  $\partial_I \bar{\Psi}$  as a base surface for our marking. This ties in with the informal description given by Theorem 1.2.1. We refer to  $\bar{\Psi}$  as the **compactification** of  $\Psi$ .

Suppose we have a homeomorphism  $f : \Psi \rightarrow \Psi'$  between two such manifolds. This will associate to each end,  $e$ , of  $\Psi$ , an end of  $\Psi'$ , which we denote by  $f(e)$ . Moreover, there is a canonical homotopy equivalence between the base surfaces, respecting their peripheral structure. Thus we get induced isomorphisms  $f_* : \mathbb{T}(e) \rightarrow \mathbb{T}(f(e))$  and  $f_* : \mathcal{G}(e) \rightarrow \mathcal{G}(f(e))$ . In fact, we only require  $f$  to be a homotopy equivalence.

Now suppose that  $M$  is a complete hyperbolic 3-manifold, and that  $\Psi(M)$  is the non-cuspidal part with respect to some Margulis constant  $\eta > 0$ . For the purposes of this section, we can regard  $\eta$  as being fixed. The choice does not

really matter, since if we choose a smaller constant, then the difference between the two non-cuspidal parts will just be a topological product with a very simple geometrical structure. (In fact, it will be convenient later to allow  $\eta$  to depend on the topological type of  $\Psi$ .)

For the purposes of this paper, we shall make the following definition:

**Definition.** Let  $e \in \mathcal{E}(M)$ . We say that  $e$  is *geometrically finite* if there is a neighbourhood of the end in  $\Psi(M)$  which meets no closed geodesic in  $M$ . Otherwise, we say that  $e$  is *degenerate*.

We partition  $\mathcal{E}(M) = \mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$  into geometrically finite and degenerate ends accordingly.

In view of the Tameness Theorem, there are several other equivalent ways of describing these two types of ends. We begin by discussing the geometrically finite case. This has long been well understood, and the observations we make below are now standard.

Let  $e \in \mathcal{E}_F(M)$ . We can find a neighbourhood,  $E$ , of the end  $e$  in  $\Psi(M)$  and a homeomorphism of  $E$  with  $\Sigma(e) \times [0, \infty)$  such that each surface  $\Sigma(e) \times \{t\}$  is convex outwards. As a result,  $E$  has a fairly simple geometry. This is described in more detail in Section 2.14. (Also as observed in Section 1.2, we can construct a neighbourhood of the end by taking a finite-sided polyhedron and identifying faces. This justifies the terminology, but is not especially useful to us here.) Recall that we can write  $M = \mathbb{H}^3/\Gamma$  where  $\Gamma \equiv \pi_1(M)$ . Let  $R(M) = D(\Gamma)/\Gamma$  be the quotient of the discontinuity domain, as described in Section 1.2. Note that  $M \cup R(M)$  carries a quotient topology as  $(\mathbb{H}^3 \cup D(\Gamma))/\Gamma$ . In this topology, the closure of  $E$  is equal to  $E \cup R(e)$ , where  $R(e)$  is a component of  $R(M)$ . In fact, we can extend the above homeomorphism to a homeomorphism of  $E \cup R(e)$  with  $(\Sigma(e) \times [0, \infty]) \setminus (\partial\Sigma(e) \times \{\infty\})$ , where  $R(e)$  corresponds to  $(\Sigma(e) \setminus \partial\Sigma(e)) \times \{\infty\}$ . Throwing in the (possibly empty) union of curves  $\partial\Sigma \times \{\infty\}$  we get a compactification of  $E$  as  $\Sigma(e) \times [0, \infty]$ . (We can identify  $\Sigma(e) \times \{\infty\}$  with a component of  $\partial_\Gamma\Psi$ , and  $\Sigma(e) \times [0, \infty]$  with regular neighbourhood thereof in  $\Psi$ .) In this way, it is natural to view  $R(e)$  as describing the structure of  $e$  at infinity. Note that this comes equipped with a marking, as described earlier, so we get a well defined point,  $a(e) \in \mathbb{T}(e)$ .

The geometric structure of a degenerate end is much more subtle. Understanding it is the main task in proving the Ending Lamination Theorem. We shall start from the following property of a degenerate end. It follows directly from the description of simple degeneracy in [Th1], and elaborated on in [Bon]. We will give a general account of it in Chapter 3.

**Proposition 1.5.2.** *Suppose that  $e \in \mathcal{E}_D(M)$ . There is a sequence  $(\gamma_i)_i$  of elements of  $\mathbf{C}(e)$  which have representatives of length at most  $l_0$  in  $\Psi(M)$  and which tend out the end of  $e$ , where  $l_0$  depends only on  $\xi(\Sigma(e))$ .*

Here the phrase “tend out” means that, given any neighbourhood,  $E$ , of  $e$ , we have  $\gamma_i \subseteq E$  for all sufficiently large  $i$ . To define “representative” in the above, choose a neighbourhood  $E$  of  $e$  in  $\Psi(M)$  with a homeomorphism of  $E$  with  $\Sigma(e) \times [0, \infty)$ . Then any element of  $\mathcal{G}(e)$  determines a free homotopy class in  $\Sigma(e)$  hence in  $E$ . Note that can assume that  $\gamma_i$  lies in  $E$  for all  $i$ . (Figure 1.3.)

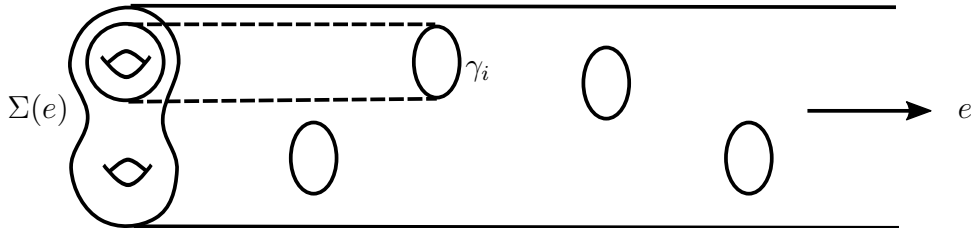


FIGURE 1.3. Curves realised in the end:  $\Sigma(e) \times [0, \infty) \hookrightarrow \Psi(M)$ .

In terms of the earlier terminology, the conclusion of the proposition means that a degenerate end is necessarily “simply degenerate”. The key point here is that each curve  $\gamma_i$  is represented by a simple closed curve in the surface  $\Sigma(e)$ . If we dropped the requirement that it be simple, then the statement would be significantly simpler. For further discussion of this, see Section 3.6 (in particular, Lemma 3.6.1).

In fact, we shall see that we could replace “representative” by “closed geodesic representative”, in the above statement.

It also turns out that the limit point of the sequence  $(\gamma_i)_i$  in  $\partial\mathcal{G}(e)$  is well defined. In fact we have the following [Th1, Bon]. It is an immediate consequence of Proposition 3.7.3 here.

**Proposition 1.5.3.** *There is some  $a \in \partial\mathcal{G}(e)$  such that if  $(\gamma_i)_i$  is any sequence with bounded length representatives going out the end  $e$ , then  $(\gamma_i)_i$  tends to  $a$  in  $\mathcal{G}(e) \cup \partial\mathcal{G}(e)$ .*

Here we can allow any bound on the length — it may depend on our sequence.

We postpone further comment on Propositions 1.5.2 and 1.5.3 until later in this section, and just note that they determine  $a$  uniquely, when  $e \in \mathcal{E}_D$ . We can therefore denote it by  $a(e)$ . (Figure 1.4.)

**Definition.** Given  $e \in \mathcal{E}(M)$ , we refer to  $a(e)$  as the *end invariant* of  $e$ .

Therefore, if  $e \in \mathcal{E}_F(M)$ , then  $a(e) \in \mathbb{T}(e)$  and if  $e \in \mathcal{E}_D(M)$ , then  $a(e) \in \partial\mathcal{G}(e)$ .

In the case where  $\Sigma(e)$  is a 3HS, then necessarily,  $e \in \mathcal{E}_F(M)$ . In fact, we can find a neighbourhood,  $E$ , of the end so that  $\partial E$  is a totally geodesic surface. (This is because a 3HS cannot admit any “bending” or “pleating” lamination: see [Th1, CanaEG].) Moreover,  $\mathbb{T}(e)$  is just a singleton. In other words, the end invariant carries no information in this case. We can effectively discard the end invariants of such ends.

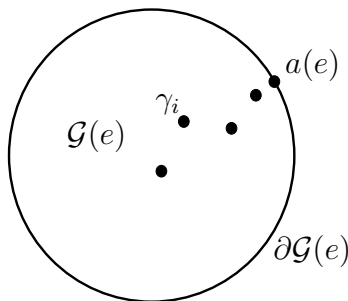


FIGURE 1.4. The limit in the curve graph:  $\gamma_i \rightarrow a(e)$ .

We are now in a position to give a formal statement of the Ending Lamination Theorem as follows:

**Theorem 1.5.4.** *Suppose that  $M$  and  $M'$  are complete hyperbolic 3-manifolds with finitely generated fundamental groups. Suppose that  $f : \Psi(M) \rightarrow \Psi(M')$  is a proper homotopy equivalence between the respective non-cuspidal parts such that for each  $e \in \mathcal{E}(M)$ , we have  $f_*(a(M, e)) = a(M', f(e))$ . Then there is an isometry  $g : M \rightarrow M'$  such that  $g|_{\Psi(M)} : \Psi(M) \rightarrow \Psi(M')$  is properly homotopic to  $f$ .*

Here, a “proper homotopy” is a continuous map,  $F : \Psi(M) \times [0, 1] \rightarrow \Psi(M')$ , with  $F^{-1}(\partial\Psi(M')) = \partial\Psi(M) \times [0, 1]$  and which is proper in the usual topological sense, i.e. the preimage of every compact set is compact. Note that  $g$  necessarily maps  $\Psi(M)$  isometrically to  $\Psi(M')$ .

Theorem 1.5.4, will be proven in the indecomposable case in Section 2.14, and in general in Section 3.8.

There is special case we should mention. Suppose that  $\Psi(M) \cong \Sigma \times \mathbb{R}$ . Then,  $\Psi(M)$  has two ends, say  $e^-$  and  $e^+$ . If these are both degenerate, we say that  $M$  is **doubly degenerate**. In that case, we have the following (see [Bon]):

**Theorem 1.5.5.** *If  $M$  is doubly degenerate with ends  $e^+$  and  $e^-$ , then  $a(M, e^-) \neq a(M, e^+)$ .*

As observed in Section 1.2, this case calls for some special attention, in that we need to keep track of which end is which. (For further discussion of Theorem 1.5.5, see Lemma 2.13.1.)

Let  $E$  be a neighbourhood of  $e$  homeomorphic to  $\Sigma \times [0, \infty)$ .

**Definition.** We say that  $e$  is **incompressible** if the inclusion of  $E$  into  $\Psi(M)$  is  $\pi_1$ -injective.

It is easily seen that this is independent of the choice of  $E$ . It is also equivalent to saying that  $\partial E$  is an incompressible surface in  $\Psi(M)$ . If  $M$  is indecomposable, then every end is incompressible (and conversely).

Suppose that  $e \in \mathcal{E}_D(M)$  is incompressible. It was shown in [Bon] that  $e$  is “simply degenerate”. (Of course, Bonahon did not assume a-priori that  $e$  is

topologically finite. That is a consequence of being simply degenerate. We shall however take that as given here.) One way of formulating simple degeneracy is to assert that there is a sequence,  $(\gamma_i)_i$ , in  $\mathbf{C}(e)$  whose geodesic representatives in  $M$  all lie in  $\Psi(M)$  and tend out the end  $e$ . (For this, we need to assume that the constant  $\eta$  defining  $\Psi(M)$  is sufficiently small in relation to the complexity of  $\Sigma$ .) Moreover, Bonahon showed that any such sequence must converge on a well defined “arational lamination”  $\lambda$  in  $\Sigma$  (as discussed below). This give rise to the rise to the traditional formulation of the “ending lamination”.

We need some other ideas, also found in [Th1] and [Bon]. First, we can extend each of the closed geodesic representatives of the  $\gamma_i$  to a “pleated surface”. This notion was originally due to Thurston — see [CanaEG] for a detailed discussion. A pleated surface is (in particular) a 1-lipschitz map of  $\Sigma$  into  $\Psi(M)$ , with respect to some hyperbolic structure,  $\sigma_i$ , in the domain. These pleated surfaces also go out the end  $e$ . (For further discussion of surfaces of this type, see Section 3.2.) We can now find a simple closed curve in  $(\Sigma, \sigma_i)$  whose length is bounded by some constant,  $l(e)$ , depending only on the complexity of  $\Sigma$ . This gives us an element  $\beta_i$  in  $\mathbf{C}(\Sigma) = \mathbf{C}(e)$  represented by a curve of bounded length at most  $l(e)$  in  $\Psi(M)$ . These curves also go out  $e$ , and in fact, so do their geodesic representatives in  $M$ . Thus, in retrospect, we could have chosen our curves  $\gamma_i$  all to have bounded length. It turns out that all these sequences tend to the same element of  $\partial\mathcal{G}(e)$ . We recover the formulation given by Propositions 1.5.2 and 1.5.3 for an incompressible end.

If  $e \in \mathcal{E}_D(M)$  is compressible, then one needs to modify the above. The essential ingredients are contained in the general proof of tameness, as in [Ag, CalG]. We give a direct proof of this (without explicit reference to laminations) in Section 3.1 here. This is based on ideas in [So1], as formulated in [Bow6].

In fact the uniqueness of the point  $a \in \partial\mathcal{G}(\Sigma)$  is quite subtle in the compressible case. The proof we give here will involve some machinery from the proof of the Ending Lamination Theorem, and is postponed until Section 3.7. It is possible to give a statement of Theorem 1.5.4 without assuming the uniqueness of  $a$  as follows. Instead of a point  $a \in \partial\mathcal{G}(\Sigma)$  we could take the end invariant to be a nonempty subset  $\mathbf{a}(e) \subseteq \partial\mathcal{G}(\Sigma)$  (namely, the set of all possible limits of curves of bounded length that go out the end). The relevant hypothesis of Theorem 1.5.4 would then become  $f_*(a) \in \mathbf{a}(f(e))$  for some  $a \in \mathbf{a}(e)$ . Given this, the fact that  $\mathbf{a}(e)$  must be a singleton becomes more apparent. This is formally proven here as Proposition 3.7.2 (see also Proposition 3.7.3). This discussion is also applicable to the incompressible case.

We have said everything we need to understand the Ending Lamination Theorem as we have formulated it. What remains can be considered a digression which relates it to other formulations elsewhere.

Our description of the end invariant of a simply degenerate end made no explicit reference to “laminations”. To relate our account to the more traditional



formulation, we begin by saying briefly what a lamination is. We will not give formal definitions here. More detailed accounts can be found in [CasB, CanaEG], and some further discussion of laminations can be found in Section 5.1 (though in the context of pleated surfaces).

A “geodesic lamination” can be viewed as a foliation of a non-empty closed subset of a surface by 1-dimensional leaves. (Given a hyperbolic structure on the surface, it can be realised so that the leaves are all geodesic.) A “projective lamination” is a geodesic lamination together with a transverse measure defined up to a scalar multiple. The set of geodesic laminations can be given a natural compact hausdorff topology. We denote resulting “projective lamination space” by  $\partial\mathbb{T}(\Sigma)$ . In fact,  $\mathbb{T}(\Sigma) \cup \partial\mathbb{T}(\Sigma)$  also has a natural topology as the “Thurston compactification” of Teichmüller space. (It is a homeomorphic to a euclidean ball, with interior  $\mathbb{T}(\Sigma)$ .) Note that we can view  $\mathbf{C}(\Sigma) = V(\mathcal{G}(\Sigma))$  as a subset of  $\partial\mathbb{T}(\Sigma)$  (since a simple closed curve is also a lamination).

An element of  $\partial\mathbb{T}(\Sigma)$  is “arational” if its support “fills” the surface: that is if it has no closed leaves, and it intersects every non-trivial non-peripheral closed curve. We give the set of arational laminations the subspace topology. We then take a quotient, by identifying two arational laminations if they have the same support; that is the same underlying geometric lamination after forgetting measures. The resulting space in the quotient topology is called the “ending lamination space” of the surface. In fact, it can be canonically identified with  $\partial\mathcal{G}(\Sigma)$  [Kla]. Under this identification, a sequence in  $\mathcal{G}(\Sigma)$  converges on a point in  $\partial\mathcal{G}(\Sigma)$  if and only if the same sequence in  $\partial\mathbb{T}(\Sigma)$  converges to the corresponding arational lamination in the quotient space. In particular, we see that the end invariants of simply degenerate ends, as we have defined them, can also be viewed as elements of ending lamination space. (The set of arational laminations could be described without references to measures: simply as geometric laminations which fill, and have no isolated leaves. However, the topology of the space is more complicated to describe in these terms.)

Although we will not be studying ending lamination space as such, it is worth remarking that it has a rich and interesting structure, which has been investigated by a number of authors. See, for example, [HenP, Ga, BesB].

Note that the mapping class group,  $\text{Map}(\Sigma)$  acts by homeomorphism on both  $\partial\mathbb{T}(\Sigma)$  and  $\partial\mathcal{G}(\Sigma)$ . Suppose that  $\Sigma$  happens to be the boundary component of a 3-manifold,  $M$ , say. Let  $\text{Map}_M(\Sigma) \leq \text{Map}(\Sigma)$  be the subgroup consisting of those mapping classes that extend to a homeomorphism of  $M$ . It turns out that there is a canonical open subset of  $\partial\mathbb{T}(\Sigma)$ , on which  $\text{Map}_M(\Sigma)$  acts properly discontinuously. This is generally referred to as the “Masur domain”. It was originally described by Masur [Mas] when  $M$  is a handlebody. The general case for a compact 3-manifold (which reduces to that of a compression body) was described by Otal [Ota1]. One can generalise further to allow  $\Sigma$  to be a subsurface of a boundary component which is incompressible in that boundary component. In particular, in

the situation where  $\Sigma$  is a component of  $\partial_I \bar{\Psi}$ , we get a “Masur domain” in  $\partial\mathbb{T}(\Sigma)$  for the corresponding end of  $\Psi$ . Generalisations have been studied by Kleineidam, Souto, Lecuire, Ohshika and Namazi. For a general account see Section 6 of [NS]. Such a domain is always closed under the above equivalence (namely forgetting transverse measures), and gives rise to an open subset of ending lamination space, also referred to as the “Masur domain”. In all cases, ending laminations of a hyperbolic 3-manifold lie in this domain. Note that we can also take the quotient of this domain by the action of  $\text{Map}_M(\Sigma)$  to give another hausdorff topological space.

One can use the Masur domain to give another way of formulating the Ending Lamination Theorem, which we now describe.

In the notation of Section 1.2, we note:

**Lemma 1.5.6.** *Any self-homeomorphism of  $\Psi$  is properly homotopic to a self-homeomorphism which extends to a homeomorphism of  $\bar{\Psi}$ .*

*Proof.* This is a simple exercise using Waldhausen’s Theorem (stated here as Theorem 1.6.2).  $\square$

**Lemma 1.5.7.** *Let  $\Sigma$  be component of  $\partial_I \bar{\Psi}$ , and let  $\phi, \phi' : \bar{\Psi} \rightarrow \bar{\Psi}$  be self-homeomorphisms both preserving setwise both  $\partial_I \bar{\Psi}$  and  $\Sigma$ . If  $\phi|_{\Psi}$  and  $\phi'|_{\Psi}$  are properly homotopic in  $\Psi$ , then  $\phi|_{\Sigma}$  and  $\phi'|_{\Sigma}$  are homotopic in  $\Sigma$ .*

*Proof.* Let  $\Sigma \times [0, \infty]$  be a collar neighbourhood, in  $\Psi$ , of  $\Sigma$  which we identify with  $\Sigma \times \{\infty\}$ . Thus,  $E \equiv \Sigma \times [0, \infty)$ , is a neighbourhood of the corresponding end of  $\Psi$ . Write  $\Sigma_t = \Sigma \times \{t\} \subseteq E$ . For sufficiently large  $t$ , the given homotopy from  $\phi|_{\Sigma_t}$  to  $\phi'|_{\Sigma_t}$  lies entirely within  $E$ ; and so also do the images  $\phi(\Sigma \times [t, \infty))$  and  $\phi'(\Sigma \times [t, \infty))$ . Therefore, combining these maps, and postcomposing with projection to the  $\Sigma$  factor, we get a homotopy from  $\phi|_{\Sigma}$  to  $\phi'|_{\Sigma}$  in  $\Sigma$  as required.  $\square$

The above tells us that the set of self-homeomorphisms of the end of  $\Psi$  modulo proper homotopy of  $\Psi$  can be identified with the subgroup,  $\text{Map}_M(\Sigma)$ , of  $\text{Map}(\Sigma)$  which extends over  $\bar{\Psi}$ .

This gives rise to another phrasing of the Ending Lamination Theorem. (It is often given in this form.) If  $M$ , is a complete hyperbolic 3-manifold, and  $e$  is a degenerate end of  $\Psi(M)$ , then one can view the end invariant of  $e$  as an element of the quotient of the Masur domain. (Of course, we may lose some marking information in this process.) If  $e$  a geometrically finite end, we view it end invariant as an element of  $\mathbb{T}(\Sigma(e))/\text{Map}_M(\Sigma(e))$ .

Suppose now that  $M$  and  $M'$  are hyperbolic 3-manifolds, and that  $f : \Psi(M) \rightarrow \Psi(M')$  is a homeomorphism which sends each end invariant of  $M$  to the end invariant of  $M'$ , viewed this time, as elements of the quotient of the respective Masur domains or of Teichmüller space. Then, there is an isometry,  $g : M \rightarrow M'$ , such that  $g|_{\Psi(M)}$  is homotopic to  $f$ . This need no longer be a proper homotopy. (For example, suppose  $D \subseteq \bar{\Phi}$  is a compressing disc, with  $\partial D \subseteq \partial_I \Psi$ . Then  $f$

and  $g$  might differ by a twist along  $D$ . These maps are homotopic in  $\Psi$ , but not properly homotopic.)

If  $M$ , and hence  $M'$ , happen to be indecomposable, then the ends are incompressible. The relevant subgroups of the mapping class groups are all trivial, and so there is no substantial difference between this formulation and the original.

We will make no direct use of the Masur domain in this paper. We do not need to know that our degenerate end invariants lie in this domain. (If they do not, the relevant statements just become vacuously true.)

We remark that there are yet other ways of interpreting degenerate end invariants, for example via “non-realisability” of laminations in  $M$  (see for example [NS]).

### 1.6. INGREDIENTS OF THE PROOF.

We list below some general topological and geometrical ingredients of our argument. First, we describe a few well known topological facts about 3-manifolds. Some general references are [Hem1, Ja, Ka, Mar2, Moi].

We have the following procedure for replacing maps of surfaces by embeddings in 3-manifolds. Let  $\Psi$  be an aspherical 3-manifold with (possibly empty) boundary,  $\partial\Psi$ . By a “proper map”  $f : \Phi \rightarrow \Psi$  of a surface  $\Phi$  into  $\Psi$  we mean that  $f$  is continuous and that  $f^{-1}(\partial\Psi) = \partial\Phi$ . We can always assume that  $f$  is in general position. By a “proper homotopy” we mean a map  $F : \Phi \times [0, 1] \rightarrow \Psi$  such that  $F^{-1}(\partial\Psi) = \partial\Phi \times [0, 1]$ .

**Theorem 1.6.1.** *Suppose that  $f : \Phi \rightarrow \Psi$  is a  $\pi_1$ -injective proper map, and that  $f$  is properly homotopic to an embedding of  $\Phi$ . Suppose that  $U \subseteq \Psi$  is any open subset containing  $f(\Phi)$ . Then  $f$  is properly homotopic in  $\Psi$  to a proper embedding  $f' : \Phi \rightarrow \Psi$  with  $f'(\Phi) \subseteq U$ .*

(See Figure 1.5.)

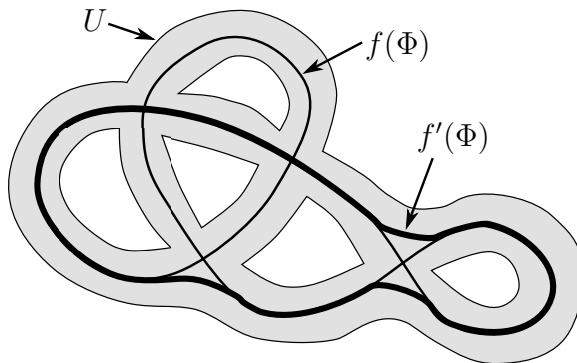


FIGURE 1.5. Obtaining an embedded surface by surgery

As observed in [Bon], this follows from the construction of [FHS]. The general result given in [FHS] makes use of minimal surfaces. If  $f$  is a homotopy equivalence, the relevant part of their argument in this case is a purely combinatorial tower construction. This works equally well in the relative case (whereas [FHS] deals with closed surfaces). We should note that [FHS] assumes that  $f$  is an immersion. However (as observed in [Bow5] for example) the combinatorial argument is readily adapted to general position maps. Alternatively, another (somewhat artificial) way to deal with this would be to lift  $f$  to the cover corresponding to  $f_*(\pi_1(\Sigma))$ , so that  $f$  becomes a homotopy equivalence. The tower argument then gives us an embedding in this cover, which projects back to an immersion in an arbitrarily small neighbourhood of the original map. We can then apply [FHS] in the original form to this immersion to give us an embedding.

In applying Theorem 1.6.1, some caution is needed in that it gives us no geometric control on the homotopy between  $f$  and  $f'$ . In principle, it could go all over the place. With more work, one could place some restrictions on the homotopy, but we will circumvent that issue here.

We also note Waldhausen's cobordism theorem (see [Ja]):

**Theorem 1.6.2.** *Let  $\Psi$  be an aspherical 3-manifold with (possibly empty) boundary,  $\partial\Psi$ . Suppose that  $\Sigma_1$  and  $\Sigma_2$  are disjoint homotopic properly embedded  $\pi_1$ -injective surfaces  $\Sigma_1 \cap \Sigma_2 = \emptyset$ . Then  $\Sigma_1 \cup \Sigma_2$  bounds a submanifold,  $Q$ , of  $\Psi$ , such that  $(Q, Q \cap \partial\Psi)$  is homeomorphic to  $(\Sigma_i \times [0, 1], \partial\Sigma_i \times [0, 1])$ .*

For product manifolds, we have the following specific results. Let  $\Sigma$  be a compact surface. By a **homotopy fibre** we mean a map  $f : \Sigma \rightarrow \Sigma \times \mathbb{R}$  with  $f^{-1}(\partial\Sigma \times \mathbb{R}) = \partial\Sigma$  such that relative homotopy class of  $(\Sigma, \partial\Sigma) \rightarrow (\Sigma \times \mathbb{R}, \partial\Sigma \times \mathbb{R})$  is the same as that of the inclusion  $[x \mapsto (x, 0)]$ .

We have the following result of Brown [Brow]:

**Theorem 1.6.3.** *An injective homotopy fibre in  $\Sigma \times \mathbb{R}$  is ambient isotopic to  $\Sigma \times \{0\}$ .*

(Note that, if  $\Psi \cong \Sigma \times \mathbb{R}$ , and  $\Sigma_1$  and  $\Sigma_2$  are the images of embedded homotopy fibres, then Theorem 1.6.2 in this case is a consequence of Theorem 1.6.3.)

There is a refinement of Theorem 1.6.1 available in this case:

**Theorem 1.6.4.** *Suppose that  $f : \Sigma \rightarrow \Sigma \times \mathbb{R}$  is a homotopy fibre, and that  $\alpha \subseteq \Sigma$  is a curve with  $f^{-1}(f(\alpha)) = \alpha$ . Let  $U \subseteq \Sigma \times \mathbb{R}$  be any open set containing  $f(\Sigma)$ . Then there is an injective homotopy fibre  $f' : \Sigma \rightarrow \Sigma \times \mathbb{R}$  with  $f'(\Sigma) \subseteq U$  and  $f'|_\alpha = f|_\alpha$ .*

The above is observed in [Ot3]. It is a consequence of the tower argument used in [FHS] — at the top of the tower, and then at each stage in descending the tower, we can assume that we can perform the surgeries so as to retain the curve  $\alpha$  as a subset of our surface.

The above will suffice for our proof of the Ending Lamination Theorem. Further discussion of these constructions will be given in Section 4.2.

Recall that a 3-manifold is “haken” if it contains an embedded incompressible surface. We have the following theorem of Waldhausen [Wal]:

**Theorem 1.6.5.** *Suppose that  $\Lambda$  and  $\Lambda'$  are compact haken 3-manifolds with boundary. Then any relative homotopy equivalence between  $(\Lambda, \partial\Lambda)$  and  $(\Lambda', \partial\Lambda')$  is relatively homotopic to a homeomorphism.*

In Section 2.14 we will be applying Theorem 1.6.5 to the complement of a set of unknotted tori in  $\Sigma \times \mathbb{R}$ . Some further properties of product manifolds are discussed there.

We will need various facts concerning the geometry of hyperbolic 3-manifolds. We have already mentioned the thick-thin decomposition in Section 1.3. Recall that the  $\eta$ -thin part of  $M$  is the set of points of systole at most  $\eta$ .

The following is a consequence of the Margulis Lemma:

**Theorem 1.6.6.** *There is a universal constant,  $\eta_0 > 0$ , such that for all  $\eta \leq \eta_0$ , each component of the thin part of  $M$  is homeomorphic to one of  $D^2 \times S^1$ ,  $S^1 \times \mathbb{R} \times [0, \infty)$  or  $S^1 \times S^1 \times [0, \infty)$ .*

Here  $D^2$  and  $S^1$  denote the unit disc and circle respectively. We refer to the three homeomorphism types as **Margulis tubes**,  **$\mathbb{Z}$ -cusps** and  **$(\mathbb{Z} \oplus \mathbb{Z})$ -cusps** respectively.

We denote by  $\Theta(M)$  the closure of the complement of the thin part, and by  $\Psi(M)$ , the closure of the complement of the union of open cusps. In other words,  $\Psi(M)$  is  $\Theta(M)$  with all the Margulis tubes glued back in. We refer to  $\Theta(M)$  and  $\Psi(M)$  respectively as the **thick part** and the **non-cuspidal part** of  $M$ . In practice, it will be convenient to generalise the construction, by allowing different components of the thin part to be defined by different “constants”, provided each of these “constants” are constrained to lie between two genuinely fixed positive constants less than  $\eta_0$ . This makes no essential difference to our arguments, and will be discussed further in Section 2.7.

For completeness, we state again the Tameness Theorem of Bonahon, Agol, Calegari and Agol [Bon, Ag, CalG]:

**Theorem 1.6.7.** *If  $\pi_1(M)$  is finitely generated, then  $\Psi(M)$  is topologically finite.*

In other words, we have a compactification  $\bar{\Psi}(M)$  of  $\Psi(M)$  as described in Section 1.2.

We can write  $M = \mathbb{H}^3/\Gamma$ , where  $\Gamma \equiv \pi_1(M)$  acts properly discontinuously on  $\mathbb{H}^3$ . We decompose  $\partial\mathbb{H}^3$  as the limit set,  $L(\Gamma)$ , and discontinuity domain,  $D(\Gamma)$ . Except in a few trivial cases,  $L(\Gamma)$  is the unique minimal non-empty closed invariant subset.

For the final part of the proof of the Ending Lamination Theorem we need the rigidity result of Sullivan [Su] which says that there is no quasi-conformal deformation supported on the limit set. More formally:

**Theorem 1.6.8.** *Suppose that  $f : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$  is a quasiconformal map, equivariant with respect to the respective actions of  $\Gamma$  and  $\Gamma'$  (so that  $f(D(\Gamma)) = D(\Gamma')$ ), and that  $f|_{D(\Gamma)}$  is conformal. Then  $f$  is conformal.*

In other words,  $f$  is a Möbius transformation, and hence extends uniquely to an isometry of  $\mathbb{H}^3$ .

(We remark that a consequence of the Tameness Theorem is the Ahlfors Measure Conjecture. This tells us that either  $D(\Gamma) = D(\Gamma')$  are both empty, or both have full measure. In the latter case, Theorem 1.6.8 is an immediate consequence of the fact that a quasiconformal map that is conformal almost everywhere is conformal everywhere [LehV]. Of course, Sullivan’s rigidity theorem predates the Tameness Theorem.)

We recall the notion of a quasi-isometry between two path-metric defined in Section 1.3. The following is well known, and is one way of interpreting the key ingredient of original proof of the Mostow Rigidity Theorem [Most].

**Theorem 1.6.9.** *Let  $f : \mathbb{H}^3 \rightarrow \mathbb{H}^3$  be a quasi-isometry, then there is a unique extension  $f : \mathbb{H}^3 \cup \partial\mathbb{H}^3 \rightarrow \mathbb{H}^3 \cup \partial\mathbb{H}^3$  which is continuous at every point of  $\partial\mathbb{H}^3$ . Moreover,  $f|_{\partial\mathbb{H}^3}$  is a quasiconformal homeomorphism of  $\partial\mathbb{H}^3$ .*

In Chapter 3, we will need a version of Thurston’s Uniform Injectivity Theorem for decomposable manifolds. This is discussed in Section 5.1.

Central to both the statement and the proof of the Ending Lamination Theorem is the curve graph,  $\mathcal{G}(\Sigma)$ , associated to a compact surface  $\Sigma$ , which we have already discussed in Section 1.5. The fact that  $\mathcal{G}(\Sigma)$  is Gromov hyperbolic is a central ingredient to the proof of the Ending Lamination Theorem. For the most part, this is already exploited in the proving other results quoted here — notably the A-priori Bounds Theorem. We will revisit this in Section 3.5 to prove a variant of the A-priori Bound Theorem applicable in the decomposable case. For this, we will need the following standard fact about Gromov hyperbolic spaces.

**Proposition 1.6.10.** *Suppose that  $(\mathcal{G}, d)$  is a  $k$ -hyperbolic space in the sense of Gromov. There exist  $\mu, c > 0$  depending only on  $k$  with the following property. Suppose that  $r \geq 0$  and  $\alpha \subseteq \mathcal{G}$  is a geodesic segment, and denote its  $r$ -neighbourhood by  $N(\alpha, r)$ . Let  $\beta$  be any path in  $\mathcal{G}$  with  $d(\alpha, \beta) \geq r$ , connecting any two points,  $x, y \in \partial N(\alpha, r)$ . Then the length of  $\beta$  is at least  $d(x, y)e^{\mu r} - c$ .*

This is a standard fact, used for example in proving the “stability of quasi-geodesics” in a Gromov hyperbolic see for example [GhH].

A key ingredient to the proof of the Ending Lamination Theorem is the “A-priori Bounds” result of Minsky [Mi4]. This uses the notion of a “tight geodesic” in  $\mathcal{G}(\Sigma)$ . This was introduced in [MasM2], and will be defined in Section 2.2 here. (We use a slightly more general notion than the original.) We quote the following results directly from [Bow3]. However, as we discuss there, they can also readily be deduced from various statements in [Mi4] (which uses rather different methods).

Let  $M$  be a complete hyperbolic 3-manifold with non-cuspidal part  $\Psi(M)$  homeomorphic to  $\Sigma \times \mathbb{R}$ , where  $\Sigma$  is a compact surface. Given a free homotopy class  $\alpha$  of closed curve in  $\Sigma$ , we write  $\alpha^*$  (or  $\alpha_M^*$ ) for its closed geodesic realisation in  $M$ . If  $\alpha$  is simple in  $\Sigma$ , then we can assume this to lie in  $\Psi(M)$ , provided the Margulis constant is chosen small enough in relation to  $\xi(\Sigma)$  (see for example, the discussion in Section 3.2). We write  $l_M(\alpha) = \text{length}(\alpha^*)$ .

**Theorem 1.6.11.** *Suppose that  $\alpha, \beta, \gamma \in \mathbf{C}(\Sigma)$ , and that  $\gamma$  lies on some tight geodesic from  $\alpha$  to  $\beta$ . Then  $l_M(\gamma)$  is bounded above in terms of  $\xi(\Sigma)$  and  $\max\{l_M(\alpha), l_M(\beta)\}$ .*

This is Theorem 1.3 of [Bow3] when  $\xi(\Sigma) \geq 2$ , and follows from Proposition 9.3 thereof, when  $\xi(\Sigma) = 1$ .

The following weakens the conditions on the endpoints of the geodesic at the cost of requiring us to stay at least a fixed distance from the endpoints. (Figure 1.6.)

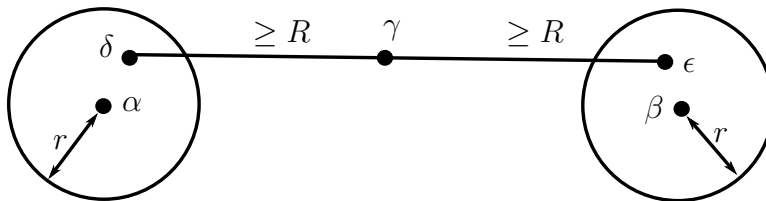


FIGURE 1.6. Illustration of Theorem 1.6.12

**Theorem 1.6.12.** *Given  $r, \xi$  there exists  $R$  with the following property. Suppose that  $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbf{C}(\Sigma)$ , and that  $\gamma$  lies on some tight geodesic from  $\delta$  to  $\epsilon$  in  $\mathcal{G}(\Sigma)$ , where  $d_{\mathcal{G}}(\alpha, \delta) \leq r$ ,  $d_{\mathcal{G}}(\beta, \epsilon) \leq r$ ,  $d_{\mathcal{G}}(\gamma, \delta) \geq R$ ,  $d_{\mathcal{G}}(\gamma, \epsilon) \geq R$ ,  $l_M(\alpha) \leq l$  and  $l_M(\beta) \leq l$ . Then  $l_M(\gamma)$  is bounded above in terms and  $l, r$  and  $\xi(\Sigma)$ .*

For  $\xi(\Sigma) \geq 2$ , this follows from Theorem 1.2 of [Bow3]. For  $\xi(\Sigma) = 1$ , it can be deduced from the discussion of Section 9 of [Bow3]. Let  $R = r + 1$ . From the combinatorial structure of the Farey graph,  $\mathcal{G}(\Sigma)$ , we see that either  $\gamma$  lies on a geodesic from  $\alpha$  to  $\beta$ , or else there are vertices,  $\zeta, \eta \in V(\mathcal{G}(\Sigma))$ , such that  $\gamma, \zeta, \eta$  form a triangle in  $\mathcal{G}(\Sigma)$ . In the former case,  $l_M(\gamma)$  is bounded by Theorem 1.6.11. In the latter case,  $l_M(\zeta)$  and  $l_M(\eta)$  are bounded, and it follows by Lemma 9.2 of [Bow3], that  $l_M(\gamma)$  is bounded. (For future reference, we note that Section 9 of [Bow3] does not require the representation to be discrete, so the statements when  $\xi(\Sigma) = 1$  hold more generally.)

(In the above to theorems we can allow for “accidental parabolics”. These correspond simple closed curves in  $\Sigma$  which give rise to cusps of  $M$  and are deemed to have 0 length. These statements were included in the description given in [Bow3], though we won’t need them here.)

There are also versions applicable to subsurfaces, see Theorem 2.13.2 here. To deal with compressible ends, we will need generalisations of these statements, see Propositions 3.5.7 and 3.5.8.

The A-priori Bounds Theorem is used to find a sequence of closed geodesic curves in the 3-manifold which tend out a given end. It means that, as curves on  $\Sigma$ , they can be chosen “a-priori”, that is, just knowing the ending lamination, and not given any other advance information about the geometry of  $M$ .

This also applies to any incompressible end of (the non-cuspidal part) of a hyperbolic manifold — by passing to the appropriate covering space. This is sufficient for the indecomposable case. However, the cover of a compressible end will not be a product manifold, and we will need to revisit the A-priori Bounds Theorem in Section 3.5. We give generalisations as Propositions 3.5.7 and 3.5.8.

We remark that we make no use in this paper of the important tool of “subsurface projections” introduced in [MasM2], and central to the original proof of [Mi4, BrocCM]. Of course, this notion finds many other applications elsewhere. A discussion of subsurface projections in connection with the models we construct here can be found in [Bow5].

Various other constructions will be outlined where needed.

## 1.7. OUTLINE OF THE PROOF OF THE ENDING LAMINATION THEOREM.

As mentioned in Sections 1.1 and 1.4, the proof involves constructing a “model” manifold,  $P$ , based on a topological data, together with a prescribed set of “end invariants”. Such a manifold will approximate the geometry of a complete hyperbolic 3-manifold,  $M$ . We assume that  $\pi_1(M)$  is finitely generated, and write  $\Psi = \Psi(M)$  for the non-cuspidal part.

We construct the model, starting with  $\Psi$ , viewed as a topological 3-manifold. We assume that we have a formal partition of its ends into two sets deemed “geometrically finite” or “degenerate”, and an assignment of an element  $a(e) \in \partial\mathcal{G}(e)$  to each degenerate end,  $e$ . From this data, we construct a “geometric model”,  $P$ . This is a riemannian manifold, with a preferred submanifold,  $\Psi(P)$ , called the “non-cuspidal part” of  $P$ . This non-cuspidal part is homeomorphic to  $\Psi(M)$  and there is a preferred proper homotopy class of homeomorphism from  $\Psi$  to  $\Psi(P)$ . This model has the following property:

**Proposition 1.7.1.** *Suppose that  $M$  is a complete hyperbolic 3-manifold, and that  $g : \Psi(P) \rightarrow \Psi(M)$  is a homeomorphism. Suppose that  $g$  respects the partition of ends into geometrically finite and degenerate, and suppose that  $g_*(a(e)) = a(M, g(e))$  for each degenerate end  $e$ . Then there is a proper lipschitz homotopy equivalence  $f : P \rightarrow M$  such that  $f^{-1}(\Psi(M)) = \Psi(P)$ , and  $f|_{\Psi(P)} : \Psi(P) \rightarrow \Psi(M)$  is properly homotopic to  $g$ , and such that a lift of  $f$  to the universal covers,  $\tilde{f} : \tilde{P} \rightarrow \tilde{M} \equiv \mathbb{H}^3$  is a quasi-isometry.*

Proposition 1.7.1 will be proven in the indecomposable case in Section 2.14, and in general in Section 3.8. (In Section 4.5 we will promote  $f$  to a bilipschitz map, see Theorem 4.5.1.)



Note that we have not used the end invariants of the geometrically finite ends in the construction of  $P$ . These are not needed to construct a quasi-isometry of the type described. In our approach, the geometrically finite end invariants are only brought into play at the final stage of the proof, see Sections 2.14 and 3.8. (Alternatively, one could use this information to construct a model with stronger properties at this stage. This is the approach taken by Minsky [Mi4]. His argument could be substituted for ours at the relevant stage.)

1.7.1. *Construction of the model space.*

We first outline how the construction of  $P$ , and the remainder of the argument, works in the case where  $\Psi(M)$  is homeomorphic to  $\Sigma \times \mathbb{R}$ . For simplicity we focus on the doubly degenerate case, i.e. where both ends of  $\Psi(M)$  are degenerate. We will assume here that  $\xi(\Sigma) \geq 2$ . Thus,  $\Psi(M) \cong \Sigma \times \mathbb{R}$  where  $\Sigma$  is a compact orientable surface. Traditionally, the  $\Sigma$ -factor is thought of as the “horizontal” direction, and the  $\mathbb{R}$ -factor as the “vertical” direction. (Though in our figures, we will represent them the other way around, so that they fit on the page.) Recall that  $\Theta(M)$  denotes the thick part of  $M$ .

Let  $\mathcal{G}(\Sigma)$  be the curve graph associated to  $\Sigma$  as described in Section 1.5. The end invariants of  $M$  give us two distinct points,  $a, b \in \partial\mathcal{G}(\Sigma)$  (Section 2.8). One can show that this is a tight bi-infinite geodesic in  $\mathcal{G}(\Sigma)$  from  $a$  to  $b$ . This consists of a sequence of curves,  $(\alpha_i)_{i \in \mathbb{Z}}$ , in  $\Sigma$  with  $\alpha_i \cap \alpha_{i+1} = \emptyset$  for all  $i$ . One way to imagine this would be to construct what we call a “ladder” in  $\Sigma \times \mathbb{R}$ . This is a bi-infinite sequence of annuli,  $\Omega_i = \alpha_i \times I_i$ , where  $I_i \subseteq \mathbb{R}$  is a closed interval, and  $I_i \cap I_j$  is a non-trivial interval if  $|i - j| \leq 1$  and  $I_i \cap I_j = \emptyset$  otherwise. (Figure 1.7.) A consequence of the A-priori Bounds Theorem is that the closed geodesics in

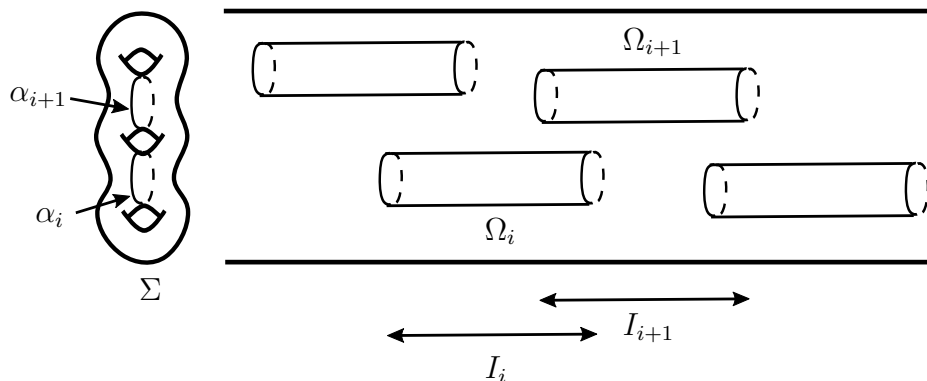


FIGURE 1.7. A ladder in  $\Sigma \times \mathbb{R}$

$M$  in the corresponding homotopy classes all have bounded length (see Theorem 1.7.2 below). Thus, the ladder in some way reflects the geometry of  $M$ .

However, we need more than this to determine a riemannian metric on our model. We can extend  $\{\Omega_i \mid i \in \mathbb{Z}\}$  to a locally finite set,  $\mathcal{W}$ , of disjoint annuli

of the form  $\alpha \times I$ , where  $\alpha \subseteq \Sigma$  is a curve and  $I \subseteq \mathbb{R}$  is a closed interval, and which is “complete”. Completeness means that for each  $t \in \mathbb{R}$ , each component of  $(\Sigma \times \{t\}) \setminus \bigcup \mathcal{W}$  has complexity at most 1. (In fact, up to sliding the annuli in the vertical direction one can suppose that each component of  $(\Sigma \times \{t\}) \cap \bigcup \mathcal{W}$  is either a pants decomposition of  $\Sigma$ , or a pants decomposition missing one curve. In the former case, each complementary component is a three-holed sphere. In the latter case there will be one component that is either a four-holed sphere or a one-holed torus. Combinatorially this is essentially the same as a path in the “pants graph” of  $\Sigma$ . In [MasM2], the analogous procedure is expressed in terms of a “resolution of a hierarchy”.)

There are many ways one might construct such a complete system of annuli. The only properties we need are laid out in Theorem 2.4.3, labelled (P1)–(P4). (Some additional conditions are added in Lemmas 2.4.1 and 2.4.2 for the purposes of giving an inductive construction, but these are not needed for applications.) Firstly, (P1) tells us that the annuli arise from a bounded iteration of the tight geodesic construction. This is needed in order to obtain the a-priori bound on the length of the corresponding curves in  $M$ . More precisely:

**Theorem 1.7.2.** *There is some  $L$  depending only on the topological type of  $\Sigma$  such that if  $\Omega \in \mathcal{W}$ , then the corresponding closed geodesic in  $M$  has length at most  $L$ .*

Next, (P2) says that no two annuli are homotopic, i.e. have the same base curve. This ensures that no two tubes in the model will correspond to the same Margulis tube in  $M$ . This is in turn needed to ensure that the map constructed between thick parts is a homotopy equivalence (see Section 2.3). We require a “tautness” condition (P3) and (P4), expressed in terms of ladders, which says that the annuli follow a geodesic in the curve graph up to bounded distance. This will ensure that the map from our model to  $M$  does not crumple up or fold back over large distances. For the inductive structure of the proof we also require this to hold on a class of subsurfaces of  $\Sigma$ , in the appropriate sense. This is incorporated into the statement of properties (P3) and (P4).

Given our annulus system  $\mathcal{W}$ , the construction of our model space,  $P$ , is relatively straightforward. First, we cut open each annulus of  $\mathcal{W}$  so as to give us a manifold  $\Lambda = \Lambda(\mathcal{W})$  with a toroidal boundary component for each annulus. The fact that the local combinatorics of  $\mathcal{W}$  are bounded means that we can give  $\Lambda$  a riemannian metric that is natural up to bilipschitz equivalence, and such that each toroidal boundary is intrinsically euclidean. The exact construction doesn’t matter, but a precise prescription is given in Section 2.5. We can now glue in a “model” Margulis tube to each toroidal boundary component. This gives the non-cuspidal part  $\Psi(P) \cong \Sigma \times \mathbb{R}$  of the model space. To obtain  $P$  from  $\Psi(P)$  we simply attach a standard cusp to each boundary component.

### 1.7.2. Construction of the map to $M$ .

The next step will be the construction of a map  $f : P \rightarrow M$ . This is where we use Theorem 1.7.2. Each model Margulis tube gets sent either to a Margulis tube in  $M$ , or to a closed geodesic of bounded length (possibly still quite long though). We write  $\mathcal{T}(P)$  for the former set of model tubes, and write  $\Theta(P) = P \setminus \text{int} \bigcup \mathcal{T}(P)$  for the “thick” part of the model space. (This partition of  $\mathcal{T}(P)$  is described by Theorem 2.8.2.) The construction tells us that  $\Theta(P) = f^{-1}\Theta(M)$  and that  $f|_{\Theta(P)}$  is lipschitz. These constructions are described in Sections 2.3 and 2.4. (It is possible that  $\Theta(P)$  may depend on  $M$ , but it doesn’t matter what strategy we adopt to construct a map from the model space, once the model space has been defined.)

Topological considerations described in Section 2.3 now imply that the map  $f : \Theta(P) \rightarrow \Theta(M)$  is a homotopy equivalence, and so we get an equivariant map  $\tilde{f} : \tilde{\Theta}(P) \rightarrow \tilde{\Theta}(M)$  of universal covers. Some effort is now invested in showing that  $\tilde{f}$  is a quasi-isometry. Given this, the argument can be completed as follows. We first use the fact (Lemma 2.5.3) that, when lifted to an appropriate cover, the boundary of a model tube is quasi-isometrically embedded. This then implies that the map between this boundary and the corresponding boundary from  $M$  is a quasi-isometry in the induced euclidean path metrics. It can now be modified to be bilipschitz on the boundary of each tube, and then extends to a bilipschitz map between the tubes themselves. We can also extend over cusps by sending geodesic rays locally isometrically to geodesics rays. We then have a lipschitz map  $P \rightarrow M$ , and the fact that the lift to the universal covers is a quasi-isometry is relatively straightforward given what we have shown. The details are described in Section 2.13.

### 1.7.3. The quasi-isometry property.

It still remains to explain why the lift,  $\tilde{f} : \tilde{\Theta}(P) \rightarrow \tilde{\Theta}(M)$  is a quasi-isometry, as we have claimed. This is where tautness comes into play. As noted in Section 2.1 one can show (see Section 2.10) that two curves of bounded length and a bounded distance apart in  $M$  are also a bounded distance apart in the curve graph,  $\mathcal{G}(\Sigma)$ . Tautness gives us some control on how far apart such curves can be in the model space. This is a start, but is not sufficient. We also need to construct topological barriers in  $P$  so that points separated by a barrier in  $P$  get mapped to points separated by a similar barrier in  $M$  — taking appropriate account of homotopy classes of paths, since we are really interested in the lift to universal covers. Our barriers are called “bands”. A band in  $\Psi(M)$  is a product,  $\Phi \times I$ , where  $\Phi$  is a subsurface of  $\Sigma$ , and  $I$  a compact interval, and such that  $\partial\Phi \times I$  lies in the boundary of a Margulis tube. Its intersection with  $\Theta(M)$  is a “band” in  $\Theta(M)$ . We shall usually insist that  $\partial\Phi \times \{0, 1\} \subseteq \Theta(M)$ . Much of the second

half of Section 2.5 and Section 2.12 are devoted to analysing such bands. (There is an analogy with the “scaffolds” used in [BrocCM].)

To give an idea of how this works, we consider a very simple case. There is a vague sense in which a point,  $x$ , of  $M$  approximately determines a “fibre” hyperbolic structure on  $\Sigma$  — the domain of a lipschitz “pleated surface” with image close to  $x$ . In the bounded geometry case, where there are no Margulis tubes, this structure progresses at a roughly uniform rate in the  $\mathbb{R}$ -direction. (Indeed it stays close to a Teichmüller geodesic [Mi1, Mi2].) A slightly more complicated situation is where we have just one (unknotted) Margulis tube,  $T$ , corresponding to a curve  $\alpha \subseteq \Sigma$ . Let us suppose that  $\alpha$  separates  $\Sigma$  into two subsurfaces  $\Phi_1$  and  $\Phi_2$ . One possibility is that the area of  $\partial T$  is bounded. As we cross the tube, we will be twisting our fibre structure along  $\alpha$ , but doing little to the structure on  $\Phi_1$  and  $\Phi_2$ . Alternatively, while we are crossing  $\alpha$ , it may be that the structure on  $\Phi_1$  changes a lot. In this case, the length of  $\partial T$  in the direction transverse to the longitude of  $T$ , becomes very large. We get a band,  $B_1 \cong \Phi_1 \times I \subseteq \Theta(M)$ , with  $B_1 \cap \partial T \equiv \partial\Phi_1 \times I$ . This will be very long in the  $I$ -direction, where the change in the structure takes place. If the structure on the other side of  $\alpha$  also changes a lot, we will get another similar band,  $B_2 \cong \Phi_2 \times I$ . In this case,  $B_1$  and  $B_2$  together act as “barriers” between the two ends of  $\Theta(M)$ . (Figure 1.8.) On the other hand, if there is no band on the  $\Phi_2$  side of  $\alpha$ , then we

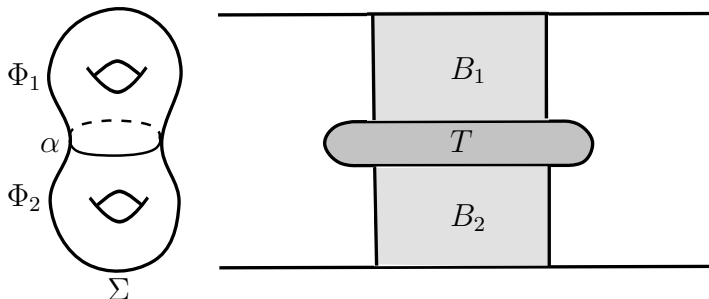


FIGURE 1.8. A Margulis tube with two bands

could sneak around  $B_1$  on the other side of  $T$ . However, this path will be in the “wrong” homotopy class, and the preimage of  $B_1$  in the universal cover,  $\tilde{\Theta}(M)$ , will still serve as a barrier there. In general we may get a very complicated system of nested bands. A general decomposition of  $M$  into bands is discussed in [Bow2], though we won’t be needing any result from that paper here. Indeed they can be recovered from the bilipschitz model space, though that is much less direct.

The logic of our argument will be somewhat different from the motivation of the last paragraph. We will construct our bands first in the model space (Section 2.4). We then show that they correspond to bands in  $M$  (Lemma 2.10.6). The manner in which they form “barriers” is rephrased in terms of pushing around paths and discs (Section 2.12). We need some general principles of bounded geometry to complete the argument. These are discussed in Section 2.9.

Finally in Section 2.14 we discuss how all this works in the general indecomposable case. The only additional ingredient needed there is an analysis of the geometry of geometrically finite ends, but this is something that has long been well understood.

#### 1.7.4. *Conclusion.*

Chapter 3 will be aimed at dealing with the compressible case. Here some of the results we cited from elsewhere, such as the A-priori Bound Theorem, and the fact that Margulis tubes are unlinked, are not immediately available, so we have to revisit these. The technique we use will be to isolate the ends of the 3-manifolds by cutting out a compact polyhedral complex. For this we use the theory of  $CAT(-1)$  spaces, described in Section 3.3. The proof of the Ending Lamination Theorem is completed in Section 3.8 (modulo the discussion of the Uniform Injectivity Theorem in Section 5.1). The promotion to a bilipschitz map in Chapter 4 is not directly relevant to this.

We remark that most of the arguments presented here should be adaptable to the case of pinched negative curvature (cf. [Cana]), though of course, the final rigidity conclusion is no longer valid. One would need to reinterpret various things, for example, the boundaries of Margulis tubes and cusps will no longer be euclidean.

In fact, as is shown in [Bow7] one can also generalise, under suitable hypotheses, to the case where one assumes only the universal cover of  $M$  is Gromov hyperbolic and that the thin part is sufficiently standard.

## 2. THE INDECOMPOSABLE CASE OF THE ENDING LAMINATION THEOREM

The Chapter is devoted to giving a proof of the Ending Lamination Theorem when  $M$  is indecomposable.

### 2.1. SURFACE GROUPS.

As mentioned above, we first consider the special case of surface groups. We describe some additional information we can obtain from the Ending Lamination Theorem, or its proof, in the case where  $\pi_1(M) \cong \pi_1(\Sigma)$ , where  $\Sigma$  is a compact surface. (This is the precursor to the general case, given as Theorem 1.4.4 here.) We will assume that the cusps of  $M$  are in bijective correspondence with the components of  $\partial\Sigma$ . We are assuming that everything is orientable, so Bonahon's Tameness Theorem tells us that  $\Psi(M) \cong \Sigma \times \mathbb{R}$ , where  $\Psi(M)$  is the  $\eta$ -non-cuspidal part of  $M$ . We write  $d$  for the path-metric on  $\Psi(M)$  induced from  $M$ .

We write  $\mathcal{T}$  for the set of  $\eta$ -Margulis tubes in  $M$ . Thus,  $\bigcup \mathcal{T} \subseteq \text{int } \Psi(M)$ , and  $\Theta(M) = \Psi(M) \setminus \text{int}(\bigcup \mathcal{T})$  is the  $\eta$ -thick part of  $M$ .

The following result of Otal [Ot3] will be reproven in Section 2.13 (though our argument will not give a computable estimate on the constant  $\eta$  which is implicit in the original).

**Theorem 2.1.1.** *There is a constant,  $\eta(\Sigma)$ , depending only on the topological type of  $\Sigma$ , such that the set of  $\eta(\Sigma)$ -Margulis tubes in  $\Psi(M)$  is topologically unlinked.*

In other words, we can choose the homeomorphism from  $\Psi(M)$  to  $\Sigma \times \mathbb{R}$  in such a way that the core of each Margulis tube lies in  $\Sigma \times \mathbb{Z}$ . (See Proposition 2.3.1 here.) Clearly the conclusion also applies for any  $\eta \leq \eta(\Sigma)$ .

For the purposes of this section, we will assume that  $\Psi(M)$  and  $\Theta(M)$  are defined by some fixed  $\eta \leq \eta(\Sigma)$ . (One can allow for some flexibility in the choice of  $\eta$ , without any essential change. This will be useful later, as discussed in Section 2.7.)

Given  $\gamma \in \mathbf{C}(\Sigma)$ , let  $\gamma_M^*$  be the geodesic representative of  $\gamma$  in  $M$ , and write  $l_M(\gamma) = \text{length}(\gamma_M^*)$ . Let  $\mathbf{C}(M, l) = \{\gamma \in \mathbf{C}(\Sigma) \mid l_M(\gamma) \leq l\}$ . Note that it is a consequence of Theorem 2.1.1, that  $\mathbf{C}(M, \eta)$  is precisely the set of core curves of Margulis tubes in  $M$ .

Recall that a subset,  $Y$ , of the curve graph  $\mathcal{G}(\Sigma)$  is called  *$r$ -quasiconvex* if any geodesic in  $\mathcal{G}(\Sigma)$  with endpoints in  $Y$  lies in the  $r$ -neighbourhood  $N(Y, r)$  of  $Y$ . The following is a consequence of the arguments used in proving the A-priori Bounds Theorem of Minsky:

**Theorem 2.1.2.** *There are constants,  $l_0$  and  $r_0$  depending only on  $\xi(\Sigma)$  such that  $\mathbf{C}(M, l_0)$  is  $r_0$ -quasiconvex in  $\mathcal{G}(\Sigma)$ , and such that for all  $l \geq l_0$  we have  $\mathbf{C}(M, l) \subseteq N(\mathbf{C}(M, l_0), t)$ , where  $t$  depends only on  $\xi(\Sigma)$  and  $l$ .*

This is given explicitly in [Bow3], though it is also a consequence of the arguments given in [Mi4]. Note that it follows that  $\mathbf{C}(M, l)$  is  $r$ -quasiconvex, where  $r$  depends only on  $\xi(\Sigma)$  and  $l$ .

One can say more [Bow3]:

**Theorem 2.1.3.** *There is some  $t_0 \geq 0$  depending only on  $\xi(\Sigma)$  such that  $\mathbf{C}(M, l_0) \subseteq N(\pi, t_0)$ , where  $\pi$  is a geodesic segment in  $\mathcal{G}(\Sigma)$ .*

Since  $\mathcal{G}(\Sigma)$  is hyperbolic,  $\pi$  is determined up to bounded Hausdorff distance, depending on  $\xi(\Sigma)$ .

Recall that  $\text{core}(M)$  denotes the convex core of  $M$ . The manifold  $\Psi(M) \cap \text{core}(M)$  is homeomorphic to  $\Sigma \times I$ , where  $I \subseteq \mathbb{R}$  is a compact interval, a ray or all of  $\mathbb{R}$ , depending on whether  $\Psi(M)$  has 0, 1 or 2 degenerate ends. These cases are termed respectively “geometrically finite” (or “quasifuchsian”), “singly degenerate” or “doubly degenerate”. In the three cases, the geodesic  $\pi$  will be respectively a compact interval, a ray, or a bi-infinite geodesic. From the discussion

of Section 1.5, we see that the unbounded ends of  $\pi$  converge to the end invariants of  $M$  in  $\partial\mathcal{G}(\Sigma)$ .

We can relate these more explicitly as follows. Given a path,  $\zeta$ , in  $\Psi(M)$ , let  $l_\rho(\zeta) = \text{length}(\zeta \cap \Theta(M))$ . Given  $x, y \in \Psi(M)$ , let  $\rho_M(x, y) = \inf\{l_\rho(\theta)\}$ , as  $\theta$  ranges over all paths from  $x$  to  $y$  in  $\Psi(M)$ . (Figure 2.1.) (In fact, the

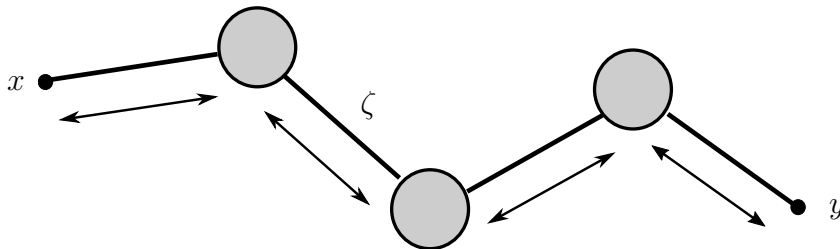


FIGURE 2.1. Definition of the electric distance  $\rho(x, y)$

minimum is attained — by a path which intersects  $\Theta(M)$  in a finite union of geodesic segments.) Thus,  $\rho_M$  is a pseudometric on  $\Psi(M)$ , with each Margulis tube having zero diameter.

**Definition.** We refer to  $\rho_M$  as the *electric pseudometric* on  $\Psi(M)$ .

We will refer to a curve,  $\gamma$ , in  $\Psi(M)$  as “simple” if it is homotopic to a simple non-peripheral curve in  $\Sigma$ . We shall write  $[\gamma] \in \mathbf{C}(\Sigma)$  for its homotopy class. Now each point of  $\Psi(M)$ , lies in some simple curve in  $\Psi(M)$  of  $d$ -length bounded by some constant depending only on  $\xi(\Sigma)$ . We may as well denote the length bound by  $l_0$  (as in Theorem 2.1.2). We shall write  $\gamma_x$  for some choice of such curve. Moreover,  $\gamma_x$ , can be taken to lie in some homotopy fibre of  $\Psi(M)$ , of bounded  $\rho_M$ -diameter. These facts follow from interpolation of pleated surfaces described by Thurston, and one can give explicit computable estimates for the bounds. They are also consequences of the results of Section 2.10 (though these do not give computable bounds). One immediate consequence is the fact that  $(\Psi(M) \cap \text{core}(M), \rho_M)$  is quasi-isometric to an interval in the real line (again, see Section 2.10). Note that the homotopy class,  $[\gamma_x]$ , gives a point in  $\mathbf{C}(\Sigma)$ .

One can elaborate on this. The following discussion briefly explains how the various pieces fit together. While it is not needed directly for the Ending Lamination Theorem, it finds application elsewhere.

One can show:

**Theorem 2.1.4.** *Suppose that  $\alpha_M^*, \beta_M^*$  are simple curves in  $M$  of  $d$ -length at most  $l \geq 0$ . Write  $\alpha = [\alpha_M^*] \in \mathbf{C}(\Sigma)$  and  $\beta = [\beta_M^*] \in \mathbf{C}(\Sigma)$ . Then:*

(1)  $\rho_M(\alpha_M^*, \beta_M^*) \leq k_1 d_G(\alpha, \beta) + k_2$

(2)  $d_G(\alpha, \beta) \leq k_3 \rho_M(\alpha_M^*, \beta_M^*) + k_4$

where the constants  $k_1, k_2, k_3, k_4$  depend only on  $\xi(\Sigma)$  and  $l$ .

Part (1) of Theorem 2.1.4 can also be proven using pleated surfaces. This arises from the fact, alluded to in Section 1.5, that any pair of disjoint simple geodesics in  $M$  can be realised in a pleated surface in  $M$ , and the intersection of such a pleated surface  $\Psi(M)$  has bounded  $\rho_M$ -diameter (see for example, the discussion in [Bow3] or Section 3.2 here). This argument gives computable bounds on  $k_1$  and  $k_2$ . Part (2) involves quite a bit more work. It follows from the existence of model spaces. In fact, in our account of the Ending Lamination Theorem, we use a closely related result as the first step in proving lower bounds in Section 2.10. As we will mention there, by a slight variation of the argument we can deduce Theorem 2.1.4 as a corollary (thereby bypassing the remainder of the proof). Unfortunately, this argument does not give us computable bounds on  $k_3$  or  $k_4$ .

Putting Theorem 2.1.4 together with the previous paragraph, we can relate these various facts. First, we can define a map from  $(\Psi(M) \cap \text{core}(M), \rho_M)$  to  $\mathcal{G}(\Sigma)$  by sending  $x$  to  $[\gamma_x]$ . By Theorem 2.1.4, this is well defined up to bounded distance. Its image lies in  $\mathbf{C}(M, l_0)$ . Moreover, every point in  $\mathbf{C}(M, l_0)$  lies a bounded distance from some point in this image. (If  $\alpha \in \mathbf{C}(M, l_0)$ , choose any point  $x$  in the closed geodesic,  $\alpha^* \subseteq M$ , then  $d_{\mathcal{G}}(\alpha, [\gamma_x])$  is bounded.) The map  $x \mapsto [\gamma_x]$  is therefore a quasi-isometric embedding of  $(\Psi(M) \cap \text{core}(M), \rho_M)$  into  $\mathcal{G}(\Sigma)$ , whose image is a bounded Hausdorff distance from  $\mathbf{C}(M, l_0)$ , and hence also from the geodesic  $\pi$  described by Theorem 2.1.3. We therefore have a quasi-isometry from  $(\Psi(M) \cap \text{core}(M), \rho_M)$  to  $\pi$ , which is natural up to bounded Hausdorff distance. This fits in with the earlier observation that  $(\Psi(M) \cap \text{core}(M), \rho_M)$  is quasi-isometric to a real interval (though the constants we get by this argument are no longer computable).

We can also tie this in with the description of the model space used in the proof of the Ending Lamination Theorem. This is a riemannian manifold,  $P$ , which a preferred “non-cuspidal” part,  $\Psi(P)$ , diffeomorphic to  $\Sigma$  times a real interval. The construction of  $P$  starts with a (tight) geodesic in  $\mathcal{G}(\Sigma)$ . One can similarly define an electric pseudometric,  $\rho_P$ , on  $P$ . It is easily seen from the construction that every point  $x \in P$ , lies in a curve  $\gamma_x$  of bounded length, which in turn lies in a fibre,  $S(x)$ , of bounded  $\rho_P$ -diameter. By the construction of  $P$ , the distance between two curves,  $[\gamma_x]$  and  $[\gamma_y]$  in  $\mathcal{G}(\Sigma)$  agrees with  $\rho_P(x, y)$  up to linear bounds. These facts can then be translated across to  $M$  via Theorem 2.13.9.

These observations have applications or potential applications beyond hyperbolic geometry. Some of these are based on the following observation. Although it is not directly relevant to the Ending Lamination Theorem, we note:

**Theorem 2.1.5.** *Given any  $\alpha, \beta$  in  $\mathbf{C}(\Sigma)$  and any  $\epsilon > 0$ , there is a complete hyperbolic 3-manifold,  $M$ , with  $\Psi(M) \cong \Sigma \times \mathbb{R}$ , and with  $\alpha, \beta \in \mathbf{C}(M, \epsilon)$ .*

In other words, we can realise any pair of curves  $\alpha, \beta$  as arbitrarily short geodesics,  $\alpha_M^*, \beta_M^*$ , in some such manifold. Theorem 2.1.5 is a simple consequence of the deformation theory of quasifuchsian groups. It is described explicitly in



[Bow4]. As a result, hyperbolic 3-manifolds can help us understand the geometry of  $\mathcal{G}(\Sigma)$ .

For example, note that by Theorem 2.1.4,  $d_{\mathcal{G}}(\alpha, \beta)$  agrees with  $\rho_M(\alpha_M^*, \beta_M^*)$  up to linear bounds depending only on  $\xi(\Sigma)$ . This finds application, for example, in [Bow3], [Bow4] and [Ta].

## 2.2. THE CURVE GRAPH.

Let  $\Sigma$  be a compact surface, and  $\xi(\Sigma)$  be its complexity, as defined in Section 1.5. We assume that  $\xi(\Sigma) \geq 1$ . Let  $\Sigma$  be the curve graph  $\mathcal{G}(\Sigma)$ , with vertex set  $\mathbf{C}(\Sigma) = V(\mathcal{G}(\Sigma))$ , as defined in Section 1.5. We write  $d = d_{\mathcal{G}}$  for the combinatorial metric on  $\mathcal{G}(\Sigma)$ . Given  $\alpha, \beta \in \mathcal{G}(\Sigma)$ , we write  $\iota(\alpha, \beta)$  for the geometric intersection number (i.e. the minimal number of intersections among realisations of  $\alpha$  and  $\beta$  in  $\Sigma$ ). It follows from work of Lickorish that  $d(\alpha, \beta)$  is bounded above in terms of  $\iota(\alpha, \beta)$ . (In fact, one can show that  $d(\alpha, \beta) \leq \iota(\alpha, \beta) + 1$ , or that  $d(\alpha, \beta) = O(\log \iota(\alpha, \beta))$ .) Recall that a subset of a graph is said to be **locally finite** if every bounded subset thereof is finite.

It is often convenient to fix some hyperbolic structure on  $\Sigma$  with geodesic boundary,  $\partial\Sigma$ . In this way, each curve is realised uniquely as a closed geodesic in  $\Sigma$ , and we can use the same notation for a curve and its realisation. This serves purely to simplify the description of certain combinatorial constructions, and bears no relation to the various geometric structures in which we have a genuine interest.

**Definition.** A **multicurve**,  $\gamma$ , in  $\Sigma$  is a non-empty disjoint union of curves.

We write  $\mathbf{C}(\gamma) \subseteq \mathbf{C}(\Sigma)$  for the set of components of  $\gamma$ . A multicurve,  $\gamma$ , is **complete** if  $\mathbf{C}(\Sigma)$  has maximal cardinality. In this case, each component of  $\Sigma \setminus \gamma$  is a 3HS. We note that the number of curves in a complete multicurve is equal to  $\xi(\Sigma)$ .

We noted in Section 1.5 that  $\mathcal{G}(\Sigma)$  is Gromov hyperbolic [MasM1], and write  $\partial\mathcal{G}(\Sigma)$  for the Gromov boundary.

Suppose, for the moment, that  $\xi(\Sigma) \geq 2$ . A **multigeodesic** is a sequence  $(\gamma_i)_i$  of multicurves such that for all  $i, j$  and all  $\alpha \in \mathbf{C}(\gamma_i)$  and  $\beta \in \mathbf{C}(\gamma_j)$ ,  $d(\alpha, \beta) = |i - j|$ . It is **tight** if for all non-terminal indices,  $i$ , any curve that crosses  $\gamma_i$  must also cross either  $\gamma_{i-1}$  or  $\gamma_{i+1}$ . A **tight geodesic**,  $(\alpha_i)_i$  is a sequence of curves such that there exists a tight multigeodesic  $(\gamma_i)_i$  such that  $\alpha_i \in \mathbf{C}(\gamma_i)$  for all  $i$ .

*Remark.* Note that this “tight” terminology is now standard, though potentially confusing: A tight geodesic is a geodesic in  $\mathcal{G}$  in the usual sense. However it need not be tight as a multigeodesic, in the sense defined. In most cases we will be talking about tight geodesics. We will only need to specify tight multigeodesic for a construction in Section 2.4: see Lemma 2.4.1.

The notion of tightness was introduced in [MasM2]. (Our definition is slightly more general, but the distinction does not matter to us here.) They show that the set of tight geodesics between any two curves is non-empty and finite. Other arguments for finiteness are given in [Bow3] and [Sh], the latter giving explicit bounds. For a closed surface, one can also get explicit bounds from [Lea]. We remark that further refinement of these results can be found in [We].

One can make a stronger statement, for example:

**Lemma 2.2.1.** *Suppose that  $(\alpha_i)_{i \in \mathbb{N}}$  and  $(\beta_i)_{i \in \mathbb{N}}$  are sequences of curves, each converging to a point of either  $\partial\mathcal{G}(\Sigma)$  or  $\mathbf{C}(\Sigma)$ . Then for any bounded set  $A \subseteq X$ , there is a finite subset,  $B \subseteq A$ , such that for all sufficiently large  $i, j$  any curve in  $A$  also lying on any tight geodesic from  $\alpha_i$  to  $\beta_j$  must lie in  $B$ .*

*Proof.* Let  $a, b \in V(\mathcal{G}) \cup \partial\mathcal{G}$  be the limits of  $(\alpha_i)_i$  and  $(\beta_i)_i$ . If  $a, b \in V(\mathcal{G}) = \mathbf{C}(\Sigma)$ , then the result is immediate from the finiteness of tight geodesics between two points [MasM2]. Suppose  $a, b \in \partial\mathcal{G}$ . If  $a = b$ , we can take  $B = \emptyset$ . If  $a \neq b$ , the result follows from other finiteness results for tight geodesics. For example, in [Bow3] (Theorem 1.2 thereof), it is shown that if  $\alpha, \beta \in \mathbf{C}(\Sigma)$  and  $r \in \mathbb{N}$ , then there is some finite  $C \subseteq \mathbf{C}(\Sigma)$  such that if  $(\gamma_i)_{i=0}^p$  is a tight geodesic with  $d(\alpha, \gamma_0) \leq r$  and  $d(\beta, \gamma_p) \leq r$ , then  $\gamma_i \in C$  for all  $i$  with  $12r \leq i \leq p - 12r$ . (See [We] for some related results.) The rest is just an exercise in hyperbolic spaces. We can take  $\alpha$  and  $\beta$  arbitrarily close to  $a$  and  $b$  respectively, and  $r \geq 0$  so that for all  $i$  and  $j$  sufficiently large, any geodesic from  $\alpha_i$  to  $\beta_j$  meets both  $N(\alpha, r)$  and  $N(\beta, r)$ . Choosing  $\alpha$  and  $\beta$  far away from our bounded set  $A$ , the result now follows. Finally the case where  $a \in V(\mathcal{G})$  and  $b \in \partial\mathcal{G}$  follows by a variation on the above result, namely if  $\gamma_0 = \alpha$  and  $d(\beta, \gamma_p) \leq r$  and then  $\gamma_i$  lies in a finite subset for all  $i \leq p - 12r$ .  $\square$

This result allows us to use diagonal sequence arguments. For example, we obtain the fact [MasM2] that any two distinct boundary points are connected by a bi-infinite tight geodesic.

In the case where  $\xi(\Sigma) = 1$  we have noted that  $\mathcal{G}(\Sigma)$  is a Farey graph. In this case, every geodesic is deemed to be tight. The above statement, in particular Lemma 2.2.1, remain valid, and can be verified directly in that case.

**Definition.** By a *subsurface* of  $\Sigma$  we mean the closure, in  $\Sigma$ , of a non-empty connected open subsurface,  $\text{int}(\Phi)$ , of  $\Sigma$  with geodesic boundary,  $\partial\Phi$ .

We write  $\partial^\Sigma\Phi = \partial\Phi \setminus \partial\Sigma$ . (Figure 2.2.) We express it in this way since we want to allow the possibility of two boundary curves in  $\partial^\Sigma\Phi$  being identified to a single curve in  $\Sigma$ . (We could homotope  $\Phi$  to an embedded surface with a complementary annulus so that these boundary curves become genuinely distinct, though for most purposes, it will be convenient to realise things with respect to some fixed hyperbolic structure.) We are allowing  $\Sigma$  as a subsurface of itself. A subsurface is not allowed to be a disc or an annulus. Note that if  $\Phi'$  is a proper subsurface of  $\Phi$  then  $\xi(\Phi') < \xi(\Phi)$ .

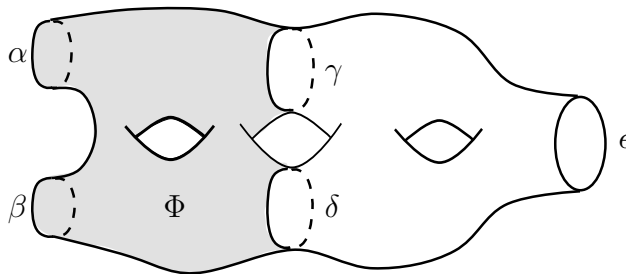


FIGURE 2.2.  $\partial^\Sigma \Phi = \gamma \cup \delta$ ,  $\partial \Phi = \alpha \cup \beta \cup \gamma \cup \delta$ ,  $\partial \Sigma = \alpha \cup \beta \cup \epsilon$

The following definitions arise out of the discussion of “hierarchies” in [MasM2]. Let  $\xi = \xi(\Sigma)$ . Given  $Q \subseteq \mathbf{C}(\Sigma)$  and  $k \in \mathbb{N}$ , let  $Y_k(Q)$  be  $Q$  together with all those curves  $\gamma \in \mathbf{C}(\Sigma)$  such that there is some subsurface  $\Phi \subseteq \Sigma$  with  $\mathbf{C}(\partial^\Sigma \Phi) \subseteq Q$  and  $2 \leq \xi(\Phi) \leq \xi - k + 1$ , and two curves  $\alpha, \beta \in Q \cap \mathbf{C}(\Phi)$ , such that  $\gamma$  lies on some tight geodesic in  $\mathcal{G}(\Phi)$  from  $\alpha$  to  $\beta$ . Note that, for  $k \geq \xi$  there is no such subsurface, so  $Y_k(Q) = Q$ . For any  $k$ , we set  $Y^k(Q) = Y_k Y_{k-1} Y_{k-2} \cdots Y_1(Q)$ , and set  $Y^\infty(Q) = Y^\xi(Q)$ .

Note that  $Y^\infty(Q)$  contains the union,  $Y_0(Q)$ , of all tight geodesics between any pair of points of  $Q \subseteq \mathbf{C}(\Sigma)$ . (For the first step, we are allowing  $\Phi = \Sigma$ .) However, all constructions involving proper subsurfaces occur in a 1-neighbourhood in  $\mathbf{C}(\Sigma)$  of a curve already constructed. In particular, we see that  $Y^k(Q) \subseteq N(Y_0(Q), k)$  and so  $Y^\infty(Q) \subseteq N(Y_0(Q), \xi)$ . If  $Q$  is locally finite, then it follows that  $Y(Q)$  is locally finite (since only finitely many subsurfaces enter into the construction in any bounded set). Thus, inductively,  $Y_k(Q)$  is locally finite, and it follows that  $Y^\infty(Q)$  is locally finite. Also, Lemma 2.2.1, implies the following:

**Lemma 2.2.2.** *Suppose that  $(\alpha_i)_{i \in \mathbb{N}}$  and  $(\beta_i)_{i \in \mathbb{N}}$  are sequences of curves, each converging to a point of either  $\partial \mathcal{G}(\Sigma)$  or  $\mathbf{C}(\Sigma)$ . Then for any bounded set  $A \subseteq X$ , there is a finite subset,  $B \subseteq X$ , such that for all sufficiently large  $i, j$ ,  $A \cap Y^\infty(\{\alpha_i, \beta_j\}) \subseteq B$ .*

Thus, after passing to a subsequence, we can assume that  $Y^\infty(\{\alpha_i, \beta_i\})$  stabilises on each bounded set to some finite set, thereby giving us a locally finite limit.

We have the following variation for subsurfaces of complexity 1. Given  $Q \subseteq \mathbf{C}(\Sigma)$ , define  $\bar{Y}(Q)$  as for  $Y_k(Q)$ , replacing the statement “ $2 \leq \xi(\Phi) \leq \xi - k + 1$ ” by the statement “ $\xi(\Phi) = 1$ ”. (All geodesics in  $\mathcal{G}'(\Phi)$  are deemed to be “tight” in this case.) We write  $\bar{Y}^\infty(Q) = \bar{Y}(Y^\infty(Q))$ . A similar discussion applies. In particular,

**Lemma 2.2.3.** *Lemma 2.2.2 holds with  $\bar{Y}^\infty$  replacing  $Y^\infty$ .*

A number of variations on the above definitions are possible, and would probably serve just as well. We have chosen a formulation that works well with our inductive constructions below, and in Section 2.4.

A **path of multicurves** is a sequence  $(\gamma_i)_i$  of multicurves such that for each  $i$ ,  $\gamma_{i+1}$  is obtained from  $\gamma_i$  by either adding or deleting one component. A path  $(\alpha_i)_i$  in  $\mathcal{G}$  determines a path of multicurves by inserting  $\alpha_i \cup \alpha_{i+1}$  between  $\alpha_i$  and  $\alpha_{i+1}$ .

**Lemma 2.2.4.** *Suppose that  $\alpha, \beta \in \mathbf{C}(\Sigma)$  with  $d(\alpha, \beta) \geq 3$ . Then there is a complete multicurve,  $\gamma$ , with  $\alpha \subseteq \gamma$  and  $\mathbf{C}(\gamma) \subseteq Y^\infty(\{\alpha, \beta\})$ .*

*Proof.* In fact, we will construct a whole path of multicurves, which connect a pair of complete multicurves which respectively contain  $\alpha$  and  $\beta$ . We will then take the multicurve containing  $\alpha$  and forget the rest. The idea is simple, start with a tight geodesic and inductively fill up complementary subsurfaces. We will express the argument as a formal induction.

Let  $\underline{\gamma} = (\gamma_i)_{i \in D}$  be a path of multicurves with indexing set  $I = \{0, \dots, m\}$ , so that  $\gamma_0 = \alpha$  and  $\gamma_m = \beta$ . Given any  $i \in I$ , write  $c_i = |\mathbf{C}(\gamma_i)|$  for the number of components of  $\gamma_i$ . Thus  $c_0 = c_m = 1$ . We say that  $i$  is a ‘‘local minimum’’ if  $c_{i-1} = c_{i+1} = c_i + 1$ . We assume:

(\*) If  $i$  is not a local minimum, then  $\mathbf{C}(\gamma_i) \subseteq Y^{c_i}(\{\alpha, \beta\})$ .

Note that such a path of multicurves exists: just take any tight geodesic from  $\alpha$  to  $\beta$  and construct a path of multicurves (each with one or two components) from it as described earlier.

Let  $\xi = \xi(\Sigma)$ . Given any  $n \in \{1, 2, \dots, \xi\}$  let  $v_n(\underline{\gamma}) = |\{i \in I \mid c_i = n\}|$  and  $v(\underline{\gamma}) = (v_1(\underline{\gamma}), \dots, v_\xi(\underline{\gamma})) \in \mathbb{N}^\xi$ . We order  $\mathbb{N}^\xi$  lexicographically. It is thus well ordered. We now choose  $\underline{\gamma}$  so as to minimise  $v(\underline{\gamma})$  among all paths of multicurves (of any length) satisfying (\*) above.

Given any  $i \in I$ , let  $F_i$  be the union of  $\gamma_i$  and all those components of  $\Sigma \setminus \gamma_i$  which are three-holed spheres.

First, we claim that if  $i$  is a local minimum, then  $F_{i-1} = F_{i+1}$ . To see this let  $\gamma_{i-1} = \gamma_i \cup \delta$  and  $\gamma_{i+1} = \gamma_i \cup \epsilon$ . Note that  $i-1$  and  $i+1$  are not local minima, and so  $\delta, \epsilon$  and all the components of  $\gamma_i$  lie in  $Y^{c_i+1}(\{\alpha, \beta\})$ . Now  $\delta$  and  $\epsilon$  must cross, otherwise we could replace  $\gamma_i$  by  $\gamma_i \cup \delta \cup \epsilon$ , thereby decreasing  $v(\underline{\gamma})$ . Thus,  $\delta$  and  $\epsilon$  lie in the same component,  $\Phi$ , of  $\Sigma \setminus \gamma_i$ . If  $\xi(\Phi) > 1$ , then we can connect  $\delta$  to  $\epsilon$  by some tight geodesic  $\delta = \delta_0, \delta_1, \dots, \delta_p = \epsilon$  in  $\mathcal{G}(\Phi)$ . Now each  $\delta_j \in Y_{c_i+2}(Y^{c_i+1}(\{\alpha, \beta\})) = Y^{c_i+2}(\{\alpha, \beta\})$ . We can now replace  $\gamma_i$  by the sequence

$$\gamma_i \cup \delta_0 \cup \delta_1, \quad \gamma_i \cup \delta_1, \quad \gamma_i \cup \delta_1 \cup \delta_2, \quad \dots, \quad \gamma_i \cup \delta_{p-1}, \quad \gamma_i \cup \delta_{p-1} \cup \delta_p.$$

To verify (\*), note that the  $\gamma_i \cup \delta_j$  are all at local minima, and that each  $\gamma_i \cup \delta_j \cup \delta_{j+1}$  has  $c_i + 2$  components. We have therefore again reduced  $v(\underline{\gamma})$ . We conclude that  $\xi(\Phi) = 1$ . From this it follows that  $F_{i-1} = F_i \cup \Phi = F_{i+1}$ , as claimed.

Next, we claim that there does not exist any  $i$  with  $c_{i+2} = c_{i+1} + 1 = c_i + 2 = c_{i-1} + 1$ . For in such a case, we have  $F_{i-1} = F_{i+1}$  by the above, and so we could replace  $\gamma_i$  by  $\gamma_{i-1} \cup \delta$  and  $\gamma_{i+1}$  by  $\gamma_i \cup \delta$ , where  $\delta = \gamma_{i+2} \setminus \gamma_{i+1}$ . Note that since  $\gamma_{i+1}$  and  $\gamma_{i+2}$  were not local minima in the original path, all curves lie in  $Y^{c_i+1}(\{\alpha, \beta\})$ .

We have reduced  $v(\gamma)$  giving a contradiction. By the same argument, we cannot have any  $i$  with  $c_{i-2} = c_{i-1} + 1 = c_i + 2 = c_{i+1} + 1$ .

We have thus shown that  $c_i$  increases monotonically from  $c_0 = 1$  up to a maximum value  $c_p = k$  and then alternates between  $k$  and  $k - 1$  before decreasing monotonically from  $c_q = k$  down to  $c_m = 1$ . By the first observation, we see that  $F_p = F_q$ . But  $\alpha \subseteq \gamma_p \subseteq F_p$  and  $\beta \subseteq \gamma_q \subseteq F_q$ . Since  $\alpha \cup \beta$  fills  $\Sigma$ , it follows that  $F_p = F_q = \Sigma$ . Thus  $\gamma_p$  is a complete multicurve (i.e.  $c_p = k = \xi$ ). Moreover,  $\alpha \subseteq \gamma_p$  and  $\mathbf{C}(\gamma_p) \subseteq Y^\xi(\{\alpha, \beta\})$ . We can thus set  $\gamma = \gamma_p$ .  $\square$

We will use this result in Section 2.4 when we construct “annulus systems”.

### 2.3. TOPOLOGY OF PRODUCTS.

In this section, we give some discussion to the topology of a product  $\Psi = \Sigma \times \mathbb{R}$ , where  $\Sigma$  is a compact surface. In particular, we study collections of unlinked curves in  $\Psi$ . The main result we are aiming for will be Proposition 2.3.2. We also give a criterion for unlinking (Proposition 2.3.7), which can be used to reprove Otal’s theorem [Ota3], and generalise to the decomposable case (see Section 2.13 and Proposition 3.7.1).

Some further discussion of product spaces will be given in Section 4.3, though that is only required for the construction of the bilipschitz map.

We will write  $\partial\Psi = \partial\Sigma \times \mathbb{R}$ . We write  $\pi_\Sigma : \Psi \rightarrow \Sigma$  and  $\pi_V : \Psi \rightarrow \mathbb{R}$  for the vertical and horizontal projections respectively.

**Definition.** A *horizontal fibre* in  $\Psi$  is a subset of the form  $\Sigma \times \{t\}$  for some  $t \in \mathbb{R}$ .

More generally we have:

**Definition.** A *fibre*,  $S \subseteq \Psi$ , in  $\Sigma$  is the image  $f(\Sigma)$  in  $\Psi$  of an embedding  $f : \Sigma \rightarrow \Psi$  with  $f^{-1}(\partial\Psi) = \partial\Sigma$  such that  $\pi_\Sigma \circ f$  is homotopic to the identity on  $\Sigma$  (rel  $\partial\Sigma$ ).

If  $S, S'$  are two fibres, we write  $S < S'$  if they are disjoint and  $S'$  can be homotoped to the positive end of  $\Psi$  in  $\Psi \setminus S$ .

The theorem of Brown [Brow] (given as Theorem 1.6.3 here) tells us that if  $S$  is any fibre in  $\Psi$ , then there is an ambient isotopy sending  $S$  to a horizontal fibre. Inductively, we see that if  $S_1, S_2, \dots, S_n$  are disjoint fibres, then there is an ambient isotopy of  $\Psi$  sending each  $S_i$  to a horizontal fibre,  $S \times \{t_i\}$ . After permuting the indices, we can assume that  $t_1 < t_2 < \dots < t_n$ , and so  $S_1 < S_2 < \dots < S_n$ . From this, we see easily that  $<$  defines a total order on any locally finite set of disjoint fibres. If  $S < S'$  we write  $[S, S']$  for the compact region bounded by  $S$  and  $S'$ .

By a *homotopy fibre*, we mean a map  $f : \Sigma \rightarrow \Psi$  with  $f^{-1}(\partial\Psi) = \partial\Sigma$  and with  $\pi_\Sigma \circ f$  homotopic to the identity. (We can always take such a map to be

in general position.) We will also sometimes refer to its image as a homotopy fibre. We can also define an order on homotopy fibres by writing  $f(\Sigma) < f(\Sigma')$  if  $f(\Sigma) \cap f(\Sigma') = \emptyset$  and  $f(\Sigma')$  can be homotoped out the positive end in  $\Psi \setminus f(\Sigma)$ . This is again a total order on any pairwise disjoint, locally finite collection of homotopy fibres. Moreover, if  $S, S'$  are fibres sufficiently close to  $f(\Sigma)$  and  $f(\Sigma')$ , then  $S < S'$  if and only if  $f(\Sigma) < f(\Sigma')$ .

**Definition.** By a *curve* in  $\Psi$  we mean an embedded simple closed curve in  $\Psi$ , that is not homotopic to a point or into  $\partial\Psi$ .

(In practice, we will only be interested in curves which are also homotopically simple in  $\Sigma$ , but we don't need to take that as a hypothesis.)

We want to define the notion of unlinking of curves in  $\Psi$ . This is based on the following:

**Proposition 2.3.1.** *Let  $\mathcal{L}$  be a locally finite disjoint collection of curves in  $\Psi = \Sigma \times \mathbb{R}$ . The following are equivalent:*

- (1) *There is a homeomorphism  $f : \Psi \rightarrow \Psi$  such that  $f(\bigcup \mathcal{L}) \subseteq \Sigma \times \mathbb{Z}$ .*
- (2) *There is a collection of pairwise disjoint embedded fibres,  $(S(\alpha))_{\alpha \in \mathcal{L}}$ , such that  $\alpha \subseteq S(\alpha)$  for all  $\alpha \in \mathcal{L}$ .*
- (3) *There is a collection,  $(f_\alpha)_{\alpha \in \mathcal{L}}$ , of homotopy fibres  $f_\alpha : \Sigma \rightarrow \Psi$ , such that  $f_\alpha|_{f_\alpha^{-1}(\alpha)}$  is a homeomorphism from  $f_\alpha^{-1}(\alpha)$  to  $\alpha$ , and such that  $f_\alpha(\Sigma) \cap \beta = \emptyset$  for all  $\beta \in \mathcal{L} \setminus \{\alpha\}$ .*

Note that, in (1), we could always take  $f$  to be properly homotopic to the identity. Also, there is no loss in assuming that for each  $i \in \mathbb{Z}$ ,  $f^{-1}(\Sigma \times \{i\})$  contains at most one element of  $\mathcal{L}$ . Moreover, it is a consequence of Proposition 2.3.1 that in (2) and (3), we can take the fibres to be locally finite in  $\Psi$ , though this is not assumed a-priori.

*Proof.*

From the above observation, it is clear that (1)  $\Rightarrow$  (2). Trivially, (2)  $\Rightarrow$  (3).

(2)  $\Rightarrow$  (1):

Note that we do not assume a-priori that the fibres are locally finite in  $\Psi$ . We claim that we can, if necessary, modify them so that they become locally finite. From this, (1) can be deduced using Waldhausen's Cobordism Theorem (stated as Theorem 1.6.2 here).

Let  $<$  be the total order on  $\mathcal{L}$  induced by the order of the fibres  $(S(\gamma))_\gamma$ . Since  $\mathcal{L}$  is locally finite, we see that this order is discrete. (There can only be finitely many curves trapped between any two homotopy fibres.) Thus, we can index  $\mathcal{L}$  by a subset  $\mathcal{I} \subseteq \mathbb{Z}$  such that  $\gamma_i < \gamma_j$  if and only if  $i < j$ . We can assume that  $0 \in \mathcal{I}$ . By Brown's result (Theorem 1.6.3 here), we can assume that  $S(\gamma_0) = \Sigma \times \{0\}$ . We will obtain  $(S(\gamma_i))_i$  locally finite for  $i \geq 0$  and for  $i \leq 0$  separately. To this end, we may as well assume that  $\mathcal{I} = \mathbb{N}$  (or an initial segment of  $\mathbb{N}$ ).

Given  $i \in \mathbb{N}$ , let  $R(i)$  be the closed unbounded subset of  $\Sigma \times [0, \infty)$  with relative boundary  $S(\gamma_i)$ . Thus  $(R(i))_i$  is a decreasing sequence of subsets of  $\Psi$ . Since  $\mathcal{L}$  is locally finite, for each  $n \in \mathbb{N}$ , there is some  $i(n)$  such that  $R(i(n)) \cap \bigcup \mathcal{L} \subseteq \Sigma \times [n+1, \infty)$ . Inductively over  $n$ , we now find ambient isotopies of the  $S(\gamma_i)$  for  $i \geq i(n)$ , supported on  $\Sigma \times [n-1, n+1]$ , which push the  $S(\gamma_i)$  into  $\Sigma \times [n, \infty)$ , while fixing the curves  $\gamma_i$ . Note that the  $S(\gamma_i)$  remain disjoint. Moreover, the process stabilises on a disjoint locally finite set of embedded fibres.

We can do the same for  $i < 0$ . We end up with each of the  $S(\gamma_i)$  embedded and locally finite. Now Waldhausen's cobordism theorem (Theorem 1.6.2) gives a homeomorphism from the region of  $\Psi$  bounded by  $S(\gamma_i)$  and  $S(\gamma_{i+1})$  to  $\Sigma \times [i, i+1]$ . Moreover, we can assume that these agree on each  $S(\gamma_i)$ , so combining them give the homomorphism required by (1).

(3)  $\Rightarrow$  (2):

Note that by Theorem 1.6.4, we can assume each of the  $f_\gamma$  to be injective. We write  $Z(\gamma) = f_\gamma(\Sigma)$ . Let  $\mathcal{L} = (\gamma_i)_{i \in I}$  be an arbitrary indexing of  $\mathcal{L}$ , where  $I$  is an initial segment (or all) of  $\mathbb{N}$ . We inductively replace  $Z(\gamma_i)$  by another fibre,  $S(\gamma_i) \supseteq \gamma_i$ , such that the  $S(\gamma_i)$  are all pairwise disjoint. Suppose that we have found disjoint  $S(\gamma_0), \dots, S(\gamma_n)$ , and that  $S(\gamma_i) \cap \gamma_j = \emptyset$  for all  $i \leq n$  and  $j > n$ . Let  $R$  be the component of  $\Psi \setminus \bigcup_{i=0}^n S(\gamma_i)$  containing  $\gamma_{n+1}$ . Using Brown's result (Theorem 1.6.3) we can isotope  $Z(\gamma_{n+1})$  into  $R$  to obtain a fibre  $S(\gamma_{n+1}) \supseteq \gamma_{n+1}$  contained in an arbitrarily small neighbourhood of  $Z(\gamma_{n+1}) \cup \partial R$ . In particular,  $S(\gamma_{n+1})$  is disjoint from  $S(\gamma_i)$  for  $i < n$  and from all  $\gamma_j$  for  $j > n$ . We now proceed inductively to give  $S(\gamma_i)$  for all  $i$ .  $\square$

**Definition.** We say that a locally finite collection,  $\mathcal{L}$ , of curves in  $\Psi$  is *unlinked* if it satisfies any (hence all) of the conditions of Proposition 2.3.1. We say that a curve  $\alpha$  in  $\Psi$  is *unknotted* if  $\{\alpha\}$  is unlinked.

Note that an unknotted curve can be isotoped to be *horizontal*, that is, of the form  $\gamma \times \{t\}$ , where  $t \in \mathbb{R}$  and  $\gamma \subseteq \Sigma$  is a simple closed curve.

Suppose  $\gamma \subseteq \Psi$  is a curve. We can locally compactify  $\Psi \setminus \gamma$  by adjoining a toroidal boundary component  $\Delta(\gamma)$ . We can think of  $\Delta(\gamma)$  as identified with the unit normal bundle to  $\gamma$ . We write  $\Lambda(\gamma)$  for the resulting manifold. Note that it comes equipped with a natural homotopy class of meridian curve,  $m(\gamma)$ . We can recover  $\Psi$  up to homeomorphism by ‘‘Dehn filling’’, that is, gluing in a solid torus,  $T(\gamma)$ , along  $\Delta(\gamma)$  so that the meridian bounds a disc in  $T(\gamma)$ . If  $\gamma$  is unknotted, it also has a natural class of longitude. It can be defined as a simple curve that can be homotoped to infinity in  $\Lambda(\gamma)$ . It can also be determined by sitting the curve in some (indeed any) fibre.

More generally, if  $\mathcal{L}$  is any locally finite set of curves, we can form the manifold  $\Lambda(\mathcal{L})$  by adding toroidal boundaries to  $\Psi \setminus \bigcup \mathcal{L}$ . Thus  $\partial \Lambda = \partial \Psi \cup \bigcup_{\gamma \in \mathcal{L}} \Delta(\gamma)$ . (We can also think  $\Lambda(\mathcal{L})$  as the complement of an open regular neighbourhood of  $\bigcup \mathcal{L}$ .) Note that, by the Sphere Theorem for 3-manifolds [Hem1, Ja],  $\pi_2(\Lambda(\mathcal{L}))$

is trivial. (Any embedded sphere in  $\Lambda(\mathcal{L})$  must bound a ball in  $\Psi$  hence also in  $\Lambda(\mathcal{L})$ .)

We will need the following:

**Proposition 2.3.2.** *Suppose that  $\mathcal{L}$  and  $\mathcal{L}'$  are unlinked collections of curves. Suppose that no two elements of  $\mathcal{L}$  are homotopic in  $\Psi$ , and similarly for  $\mathcal{L}'$ . Suppose that  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}')$  is a proper degree-1 map with  $f^{-1}\partial\Lambda(\mathcal{L}') = \partial\Lambda(\mathcal{L})$ . Suppose that  $f|_{\partial\Lambda(\mathcal{L})}$  is a homotopy equivalence and sends the meridian and longitude of each toroidal boundary component to the meridian and longitude of its image. Then  $f$  is homotopic to a homeomorphism from  $\Lambda(\mathcal{L})$  to  $\Lambda(\mathcal{L}')$ .*

Note that  $f$  necessarily extends to a degree-1 map from  $\Psi$  to itself. Since the induced homomorphism on the surface group  $\pi_1(\Psi) \cong \pi_1(\Sigma)$  is surjective, it follows from the residual finiteness for such groups [Sc3] (via the hopfian property) that this extension is a homotopy equivalence of  $\Psi$ . (In fact, this will be immediate from the construction in our applications.) Thus, there is no loss in taking the extension to be homotopic to the identity on  $\Psi$ .

Degree-1 maps between 3-manifolds have been investigated by a number of authors (see, for example, [Wan] for a survey). There are certainly many examples which are not homotopy equivalences. Some positive results are also known, but I know of no result that directly implies the statement given above.

If we can show that  $\Lambda(\mathcal{L})$  and  $\Lambda(\mathcal{L}')$  are homeomorphic, then we are in reasonably good shape. Consider the case where  $\mathcal{L}$  is finite. Then  $\Lambda$  is a topologically finite and admits a hyperbolic structure hyperbolic by the work of Thurston [Ot1, Ka]. Thus  $\pi_1(\Lambda)$  is residually finite. Since it is finitely generated it is hopfian by a result of Malcev. It follows that  $f$  induces an isomorphism of fundamental groups (see [Hem2]). Since  $\pi_2(\Lambda)$  is trivial (by the Sphere Theorem),  $f$  is a homotopy equivalence. Now using the work of Waldhausen [Wal], it follows that  $f$  is homotopic to a homeomorphism. In the case where  $\mathcal{L}$  is infinite, we will need a bit more explanation as to why the map on fundamental groups is injective, but that is relatively simple. Of course, there is a highly non-trivial input into this. An argument, suggested by Gabai, that bypasses hyperbolisation will be outlined at the end of this section.

Thus, most of the additional work is involved in showing that  $\Lambda(\mathcal{L})$  and  $\Lambda(\mathcal{L}')$  are indeed homeomorphic. For this we need to define a partial order on the link components, and show that this is preserved. The idea is fairly intuitive, but the details are a bit subtle.

Suppose  $\alpha, \beta \subseteq \Psi$  are unlinked curves. Write  $\alpha \approx \beta$  if they do not cross homotopically  $\Sigma$ , in other words,  $\pi_\Sigma\alpha$  and  $\pi_\Sigma\beta$  can be homotoped to be disjoint in  $\Sigma$ . This is equivalent to asserting that there is some fibre of  $\Psi$  containing both  $\alpha$  and  $\beta$ . We will write  $\alpha \preceq \beta$  (respectively  $\alpha \succeq \beta$ ) to mean that  $\beta$  can be homotoped out the positive (respectively negative) end of  $\Psi$  in  $\Psi \setminus \alpha$ .

Suppose that  $\alpha \preceq \beta$  and  $\alpha \succeq \beta$ . Then  $\beta$  can be homotoped from the negative to the positive end of  $\Psi$  without ever meeting  $\alpha$ . Such a homotopy must intersect



any fibre of  $\Psi$  in at least some (not necessarily embedded) curve homotopic to  $\beta$ . From this one can see that  $\alpha \approx \beta$  by the above definition.

**Lemma 2.3.3.** *Suppose  $\alpha$  and  $\beta$  are unlinked. Then the following are equivalent:*

- (1)  $\alpha \preceq \beta$
- (2)  $\beta \succeq \alpha$
- (3) *There are disjoint fibres  $S \supseteq \alpha$  and  $S' \supseteq \beta$  with  $S < S'$ .*

*Proof.* It's clearly enough to show that (1) implies (3). By hypothesis we have disjoint fibres,  $Z \supseteq \alpha$  and  $Z' \supseteq \beta$ . If  $Z < Z'$ , we are done. If  $Z' < Z$ , then  $\alpha \succeq \beta$ , and so by the above observation,  $\alpha \approx \beta$ . It follows that  $\alpha$  and  $\beta$  are contained in some common fibre. We can now push these fibres slightly so that they become disjoint in the order required.  $\square$

We write  $\alpha \prec \beta$  to mean that  $\alpha \preceq \beta$  and  $\alpha \not\approx \beta$ . Thus, for any two unlinked curves,  $\alpha$  and  $\beta$ , exactly one of the relations  $\alpha \prec \beta$ ,  $\beta \prec \alpha$  or  $\alpha \approx \beta$  holds.

Suppose that  $\gamma_1, \gamma_2, \dots, \gamma_n$  are unlinked curves, with  $\gamma_i \prec \gamma_{i+1}$  for all  $i$ . By definition, we have disjoint fibres,  $S_i \supseteq \gamma_i$ , and by the above observation, we must have  $S_i < S_{i+1}$  for all  $i$ . From this we deduce that there does not exist any finite cycle of unlinked curves,  $\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0$  with  $\gamma_i \prec \gamma_{i+1}$  for all  $i$ .

If  $\mathcal{L}$  is any unlinked set of curves, we let  $<$  be the transitive closure of the relation  $\prec$  on  $\mathcal{L}$ . We see that this is a strict partial order on  $\mathcal{L}$ . Moreover,  $\alpha < \beta$  implies  $\alpha \preceq \beta$ , and so if  $\alpha \not\approx \beta$  we also get  $\alpha \prec \beta$ .

We aim to show that this order on  $\mathcal{L}$ , together with the natural map of  $\mathcal{L}$  to  $\mathbf{C}(\Sigma)$  determines  $\mathcal{L}$  up to ambient isotopy, and so, in particular,  $\Lambda(\mathcal{L})$  up to homeomorphism. We use the following observation.

We say that a strict total order  $\ll$  is **compatible** with the partial order  $<$  if  $\alpha < \beta$  implies  $\alpha \ll \beta$ . We say that it is **discrete** if all intervals are finite.

**Lemma 2.3.4.** *Suppose that  $\mathcal{L}$  is an unlinked set of curves. Suppose that  $\ll$  is a discrete total order on  $\mathcal{L}$  compatible with  $<$ . Then we can find a set of disjoint fibres,  $(S(\gamma))_{\gamma \in \mathcal{L}}$  with  $\gamma \subseteq S(\gamma)$  for all  $\gamma$ , and  $S(\alpha) < S(\beta)$  if and only if  $\alpha \ll \beta$ .*

*Proof.* Since  $\mathcal{L}$  is unlinked, we can find a locally finite disjoint collection of fibres  $Z(\gamma) \supseteq \gamma$ . Write  $\alpha \ll' \beta$  to mean that  $Z(\alpha) < Z(\beta)$ . This defines another discrete total order compatible with  $<$ . We can now find a sequence,  $(\ll_n)_n$ , of discrete total orders, all compatible with  $<$ , with  $\ll_0 = \ll'$ , with  $\ll_n$  stabilising on  $\ll$  on any finite subset of  $\mathcal{L}$ , and with  $\ll_{n+1}$  obtained from  $\ll_n$  by interchanging the order on a pair of  $\ll_n$ -consecutive elements of  $\mathcal{L}$ . We suppose inductively that  $(Z_n(\gamma))_n$  is a collection of disjoint fibres, with  $\gamma \subseteq Z_n$  for all  $n$ , and inducing the order  $\ll_n$ . Suppose that  $\ll_{n+1}$  is obtained by interchanging the order on  $\alpha$  and  $\beta$ . These are consecutive, which means that the region  $[Z_n(\alpha), Z_n(\beta)]$  contains no other curve of  $\mathcal{L}$ . Since both orders are compatible with  $\leq$ , we must have  $\alpha \approx \beta$ . We can now construct two new disjoint fibres,  $Z_{n+1}(\alpha) \supseteq \alpha$  and  $Z_{n+1}(\beta) \supseteq \beta$ ,

both in  $[Z_n(\alpha), Z_n(\beta)]$ , but with the opposite order. We set  $Z_{n+1}(\gamma) = Z_n(\gamma)$  for all  $\gamma \neq \alpha, \beta$ . Now the above process stabilises on any compact subset of  $\Psi$ , and so we eventually end up with a collection of fibres,  $(S(\gamma))_\gamma$  inducing  $\ll$  as required.  $\square$

**Lemma 2.3.5.** *Suppose  $\mathcal{L}$  and  $\mathcal{L}'$  are unlinked sets of curves with a bijection  $[\gamma \mapsto \gamma']$  from  $\mathcal{L}$  to  $\mathcal{L}'$ . Suppose that  $\gamma$  and  $\gamma'$  are homotopic in  $\Psi$  for all  $\gamma$ . Suppose also that  $\alpha \preceq \beta$  implies  $\alpha' \preceq \beta'$ . Then there is an end-preserving self-homeomorphism of  $\Psi$  sending  $\gamma$  to  $\gamma'$  for all  $\gamma$ .*

*Proof.* First note that by the condition on homotopies, we have  $\alpha \approx \beta$  if and only if  $\alpha' \approx \beta'$ . By the trichotomy, we see that  $\alpha \prec \beta$  if and only if  $\alpha' \prec \beta'$ , and so  $\alpha < \beta$  if and only if  $\alpha' < \beta'$ .

Now, let  $S'(\gamma') \supseteq \gamma'$  be a locally finite disjoint set of fibres of  $\Psi$ . Applying Lemma 1.6.4, we can find a locally finite disjoint set of fibres  $S(\gamma) \supseteq \gamma$  such that  $S(\alpha) < S(\beta)$  if and only if  $S'(\alpha') < S'(\beta')$ . We can now find an isotopy of  $\Psi$  sending each  $S(\gamma)$  to  $S'(\gamma')$ . We can therefore assume that  $S(\gamma) = S'(\gamma')$ . Since  $\gamma'$  is homotopic to  $\gamma$  in  $\Psi$  and hence in  $\Sigma$ , we can isotope  $\gamma$  to  $\gamma'$  in  $S(\gamma)$  and extend to an ambient isotopy in a small neighbourhood of  $S(\gamma)$ . The resulting homeomorphism sends  $\mathcal{L}$  to  $\mathcal{L}'$ , as required.  $\square$

**Lemma 2.3.6.** *Suppose that  $\mathcal{L}$  and  $\mathcal{L}'$  are unlinked sets of curves, and that  $f : \Psi \rightarrow \Psi$  is an end-preserving proper homotopy equivalence with  $f^{-1}(\bigcup \mathcal{L}') = \bigcup \mathcal{L}$ , and with  $f|_{\bigcup \mathcal{L}}$  a homeomorphism to  $\bigcup \mathcal{L}'$ . Then there is an end-preserving homeomorphism  $g : \Psi \rightarrow \Psi$  homotopic to  $f$  in  $\Psi$  with  $g|_{\bigcup \mathcal{L}} = f|_{\bigcup \mathcal{L}}$ .*

*Proof.* We may as well suppose that  $f$  is homotopic to the identity on  $\Psi$ . Suppose  $\alpha, \beta \in \mathcal{L}$ . If  $\alpha \preceq \beta$ , then we can homotope  $\beta$  out the positive end of  $\Psi$  in  $\Psi \setminus \alpha$ . The image of this homotopy under  $f$  sends  $f(\beta)$  out the positive end in  $\Psi \setminus f(\alpha)$ . Thus  $f(\alpha) \preceq f(\beta)$ . Lemma 1.6.5 now gives us a homeomorphism of  $\Psi$  sending each  $\gamma \in \mathcal{L}$  to  $f(\gamma)$ . By isotopy in a neighbourhood of  $\gamma$  we can assume that  $f|_\gamma = g|_\gamma$ .  $\square$

We remark that, in Lemma 2.3.6, we do not in fact need to assume that  $\mathcal{L}'$  is unlinked in  $\Psi$ . This is a consequence of the other hypotheses:

**Proposition 2.3.7.** *Suppose that  $f : \Psi \rightarrow \Psi$  is an end-preserving homotopy equivalence and that  $\mathcal{L}$  is an unlinked collection of curves in  $\Psi$ . Suppose that  $f^{-1}(f(\bigcup \mathcal{L})) = \bigcup \mathcal{L}$ , and that  $f|_{\bigcup \mathcal{L}}$  is injective. Then  $\{f(\gamma) \mid \gamma \in \mathcal{L}\}$  is an unlinked collection of curves in  $\Psi$ .*

*Proof.* We take a collection of disjoint fibres for  $\mathcal{L}$  as given by (2) of Proposition 2.3.1, and map them by  $f$  to give us a collection of homotopy fibres in  $\Psi$  satisfying (3) for the collection  $\{f(\gamma) \mid \gamma \in \mathcal{L}\}$ .  $\square$

**Lemma 2.3.8.** *Let  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}')$  be as in the hypothesis of Proposition 2.3.2. Then  $f$  is a homotopy equivalence.*

*Proof.* By extending over the tori  $T(\gamma)$  we get a map satisfying the hypotheses of Lemma 2.3.6, and so it follows that  $\Lambda(\mathcal{L})$  and  $\Lambda(\mathcal{L}')$  are homeomorphic. Let  $\Gamma = \pi_1(\Lambda(\mathcal{L}))$ . Moreover, the map  $f$  induces an epimorphism of  $\Gamma$ . We need to show that this is also injective.

If  $\mathcal{L}$  were finite, then this follows immediately from the fact that  $\pi_1(\Lambda(\mathcal{L}))$  satisfies the hopfian property as described earlier.

In general suppose  $g \in \Gamma$  were in the kernel. We can represent it by a closed curve  $\delta \subseteq \Lambda(\mathcal{L})$ , which lies between two fibres, say  $S < Z$  in  $\Psi$ . We can take these disjoint from  $\bigcup \mathcal{L}$ . Let  $\mathcal{L}_0 = \{\gamma \in \mathcal{L} \mid \gamma \subseteq [S, Z]\}$ , and write  $\mathcal{L}'_0$  for the corresponding subset of  $\mathcal{L}'$ . Now  $f$  extends to degree one map between  $\Lambda(\mathcal{L}_0)$  and  $\Lambda(\mathcal{L}'_0)$ . By the finite case, it now follows that  $\delta$  bounds a disc in  $\Lambda(\mathcal{L}_0)$ . But we can now push that disc into the region  $[S, Z]$ , and so we see that  $\delta$  bounds a disc in  $\Lambda(\mathcal{L})$ . In other words,  $g$  is trivial in  $\Gamma$ . Therefore  $f$  induces an isomorphism on  $\pi_1$  as claimed.

The result now follows by Whitehead's theorem, given that the higher homotopy groups are trivial (by the Sphere Theorem for 3-manifolds).  $\square$

*Proof of Proposition 2.3.2 :* We have shown (Lemma 2.3.8) that  $f$  is a homotopy equivalence. Suppose first that  $\mathcal{L}$  is finite. We can compactify  $\Psi$  to  $\Sigma \times [-\infty, \infty]$ , and we can modify  $f$  in a neighbourhood of the ends of  $\Psi$  so that it extends to a map from  $\Sigma \times [-\infty, \infty] \rightarrow \Sigma \times [-\infty, \infty]$ , without changing it on  $\bigcup \mathcal{L}$ . Now the compactified manifold  $\Lambda(\mathcal{L}) \cup (\Sigma \times \{-\infty, \infty\})$  is Haken. If  $\mathcal{L}$  is finite, since  $\Lambda(\mathcal{L}) \cong \Lambda(\mathcal{L}')$  is Haken, the result then follows by the result of Waldhausen [Wal] (stated here as Theorem 1.6.5).

To deal with the general case, we first show that if  $F \subseteq \Lambda(\mathcal{L})$  is a surface with  $\partial F = F \cap \partial \Lambda(\mathcal{L})$ , then  $f|_F$  is homotopic to an embedding in  $\Lambda(\mathcal{L}')$  relative to  $\partial \Lambda(\mathcal{L}')$ . (In fact, this is all we need for our applications in this paper.) To see this, choose fibres  $S, Z \subseteq \Lambda(\mathcal{L}')$ , disjoint from  $\bigcup \mathcal{L}'$ , so that  $f(F) \subseteq [S, Z]$ . Let  $\mathcal{L}'_0 = \{\gamma \in \mathcal{L}' \mid \gamma \subseteq [S, Z]\}$ , and let  $\mathcal{L}_0 \subseteq \mathcal{L}$  be the corresponding curves in  $\mathcal{L}$ . Now  $f$  is a homotopy equivalence from  $\Lambda(\mathcal{L}_0)$  to  $\Lambda(\mathcal{L}'_0)$ . By the finite case,  $f|_F$  is homotopic to an embedding in  $\Lambda(\mathcal{L}'_0)$ . By Theorem 1.6.4, we can take this embedding in a small neighbourhood of  $f(F)$ , and so, in particular, in  $[S, Z]$ . But  $\Psi$  retracts onto  $[S, Z]$ , so we can now push the homotopy into  $[S, Z]$ , and so these surfaces are also homotopic in  $\Lambda(\mathcal{L}')$ .

To complete the proof, we can apply this result to a sequence of fibres,  $(S_i)_{i \in \mathbb{Z}}$  in  $\Lambda(\mathcal{L})$ , whose images are disjoint, so as to find disjoint fibres  $Z_i$  in  $\Lambda(\mathcal{L}')$ , homotopic to  $f(S_i)$ . We can now apply the finite case to the regions between these fibres. We omit the detail of this last step, since we have already shown what we need for the following corollary.  $\square$

As we have noted, the fact we really need is the following. It was proven in the course of Proposition 2.3.2, and can also be viewed as a combination of Proposition 2.3.2 and Theorem 1.6.1.

**Corollary 2.3.9.** *Let  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}')$  satisfy any of the equivalent conditions of Proposition 2.3.1. Suppose that  $F \subseteq \Lambda(\mathcal{L})$  is a properly embedded  $\pi_1$ -injective compact surface (so that  $F \cap \partial\Lambda(\mathcal{L}) = \partial F$ ). Let  $U$  be any neighbourhood of  $f(F)$  in  $\Lambda(\mathcal{L}')$ . Then there is a proper embedding  $g : F \rightarrow U$  such that  $f|_F$  is homotopic in  $\Lambda(\mathcal{L}')$  to  $g$  relative to  $\partial F$ .*

We will give a refinement of this statement Section 4.3, though that is not needed for the proof of the Ending Lamination Theorem.

We conclude this section with an outline of how one can bypass the use of hyperbolisation in the proof of Proposition 2.3.2. It elaborates on a suggestion of Dave Gabai, and I thank him for his permission to include it here.

*Proof of Proposition 2.3.2 bypassing hyperbolisation.* We can suppose that  $\mathcal{L} = \mathcal{L}'$ , and that  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L})$  extends to a map homotopic to the identity on  $\Psi$ . We claim that  $f$  is homotopic to a homeomorphism (in fact, the identity) on  $\Lambda(\mathcal{L})$ .

We write  $\Sigma_t = \Sigma \times \{t\}$ . We can assume that  $\mathcal{L} = \{\alpha_i \mid i \in I\}$ ,  $I \subseteq \mathbb{Z}$  and each  $\alpha_i$  is a curve in  $\Sigma_i$ . We can also assume that  $\Sigma \times (\mathbb{Z} + \frac{1}{2}) \subseteq \Lambda(\mathcal{L})$ .

We will first show that if  $t \in \mathbb{Z} + \frac{1}{2}$ , then  $f|_{\Sigma_t}$  is homotopic in  $\Lambda(\mathcal{L})$  to the inclusion  $\Sigma_t \hookrightarrow \Lambda(\mathcal{L})$ . Let  $J$  be the vertical range of  $f(\Sigma_t)$ , i.e. the compact interval  $\{u \in \mathbb{R} \mid \Sigma_u \cap f(\Sigma_t) \neq \emptyset\}$ . Suppose first that  $I \cap J \cap [t, \infty) \neq \emptyset$ . Let  $i$  be its maximal element, and let  $A = \alpha_i \times [i, \infty)$ . We can assume that  $(f|_{\Sigma_t})^{-1}A$  consists of simple closed curves. These will be either trivial or homotopic to  $\alpha_i$ . Since  $\Lambda(\mathcal{L})$  is aspherical, after homotopy in  $\Lambda(\mathcal{L})$ , we can get rid of trivial curves, and since it is atoroidal, we can get rid of pairs of non-trivial curves with opposite orientations. We are left with either  $p$  positively oriented curves or  $-p$  negatively oriented curves homotopic to  $\alpha$ , where  $p \in \mathbb{Z}$  is the number of times  $f(\Sigma_t)$  wraps around  $\alpha_i$ . More precisely,  $p = \langle \omega, f(\Sigma_t) \rangle$ , where the cohomology class,  $\omega \in H^2(\Lambda(\{\alpha_i\}))$ , measures the intersection with a ray in  $\Lambda(\{\alpha_i\})$  from  $\alpha_i$  to  $+\infty$ . For sufficiently negative  $u$ ,  $f(\Sigma_u) \subseteq \Sigma \times (-\infty, n)$  and so  $\langle \omega, f(\Sigma_u) \rangle = 0$ . Since  $f(\Sigma_u)$  is homotopic to  $f(\Sigma_t)$  in  $\Lambda(\{\alpha_i\})$  it follows that  $p = \langle \omega, f(\Sigma_t) \rangle = 0$ . In other words, we have pushed  $f(\Sigma_t)$  off  $A$ . We can now homotope it (in  $\Lambda(\mathcal{L})$ ) below  $\Sigma_i$ . We can continue inductively until  $I \cap J \cap [t, \infty) = \emptyset$ . Proceeding similarly below, we push  $f(\Sigma_t)$  so that its vertical range lies in the component of  $\mathbb{R} \setminus I$  containing  $t$ . After a further homotopy, we will get  $f(\Sigma_t) \subseteq \Sigma_t$ . Now,  $f|_{\Sigma_t} : \Sigma_t \rightarrow \Sigma_t$  is homotopic to the identity in  $\Sigma \times \mathbb{R}$  and hence in  $\Sigma_t$ . This proves the claim.

Performing such homotopies for all  $t \in \mathbb{Z} + \frac{1}{2}$ , we can assume that  $f|_{\Sigma \times (\mathbb{Z} + \frac{1}{2})}$  is just the inclusion  $\Sigma \times (\mathbb{Z} + \frac{1}{2}) \hookrightarrow \Sigma \times \mathbb{R}$ .

Now let  $P_n = \Sigma \times [n - \frac{1}{2}, n + \frac{1}{2}]$ , and let  $R_n = P_n \cap \Lambda(\mathcal{L})$ . (In other words,  $R_n$  is  $P_n$  with at most one tube drilled out.) We next homotope  $f$  so that  $f(R_n) = R_n$ . The idea is to proceed as we did for the surfaces  $\Sigma_t$ . Given  $i \in I \cap J \cap [n + 1, \infty)$  maximal, let  $A_i = \alpha_i \times [i, \infty)$  as before. This time, we can assume that each

component of  $f^{-1}(A_i)$  is a surface in the interior of  $R_n$ . Moreover, by standard 3-manifold topology, we can assume it to be incompressible. It's not hard to see that any incompressible surface in  $R_n$  must be boundary parallel. Thus it must either be a fibre or be homotopic in  $P_n$  to  $\alpha_n$ . But it must be homotopic in  $\Sigma \times \mathbb{R}$  to  $\alpha_i$ , giving a contradiction. We have arranged that  $f(R_n) \cap A_i = \emptyset$ . Continuing as with  $\Sigma_t$ , we eventually homotope  $f(R_n)$  into  $R_n$  as claimed.

Next, if  $n \notin I$ , then  $R_n = P_n$ , and by Waldhausen [Wal], we can homotope  $f|_{R_n}$  fixing  $R_{n-\frac{1}{2}} \cup R_{n+\frac{1}{2}}$  to a homeomorphism (in fact the identity). If  $n \in I$ , let  $B = (\alpha_n \times [n, n+\frac{1}{2}]) \cap R_n$ . Since  $R_n$  is atoroidal, we can assume  $f(B) = B$ . Again, we can assume that  $f^{-1}(B) \setminus B$  consists of incompressible surfaces in  $R_n \setminus B$ , and therefore empty. In other words  $f^{-1}(B) = B$ . After a homotopy, holding the curve  $\Sigma_{n+\frac{1}{2}} \cap B$  fixed (though perhaps rotating the other boundary component of  $B$ ) we get  $f|_B$  to be inclusion. Cutting  $R_n$  along  $B$ , we get a homeomorphic copy of  $\Sigma \times [0, 1]$  and so, again, we are done by Waldhausen. Doing this for all  $n$ , we homotope  $f$  to the identity in  $\Lambda(\mathcal{L})$ .  $\square$

#### 2.4. ANNULUS SYSTEMS.

In this section, we will talk about “annulus systems” in  $\Psi \cong \Sigma \times \mathbb{R}$ . This is the combinatorial structure we use to construct the model manifold. It can be thought of as a simplified version of the “hierarchy” machinery introduced in [MasM2] and applied in [Mi1]. As noted earlier, we make no use of “subsurface projections” of [MasM2] which were key to Minsky’s construction. (A discussion of subsurface projections and their relation to the annulus systems described here can be found in [Bow5].) The main result of the section will be Theorem 2.4.3.

Let  $\Sigma$  be a compact surface. Let  $\Psi = \Sigma \times \mathbb{R}$ . We write  $\pi_\Sigma : \Psi \rightarrow \Sigma$  and  $\pi_V : \Psi \rightarrow \mathbb{R}$  be vertical and horizontal projections respectively. We refer to the two ends of  $\Psi$  as the *positive* and *negative* ends. We are only really interested in the total order on the vertical coordinate in  $\mathbb{R}$ . We will thus be free to adjust by any orientation preserving homeomorphism of  $\mathbb{R}$ .

We recall the notation,  $I = [\partial_- I, \partial_+ I] \subseteq \mathbb{R}$ , and  $\partial I = \{\partial_- I, \partial_+ I\}$ , where  $\partial_- I < \partial_+ I \in \mathbb{R}$ .

**Definition.** A *horizontal curve* in  $\Psi$  is a subset of the form  $\gamma \times \{t\}$  for some curve  $\gamma \subseteq \Sigma$ , and some  $t \in \mathbb{R}$ .

A *horizontal surface* in  $M$  is a subset of the form  $\Phi \times \{t\}$  for some subsurface  $\Phi \subseteq \Sigma$  and some  $t \in \mathbb{R}$ . If  $\Phi = \Sigma$ , it is called a *horizontal fibre*.

A *vertical annulus* is a subset of the form  $\gamma \times I$  where  $\gamma \subseteq \Sigma$  is a curve, and  $I \subseteq \mathbb{R}$  is a non-trivial compact interval.

A *strip* is a subset of the form  $\Phi \times I$  where  $\Phi \subseteq \Sigma$  is a subsurface, and  $I \subseteq \mathbb{R}$  is a non-trivial compact interval.

(Figure 2.3.)

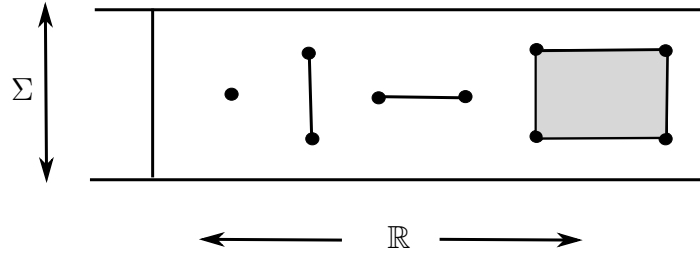


FIGURE 2.3. A schematic picture: a horizontal fibre, a horizontal curve, a horizontal surface, a vertical annulus and a strip. These are represented 1-dimension down. Following tradition, the horizontal direction is represented vertically and the vertical direction is represented horizontally.

More generally we speak about a subset or path etc. being *horizontal* if it is entirely contained in a horizontal fibre.

As before, in what follows we shall fix some hyperbolic structure on  $\Sigma$  with geodesic boundary, and realise curves as geodesics. In this way, curves will automatically intersect minimally, so it will simplify the combinatorial arguments.

Let  $\Omega = \gamma \times I$  be a vertical annulus. We write  $\partial_{\pm}\Omega = \gamma \times \{\partial_{\pm}I\}$ . These are horizontal curves. We write  $\partial_H\Omega = \gamma \times \partial I = \partial_-\Omega \sqcup \partial_+\Omega$ .

Let  $B = \Phi \times I$  be a strip. We write  $\partial_{\pm}B = \Phi \times \partial_{\pm}I$ ,  $\partial_H B = \Phi \times \partial I = \partial_-B \sqcup \partial_+B$ ,  $\partial_V B = \partial\Phi \times I$  and  $\partial_V^{\Sigma} B = \partial^{\Sigma}\Phi \times I$ . (Recall that  $\partial^{\Sigma}\Phi = \partial\Phi \setminus \partial\Sigma$  is the relative boundary of  $\Phi$  in  $\Sigma$ .) We refer to  $\partial_H B$  as the *horizontal boundary* of  $B$ . It consists of two horizontal surfaces. We refer to  $\partial_V B$  as the *vertical boundary*. (Figure 2.4.) Note that the relative boundary of  $B$  in  $\Psi$  is  $\partial_V^{\Sigma} B \cup \partial_H B$ . We define

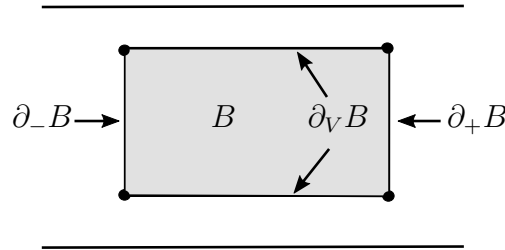


FIGURE 2.4. A strip,  $B$ , with horizontal and vertical boundaries

the complexity  $\xi(B)$  as  $B$  as  $\xi(\Phi)$ . We refer to  $\pi_{\Sigma}B$  as the *base surface* of  $B$ .

**Definition.** An *annulus system*,  $\mathcal{W}$ , in  $\Psi$  is a locally finite collection of disjoint vertical annuli.

Let  $W = \bigcup \mathcal{W}$ . Given  $t \in \mathbb{R}$ , let  $\gamma_t = \pi_{\Sigma}(W \cap (\Sigma \times t))$ . This is either empty or a multicurve in  $\Sigma$ . Clearly,  $W$  is completely determined by the piecewise constant

map  $[t \mapsto \gamma_t]$ . For many purposes, it will be convenient to assume that  $\mathcal{W}$  is **vertically generic**, that is, if  $\Omega, \Omega' \in \mathcal{W}$  with  $\pi_V \partial_H \Omega \cap \pi_V \partial_H \Omega' \neq \emptyset$ , then  $\Omega = \Omega'$ . This is easily achieved by pushing the horizontal boundaries of annuli up or down a little. In this case,  $\Omega$  is combinatorially equivalent to a sequence  $(\gamma_i)_i$  where each  $\gamma_i$  is either empty or a multicurve, and  $\gamma_{i+1}$  is obtained from  $\gamma_i$  by either adding or deleting a curve.

We say that  $W$  is **vertically full** if  $\mathbb{R} = \pi_V W$ . In this case, the  $\gamma_i$  are all multicurves. (Figure 2.5.) In Section 2.2, we referred to such a sequence  $(\gamma_i)_i$

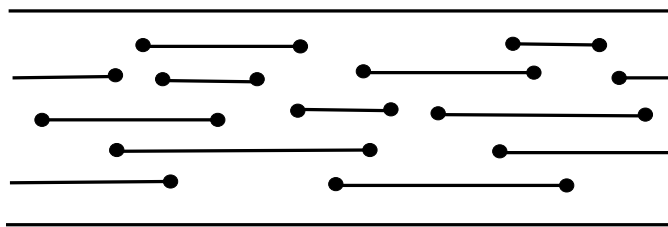


FIGURE 2.5. A vertically full annulus system in  $\Psi$ .

as a **path of multicurves**. In other words, there is a bijective correspondence between paths of multicurves and vertically full (and generic) annulus systems up to vertical reparametrisation (i.e. an orientation preserving self-homeomorphism of the  $\mathbb{R}$ -factor).

In general we are only interested in annulus systems up to this equivalence (vertical reparametrisations), and so they can be viewed as essentially combinatorial objects.

A **ladder** is a minimal vertically full annulus system. (This agrees with the notion as defined earlier.) (Figure 2.6.) This corresponds to a path of multicurves

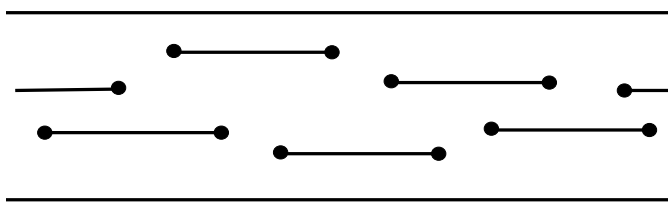


FIGURE 2.6. A schematic picture of a ladder.

where the number of components alternates between 1 and 2. Those with one component constitute a path in the curve complex. Thus a ladder is combinatorially equivalent to a (bi-infinite) path in  $\mathcal{G}(\Sigma)$ . The **length** of a ladder is the length of the path: i.e. the number of 2-component multicurves.

**Definition.** A **band** (with respect to  $\mathcal{W}$ ) is a strip  $B \subseteq \Psi$  such that  $\partial_V^\pm B \subseteq W$ .

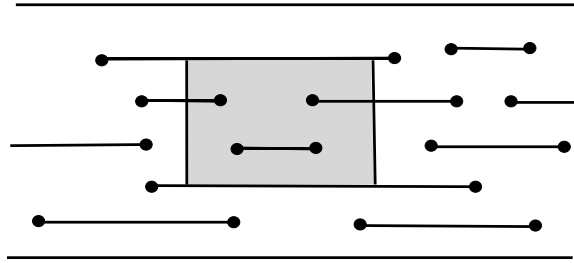


FIGURE 2.7. A band

(Figure 2.7.)

We can view  $W_B = (W \cap B) \setminus \partial_V B$  as a finite annulus system on  $B$  (at least if  $\partial_H B \cap \partial_H \Omega = \emptyset$  for all  $\Omega \in \mathcal{W}$ ), and a similar discussion applies. In this case if  $W_B$  is full then it corresponds to a finite path of multicurves in  $\Phi$ . We write  $\mathcal{W}_B$  for the set of components of  $W_B$ .

**Definition.** An annulus system is *complete* if for any horizontal fibre  $S \subseteq \Psi$ , each component of  $S \setminus W$  has complexity at most 1.

Let  $\mathcal{W}$  be a complete annulus system, and let  $W = \bigcup \mathcal{W}$ .

**Definition.** A *brick* is a maximal band whose interior does not meet  $W$ .

Any such brick has complexity at most 1. We refer a brick as *type 0* or *type 1* depending on whether its complexity is 0 or 1, i.e. its base surface is a 3HS or a 1HT or 4HS. Let  $\mathcal{D} = \mathcal{D}(\mathcal{W})$  be the set of all bricks. One sees easily that this is a locally finite collection of bands with disjoint interiors, and that  $\Psi = \bigcup \mathcal{D}$ . (Figure 2.8.) If two bands meet in a horizontal surface, then one is of type 0 and the other

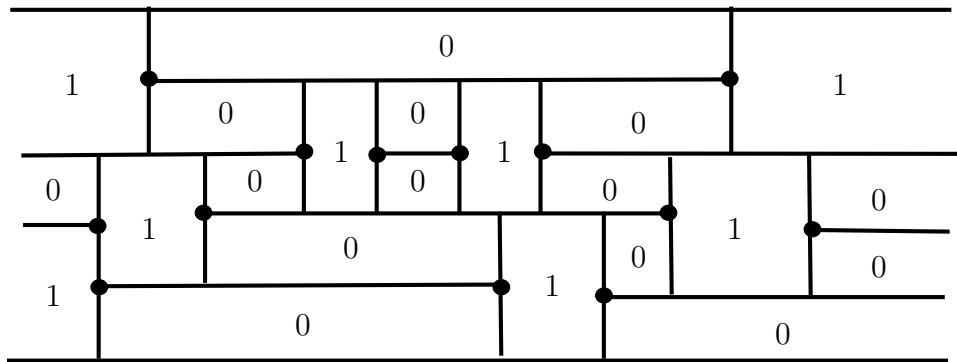


FIGURE 2.8. A decomposition of  $\Psi$  into bricks of type 0 and 1.

of type 1. We can recover  $\mathcal{W}$  from  $\mathcal{D}$  as the set of components of  $W = \bigcup_{B \in \mathcal{D}} \partial_V^\Sigma B$ . We write  $\mathcal{W} = \mathcal{W}(\mathcal{D})$ . We refer to  $\mathcal{D}$  as a *brick decomposition* of  $\Psi$ .



*Remark.* The notion of a complete annulus system is closely related to that of a path in the “pants graph”. Recall that this is a graph whose vertex set is the set of complete multicurves, and adjacency is defined by removing one curve, and replacing it with another curve which it intersects minimally, so as to give another complete multicurve. Suppose that  $\mathcal{W}$  is a complete annulus system. Suppose moreover that the horizontal projections of the type 1 bricks are pairwise disjoint. (This can always be achieved by vertical isotopy of the annuli.) Let  $(\gamma_i)_i$ , as described as described above. For even indices  $\gamma_i$  is a complete multicurve, (or “pants decomposition”) and for odd indices it is a complete multicurve with one curve removed. Thus, at odd indices we have exactly one complementary component of complexity 1 (a 1HT or 4HS). By interpolating additional curves in this component, we arrive at the situation where consecutive complete multicurves differ by replacing one curve (say  $\alpha$ ) by another (say  $\beta$ ) such that a regular neighbourhood of  $\alpha \cup \beta$  is either a 1HT or a 4HS. (A particular case of this interpolation process is used again below.) The sequence of complete multicurves is then a path in the pants graph. (Since this is how adjacency is defined in this graph.) Conversely, a path in the pants graph gives rise to a complete annulus system of this sort. We will not have any formal use for the pants graph in this paper.

We want to describe a particular construction of complete annulus systems. Note that Lemma 2.1.3 gave us a means of connecting two curves by a path of multicurves giving us a complete annulus system in some compact region of  $\Psi$ . However this will not be sufficient for our purposes here. We will require some additional properties. To describe these, we need some further definitions.

Let  $\mathcal{W}$  be an annulus system and  $B \subseteq \Psi$  a band. The annuli at the “top” and “bottom” of the band determine subsets of the curve graph. More precisely, we write  $\mathbf{C}_\pm(B) = \pi_\Sigma(W \cap \partial_\pm B \setminus \partial_V B) \subseteq \mathbf{C}(\Phi) \subseteq \mathbf{C}(\Sigma)$ . Recall that a ladder in  $B$  is minimal set of annuli in  $\mathcal{W}_B$  which is vertically full in  $B$  (i.e.  $\pi_V W_B = \pi_V B$ ).

**Definition.** The *height*,  $H(B)$  of  $B$  is the minimal length of a ladder in  $B$ .

More intuitively,  $H(B) + 1$  is the minimal number of annuli of  $\mathcal{W}_B$  we need to cross get from one horizontal boundary component of  $B$  to the other, where we are allowed to jump between annuli along horizontal paths. Thus  $H(B)$  is the number of jumps. (Figure 2.9.)

We also write  $H_0(B) = d_{\mathcal{G}(\Phi)}(\mathbf{C}_-(B), \mathbf{C}_+(B))$ . Note that  $H_0(B) \leq H(B)$ . (Since any ladder gives us a path in  $\mathcal{G}(\Phi)$ .) In the case where  $\mathcal{W}$  is complete and  $\xi(B) \geq 2$ , these quantities are finite.

**Definition.** We say that  $B$  is *k-taut* if  $H(B) \leq H_0(B) + k$ .

In the case where  $\xi(B) = 1$ ,  $\mathcal{W}_B$  consists of a sequence  $\Omega_0, \dots, \Omega_n$  of annuli whose vertical projections are disjoint and occur in this order. In this case, we say that  $B$  is *taut* if  $(\pi_\Sigma \Omega_i)_i$  is a geodesic segment in  $\mathcal{G}(\Phi)$ .

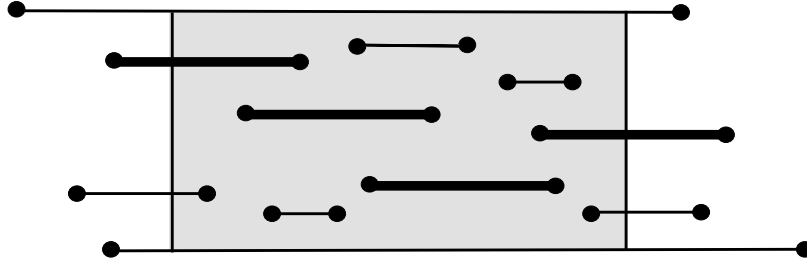


FIGURE 2.9. A band,  $B$ , with  $H(B) = 3$ , with a ladder of minimal length 3 highlighted.

We now have the basis for proving Theorem 2.4.3. The argument will be by induction on complexity. We need to state the induction hypothesis in a different way. This formulation is mostly stronger than that already given. However, it is weaker in the sense that we are assuming we are given an initial complete multicurve, and we will also forget, for the moment about bands of complexity 1.

To this end, let  $I \subseteq \mathbb{R}$  be a non-trivial compact interval, and let  $O = \Sigma \times I$ . For the moment, we restrict to annulus systems on  $O$ .

**Definition.** We say that a horizontal curve  $\gamma \subseteq O$  is *compatible* with an annulus system  $W = \bigcup \mathcal{W}$  if either  $\gamma \subseteq W$  or  $\gamma \cap W = \emptyset$ . We say that a vertical annulus  $\Omega \subseteq O$  is *compatible* with  $W$  if  $\Omega \cap W$  is empty, a single vertical annulus or a boundary curve of  $\Omega$ . (This implies that  $W \cup \Omega$  is also an annulus system.)

**Lemma 2.4.1.** *Suppose that  $\xi(\Sigma) \geq 1$ , and that  $\alpha, \beta$  are multicurves in  $\Sigma$  with  $\alpha$  complete. Then there is a complete (vertically generic) annulus system,  $W = \bigcup \mathcal{W} \subseteq O = \Sigma \times I$  satisfying:*

**(R1)**  $\alpha = \mathbf{C}_-(\mathcal{W})$  and  $\beta \subseteq \mathbf{C}_+(\mathcal{W})$ .

**(R2)** If  $\Omega \subseteq O$  is a vertical annulus with  $\partial_-\Omega$  and  $\partial_+\Omega$  both compatible with  $W$ , then  $\Omega$  is compatible with  $W$ .

**(R3)** If  $\Omega$  is a vertical annulus with  $\partial_-\Omega$  compatible with  $W$  and with  $\partial_+\Omega \subseteq \partial_+O$  and not crossing  $\beta$ , then  $\Omega$  is compatible with  $W$  (so that by completeness,  $\partial_+\Omega \subseteq W$ ).

**(R4)** If  $B = \Phi \times J \subseteq O$  is a band of complexity at least 2, then  $B$  is  $2(\xi(\Sigma) - \xi(\Phi) + 1)$ -taut.

**(R5)**  $\mathbf{C}(\mathcal{W}) \subseteq Y^\infty(\mathbf{C}(\alpha) \cup \mathbf{C}(\beta))$ .

*Proof.* The proof will be induction on  $\xi(\Sigma)$ .

In the case where  $\xi(\Sigma) = 1$ , there is not much to be said. Here,  $\alpha, \beta$  are just single curves, and we enlarge each of them slightly to be vertical annuli. We put them in a single annulus if they happen to be equal.

The case where  $\xi(\Sigma) = 2$  is not much harder. We choose  $\gamma_0 \in \mathbf{C}(\alpha)$  and  $\gamma_n \in \mathbf{C}(\beta)$  with  $d(\gamma_0, \gamma_n) = d(\mathbf{C}(\alpha), \mathbf{C}(\beta)) = n$ , say, and connect  $\gamma_0$  to  $\gamma_n$  by a

tight geodesic  $\gamma_0, \gamma_1, \dots, \gamma_n$  in  $\mathcal{G}(\Sigma)$ . The path of multicurves,

$$\alpha, \quad \gamma_0, \quad \gamma_0 \cup \gamma_1, \quad \gamma_1, \quad \dots, \quad \gamma_{n-1} \cup \gamma_n \quad [, \quad \gamma_n, \quad \beta]$$

gives us a ladder in  $O$ , which in this case, is a complete annulus system. (The last two bracketed terms are omitted if  $\beta$  is a single curve.) The properties stated are all easily verified in this case.

Now suppose that  $\xi(\Sigma) \geq 3$  and that the lemma holds for all surfaces of smaller complexity. To apply the induction hypothesis, first consider the special case where  $\mathbf{C}(\alpha) \cap \mathbf{C}(\beta) \neq \emptyset$ . Let  $\gamma \subseteq \alpha$  be the union of all components of  $\alpha$  that do not cross any curve of  $\beta$ . Thus,  $\gamma$  and  $\gamma \cup \beta$  are multicurves. Now,  $\gamma \times I$  cuts  $O$  into subsets of the form  $\Phi \times I$  where  $\Phi$  is (the completion of) a component of  $\Sigma \setminus \gamma$ . If  $\beta \cap \text{int } \Phi = \emptyset$ , let  $W_\Phi = (\alpha \cap \text{int } \Phi) \times I$ . Otherwise, we apply the inductive hypothesis to  $\Phi \times I$  and the curves  $\alpha \cap \text{int } \Phi$  and  $\beta \cap \text{int } \Phi$  to give us a complete annulus system  $W_\Phi$  in  $\Phi \times I$ . We now set  $W = (\gamma \times I) \cup \bigcup_\Phi W_\Phi$  as  $\Phi$  ranges over all such subsurfaces. (We can modify  $W$  slightly so that it becomes vertically generic.) It is complete, and all the above properties are easily verified. For (R4), note that if a band  $B$  does not lie in any of the components,  $\Phi \times I$ , then it is crossed by a single annulus and so  $H(B) = H_0(B) = 0$ , so it is 0-taut.

The general construction (when  $\mathbf{C}(\alpha) \cap \mathbf{C}(\beta) = \emptyset$ ) is as follows. Let  $n = d(\mathbf{C}(\alpha), \mathbf{C}(\beta))$ . Similarly as in the complexity 2 case, we choose  $\gamma_0 \in \mathbf{C}(\alpha)$  and  $\gamma_n \in \mathbf{C}(\beta)$  with  $d(\gamma_0, \gamma_n) = n$ . Now connect  $\gamma_0$  to  $\gamma_n$  by a tight multigeodesic  $\gamma_0, \gamma_1, \dots, \gamma_n$ . (This needs to be a bona fide tight multigeodesic, as discussed in Section 2.2.) Now write  $I = [0, n+1]$  and set  $O_i = \Sigma \times [i, i+1]$  for  $i \in \{0, 1, \dots, n\}$ . Thus  $O = \bigcup_i O_i$  and  $\partial_+ O_i = \partial_- O_{i+1}$ . We apply the above construction to give us a complete annulus system,  $W_0 \subseteq O_0$  with  $\alpha = \mathbf{C}_-(W_0)$  and  $\gamma_0 \cup \gamma_1 \subseteq \mathbf{C}_+(W_0)$ . Let  $\alpha_1$  be the complete multicurve  $\mathbf{C}_+(W_0)$ . We now do the same thing in  $O_1$  to get a complete annulus system,  $W_1 \subseteq O_1$  with  $\alpha_1 = \mathbf{C}_-(W_1)$  and  $\gamma_1 \cup \gamma_2 \subseteq \mathbf{C}_+(W_1)$ . Set  $\alpha_2 = \mathbf{C}_+(W_1)$ , and continue inductively. For the final step,  $O_n$ , we get  $W_n \subseteq O_n$  with  $\gamma_{n-1} \cup \gamma_n \subseteq \alpha_n = \mathbf{C}_-(W_n)$  and  $\beta \subseteq \mathbf{C}_+(O_n)$ . Note that for all  $i$ ,  $\mathbf{C}_+(O_i) = \mathbf{C}_-(O_{i+1})$  and so the annulus systems match up. We set  $W = \bigcup_{i=0}^n W_i \subseteq O$ , and let  $\mathcal{W}$  be the set of components of  $W$ .

By construction,  $W$  is complete annulus system satisfying (R1). We verify the remaining properties in turn.

We need the following observation. Suppose  $\Omega$  is any vertical annulus with  $\partial_H \Omega$  compatible with  $W$ . If  $\partial_- \Omega \subseteq O_i$  and  $\partial_+ \Omega \subseteq O_j$ , then  $d(\mathbf{C}(\gamma_i), \pi_\Sigma \Omega) \leq 1$  and  $d(\mathbf{C}(\gamma_j), \pi_\Sigma \Omega) \leq 1$  and so  $j - i = d(\mathbf{C}(\gamma_i), \mathbf{C}(\gamma_j)) \leq 2$ . In other words,  $\Omega$  can enter at most three of the  $O_i$ .

(R2) Suppose that  $\partial_- \Omega \subseteq O_i$  and  $\partial_+ \Omega \subseteq O_j$  are both compatible with  $W$ . By the above observation, there are three cases:

Case (R2a):  $j = i$  so  $\Omega \subseteq O_i$ , and we are done by the inductive procedure, i.e. (R1) applied to  $W_i$ .

Case (R2b):  $j = i + 1$ . Note that  $\pi_\Sigma \Omega$  does not cross either  $\gamma_i$  or  $\gamma_{i+1}$  (by

compatibility of  $\partial_- \Omega$  with  $W_i$  and  $\partial_- \Omega$  with  $W_{i+1}$  respectively). By (R3) applied to  $W_i$ , we see that  $\Omega \cap O_i$  is compatible with  $W_i$ , and that  $\Omega \cap \partial_+ O \subseteq W$ . We now apply (R2) to  $W_{i+1}$ , showing that  $\Omega \cap O_{i+1}$  is compatible with  $W_{i+1}$ . Thus  $\Omega$  is compatible with  $W$  as required.

Case (R2c):  $j = i + 2$ . In this case,  $\pi_\Sigma(\Omega)$  does not cross either  $\gamma_i$  or  $\gamma_{i+2}$ . By tightness of the multigeodesic  $(\gamma_i)_i$  we see that it cannot cross  $\gamma_{i+1}$  either. As in (R2b), we see that  $\Omega \cap O_i$  is compatible with  $W_i$ , and that  $\Omega \cap \partial_+ O_i \subseteq W$ . Applying the inductive hypothesis (R3) to  $W_{i+1}$ , we see that  $\Omega \cap O_{i+1} \subseteq W$ . Finally applying (R2) to  $W_{i+2}$ ,  $\Omega \cap O_{i+2}$  is compatible with  $W_{i+2}$ . Thus  $\Omega$  is compatible with  $W$  as required.

(R3) Suppose  $\Omega$  is a vertical annulus with  $\partial_- \Omega$  compatible with  $W$  and  $\partial_+ \Omega \subseteq \partial_+ O$  and compatible with  $\beta$ . Let  $\partial_- \Omega \subseteq O_i$ . As with (R2), there are three possibilities.

Case (R3a)  $i = n$ . We just apply (R3) to  $W_n$ .

Case (R3b)  $i = n - 1$ . As in (R2b), we see that  $\Omega \cap W_i$  is a vertical annulus meeting  $\partial_+ O_i = \partial_- O_n$ , and applying (R3) to  $W_n$ , we see that  $\Omega \cap O_n \subseteq W$ .

Case (R3c)  $i = n - 2$ . We argue as in (R2c). This time, we get  $\Omega \cap (O_{n-1} \cup O_n) \subseteq W$ .

(R4) Let  $B$  be a band. First, consider the case where  $\pi_\Sigma(B) = \Sigma$ . Let  $\partial_- B \subseteq O_i$  and  $\partial_+ B \subseteq O_j$ . Thus,  $\mathbf{C}(\gamma_i) \subseteq \mathbf{C}_-(W \cap B)$  and  $\mathbf{C}(\gamma_j) \subseteq \mathbf{C}_+(W \cap B)$ , and so  $H_0(B) = d(\mathbf{C}_-(W \cap B), \mathbf{C}_+(W \cap B)) \geq d(\mathbf{C}(\gamma_i), \mathbf{C}(\gamma_j)) - 2 = j - i + 2$ . By construction we have a ladder crossing in  $W \cap B$  of length at most  $j - i$ . Thus  $H(B) - H_0(B) \leq 2$ . In other words this is 2-taut, so we are done.

For the case where  $\pi_\Sigma B \neq \Sigma$ , we make the following observation. Suppose that a band  $B$  is a union of two subbands,  $B = B_1 \cup B_2$ , meeting at a common horizontal boundary. If  $H(B_2) = 0$  (i.e. some annulus in  $\mathcal{W}$  crosses  $B_2 \setminus \partial_V B_2$ ) then  $H(B) = H(B_1)$  and  $H_0(B) \geq H_0(B_1) - 1$ . Thus, if  $B_1$  is  $k$ -taut, then  $B$  is  $(k + 1)$ -taut.

Suppose now that  $B$  is a band with  $\Phi = \pi_\Sigma B \neq \Sigma$ . Since  $\partial_V^\Sigma B$  lies in  $W$ ,  $B$  can meet at most three  $O_i$ . Suppose  $\partial_- B \subseteq O_i$  and  $\partial_+ B \subseteq O_j$ . We have three cases.

Case (R4a)  $j = i$ .  $B \subseteq O_i$ , so we apply the inductive hypothesis (R4) to  $W_i$ .

Case (R4b)  $j = i + 1$ . If  $\gamma_{i+1} \cap \text{int } \Phi = \emptyset$ , then (by (R3) applied to  $W_i$ )  $B \cap W_i$  is just a product:  $(B \cap W \cap \partial_+ O_i) \times \pi_V(B \cap O_i)$  and so  $B \cap W$  is combinatorially identical to  $B \cap W_{i+1}$ . By (R4) applied to  $W_{i+1}$ ,  $B \cap W_{i+1}$  is  $2(\xi(\Sigma) - \xi(\Phi))$ -taut, and so the same applies to  $B \cap W$ , and we are happy. In the case where  $\gamma_{i+1} \cap \text{int } \Phi \neq \emptyset$ , then  $H(B \cap O_i) = 0$ . By (R4) applied to  $W_i$ ,  $B \cap W_i$  is  $2(\xi(\Sigma) - \xi(\Phi))$ -taut, and so by above observation,  $B = (B \cap O_i) \cup (B \cap O_{i+1})$  is  $(2(\xi(\Sigma) - \xi(\Phi)) + 1)$ -taut.

Case (R4c)  $j = i + 2$ . Each curve of  $\gamma_i$  is at distance 2 from each curve of  $\gamma_{i+1}$ . Therefore,  $\gamma_i \cup \gamma_{i+2}$  is connected. Since neither multicurve can cross  $\partial^\Sigma \Phi$ ,  $(\gamma_i \cup \gamma_{i+2}) \cap \partial \Phi = \emptyset$ . We are again reduced to two subcases. First, if  $(\gamma_i \cup \gamma_{i+2}) \cap \text{int } \Phi = \emptyset$ , then by tightness of  $\gamma_i$ , we also have  $\gamma_i \cap \text{int } \Phi = \emptyset$ . Applying (R3) to  $B \cap O_i$ , and then again to  $B \cap O_{i+1}$ , we see that  $B \cap W \cap (O_i \cup O_{i+1})$  is just a product. Thus,  $B \cap W$  is combinatorially identical to  $B \cap W_{i+2}$ . By (R4)

applied to  $W_{i+2}$ , the latter is  $2(\xi(\Sigma) - \xi(\Phi))$ -taut. The second subcase is when  $\gamma_i \cup \gamma_{i+2} \subseteq \text{int } \Phi$ . In this case,  $H(B \cap O_i) = H(B \cap O_{i+2}) = 0$ . Also, by (R4) applied to  $W_{i+1}$ ,  $B \cap W_{i+1}$  is  $2(\xi(\Sigma) - \xi(\Phi))$ -taut. Thus, the above observation applied twice tells us that  $B$  is  $(2(\xi(\Sigma) - \xi(\Phi)) + 2)$ -taut, as required.

(R5) By construction,  $\mathbf{C}(\gamma_i) \subseteq Y_1(\mathbf{C}(\alpha) \cup \mathbf{C}(\beta))$  for all  $i$ . By (R5) applied to  $W_i$ , we have  $\mathbf{C}(W_i) = Y_\xi Y_{\xi-1} \cdots Y_3 Y_2 (\mathbf{C}(\gamma_{i-1}) \cup \mathbf{C}(\gamma_i) \cup \mathbf{C}(\gamma_{i+1}))$ . Thus  $\mathbf{C}(W) = \bigcup_i \mathbf{C}(W_i) \subseteq Y_\xi \cdots Y_2 Y_1 (\mathbf{C}(\alpha) \cup \mathbf{C}(\beta)) = Y^\infty(\mathbf{C}(\alpha) \cup \mathbf{C}(\beta))$  as required.  $\square$

We can now include the case of complexity 1 bands as an afterthought.

The construction is now very simple, but we need to check that it does not mess up what we have already achieved. Namely, that no two annuli have the same base curves, and that bands are still taut.

**Lemma 2.4.2.** *Suppose that  $\alpha, \beta \subseteq \Sigma$  are multicurves with  $\alpha$  complete. Then there is a complete annulus system,  $W = \bigcup \mathcal{W} \subseteq O$ , satisfying:*

(S1)  $\alpha = \pi_\Sigma(W \cap \partial_- O)$  and  $\beta = \pi_\Sigma(W \cap \partial_- O)$ ,

(S2) If  $\Omega, \Omega' \in \mathcal{W}$  with  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$ , then  $\Omega = \Omega'$ .

(S3) If  $B \subseteq O$  is a band with  $\xi(B) \geq 2$ , then  $B$  is  $2(\xi(\Sigma) - \xi(B) + 2)$ -taut.

(S4) If  $B \subseteq O$  is a band with  $\xi(B) = 1$ , then  $B$  is taut.

(S5)  $\mathbf{C}(\mathcal{W}) \subseteq Y^\infty(\mathbf{C}(\alpha) \cup \mathbf{C}(\beta))$ .

*Proof.* Let  $\hat{W}$  be the complete annulus system given by Lemma 2.4.1. Suppose that  $A$  is a type-1 brick with respect to  $\hat{W}$ , in other words, a band with  $\xi(A) = 1$  and with  $W \cap A \setminus \partial_V A$  consisting of two curves,  $\delta \subseteq \partial_- A$  and  $\epsilon \subseteq \partial_+ A$ . By Lemma 2.4.1 (R2),  $\pi_\Sigma \delta \neq \pi_\Sigma \epsilon$ . Let  $\delta = \gamma_0, \gamma_1, \dots, \gamma_n = \epsilon$  be a geodesic in  $\mathcal{G}(\pi_\Sigma A)$ . Let  $\Omega_i = \gamma_i \times I_i \subseteq A$  be disjoint annuli occurring in this order vertically. This gives an annulus system,  $W_A = \bigcup_i \Omega_i$  in  $A$ . We perform this construction for all type-1 bricks. Since these bricks are disjoint, we get an annulus system  $W = \hat{W} \cup \bigcup_A W_A$ , as  $A$  varies over all type 1 bricks. We need to verify the above properties.

(S1) Since  $W \cap \partial_H O = \hat{W} \cap \partial_H O$ , this follows by construction.

(S2) Suppose that  $\Omega, \Omega' \in \mathcal{W}$ , with  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$ . Let  $\Omega''$  be a vertical annulus connecting  $\Omega$  to  $\Omega'$  (so that  $\Omega \cup \Omega'' \cup \Omega'$  is a vertical annulus). Now  $\partial_H \Omega'' \subseteq W$ , so  $\partial_\pm \Omega''$  are compatible with  $\hat{W}$ . It follows by 2.4.1 (R2) that  $\Omega''$  is compatible with  $\hat{W}$  and so by the construction, the only way that can happen is if  $\Omega = \Omega'$ .

(S3) Let  $B$  be a band with  $\xi(B) \geq 2$ . We can assume that  $\partial_V^\Sigma B \subseteq \hat{W}$ , for if a boundary component were in  $W \setminus \hat{W}$ , then  $B$  would be crossed by an annulus of  $W$  and so it would have height 0, and there is nothing to prove. In other words,  $B$  is a band with respect to  $\hat{W}$ . With respect to  $\hat{W}$ ,  $B$  is  $2(\xi(\Sigma) - \xi(B) + 1)$ -taut. In passing to  $W$ ,  $H(B)$  can only decrease. It is possible there may be some new curves in  $\partial_\pm B \cap W$ , but this could decrease  $H_0(B)$  by at most 2. It follows that, with respect to  $W$ ,  $B$  is  $2(\xi(\Sigma) - \xi(B) + 2)$ -taut.

(S4) Let  $B$  be a band with  $\xi(B) = 1$ . As in (S3), we need only consider the

case where  $\partial B \subseteq \hat{W}$ . We note that there is no component of  $\hat{W}$  contained in the interior of  $B$ . For if  $\Omega$  were such a component, we could construct another annulus,  $\Omega'$  in  $B$  with  $\pi_\Sigma \Omega' \neq \pi_\Sigma \Omega$ , so that  $\partial_H \Omega' \cap \mathcal{W} = \emptyset$ , and with  $\pi_V \Omega$  contained in the interior of  $\pi_V \Omega'$ . Thus  $\Omega'$  crosses  $\Omega$ . But  $\partial_\pm \Omega$  is compatible with  $\hat{W}$ , so this contradicts Lemma 2.4.1(R2). We see that the only element of  $\mathcal{W}$  in the interior of  $B$  were those added in some brick,  $A$ , of our construction. Since these were made out a geodesic in  $\mathcal{G}'(\pi_\Sigma B)$ , it follows that  $B$  is, by definition, taut.

(S5) Clearly  $\mathbf{C}(\mathcal{W}) \subseteq \bar{Y}(\mathbf{C}(\hat{W}))$ . Since  $\mathbf{C}(\hat{W}) \subseteq Y^\infty(\mathbf{C}(\alpha) \cup \mathbf{C}(\beta))$ , the result follows.  $\square$

The main result about existence of complete annulus systems in the bi-infinite case can be stated as follows.

Let  $\Sigma$  be a compact surface with complexity,  $\xi(\Sigma) \geq 2$ . Suppose  $a, b \in \partial \mathcal{G}(\Sigma)$  are distinct and that  $(\alpha_i)_i$  and  $(\beta_i)_i$  are sequences of curves converging on  $a$  and  $b$  respectively. As observed in Section 2.2, the sets  $Y^\infty(\{\alpha_i, \beta_i\})$  converge locally on some locally finite subset  $Y \subseteq B$ . This lies a bounded distance (depending only on  $\xi(\Sigma)$ ) from any bi-infinite geodesics from  $a$  to  $b$  in  $\mathcal{G}(\Sigma)$ .

We will prove:

**Theorem 2.4.3.** *There is a constant,  $c$ , depending only on  $\xi(\Sigma)$  such that for any  $a, b \in \partial \mathcal{G}(\Sigma)$  and  $Y \subseteq V(\mathcal{G})$  the locally finite set constructed as above, we can find a complete annulus system  $\mathcal{W}$  in  $\Phi = \Sigma \times \mathbb{R}$  such that*

**(P1)**  $\mathbf{C}(\mathcal{W}) \subseteq \bar{Y}^\infty(Y)$ .

**(P2)** If  $\Omega, \Omega' \in \mathcal{W}$  with  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$ , then  $\Omega = \Omega'$ .

**(P3)** Every band in  $\Psi$  of complexity at least 2 in  $\Psi$  is  $c$ -taut.

**(P4)** Every band of complexity 1 is taut.

Note that in (P3) we are allowing bands with base surface  $\Sigma$ . Tautness then tells us that the path of multicurves associated to  $\mathcal{W}$  is quasi-geodesic.

Property (P1) will eventually serve to show that the map from the model space is lipschitz (using the A-priori Bounds Theorem 2.1.2). Property (P2) is needed to show that the map has degree one on the “thick parts” of these space. Properties (P3) and (P4) are needed for the reverse coarse inequalities, to show that our map is a quasi-isometry.

Using the local finiteness of our sets,  $Y$  (Lemma 2.2.3), we can reduce to a finite case of Theorem 2.4.3. Here we consider only a finite band,  $O = \Sigma \times I$ . We interpret “completeness” to include the statement that  $\mathbf{C}_-(\mathcal{W})$  and  $\mathbf{C}_+(\mathcal{W})$  are both complete multicurves. In this case, we started with two curves  $\alpha, \beta$  which fill  $\Sigma$ . We can replace  $Y$  by  $Y^\infty(\{\alpha, \beta\})$ , and insist that  $\alpha \in \mathbf{C}_-(\mathcal{W})$  and  $\beta \in \mathbf{C}_+(\mathcal{W})$ . To get us started, we can apply Lemma 2.2.4 to give us our a multicurve that will serve as  $\mathbf{C}_-(\mathcal{W})$ . (This is the only reason we need the two  $Y$ ’s in property (P1). One can clearly formulate other versions that would not involve us in constructing quite so many tight geodesics, but there seems little point for our purposes.)

Putting Lemma 2.2.4 together with Proposition 2.4.2, we see that we have proven the finite analogue of Theorem 2.4.3. The bi-infinite case now follows using Lemma 2.2.3, as discussed earlier.

This proves Theorem 2.4.3.

There are various combinatorial properties of bands that we will need.

**Definition.** The *height*  $H(B)$  of a complexity 1 (4HS or 1HT) band to be the number of elements of  $\mathcal{W}$  it contains.

(The height of a band of higher complexity was defined earlier.)

**Definition.** We shall say that two bands,  $A, B \subseteq \Phi$  are *parallel* if they have the same base surface,  $\pi_\Sigma A = \pi_\Sigma B$ . We say that  $B$  is a parallel subband if also  $B \subseteq A$ . Note that in this case, the closures of  $B \setminus A$  are parallel bands which we refer to as the *collars* of  $B$  (in  $A$ ). We denote them by  $B_-$  and  $B_+$ . (Figure 2.10.) We

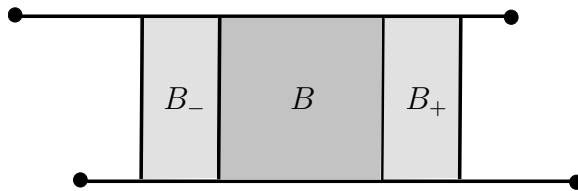


FIGURE 2.10. A band  $B$ , with collars  $B_-$  and  $B_+$

write  $D(B, A) = \max\{H(B_-), H(B_+)\}$  for the (*combinatorial*) *depth* of  $B$  in  $A$ . We say that a band  $A$  is *maximal* if it is not contained in any larger parallel band. Every band  $B$  is contained in a unique maximal parallel band,  $M(B)$ . We write  $D(B) = D(B, M(B))$ . We say that  $B$  is  *$r$ -collared* if  $D(B) \geq r$ .

**Lemma 2.4.4.** *Suppose that  $B, B'$  are 1-collared with base surfaces  $\Phi$  and  $\Phi'$  respectively. If  $\text{int } B \cap \text{int } B' \neq \emptyset$ , then  $\Phi$  and  $\Phi'$  are nested, i.e. either  $\Phi \subseteq \Phi'$  or  $\Phi' \subseteq \Phi$ .*

*Proof.* Let  $S \subseteq \Psi$  be a horizontal fibre through a point of  $\text{int } B \cap \text{int } B'$ . Now  $S \cap B$  and  $S \cap B'$  are fibres of  $B$  and  $B'$  respectively, and so the boundaries of  $\Phi$  and  $\Phi'$  cannot cross. If  $\Phi$  and  $\Phi'$  are not nested, then we can find curves  $\alpha \subseteq \partial\Phi \setminus \partial\Phi'$  and  $\alpha' \subseteq \partial\Phi' \setminus \partial\Phi$ . Let  $\Omega, \Omega' \in \mathcal{W}$  be the vertical annuli with  $\pi_\Sigma \Omega$  and  $\pi_\Sigma \Omega'$ . Thus  $\Omega$  and  $\Omega'$  contain boundary components of the maximal bands,  $M(B)$  and  $M(B')$  respectively. By considering the horizontal projections of these bands to  $\mathbb{R}$ , we see easily that either  $\Omega$  crosses one of the collars  $M(B) \setminus B$  or  $\Omega'$  crosses one of the collars  $M(B') \setminus B'$ . This contradicts the assumption that  $B$  and  $B'$  are 1-collared.  $\square$

Let  $\mathcal{B}$  be the set of maximal bands in  $\Psi$  with  $H(B) > 0$ . Given  $n \in \mathbb{N}$ , if  $A \in \mathcal{B}$  and  $r \in \mathbb{N}$  with  $H(A) \geq 2r + 1$ , we can find a parallel subband,  $B \subseteq A$ , so that each of the collars,  $B_\pm$ , has height  $H(B_\pm)$  exactly  $r$ . For each  $A \in \mathcal{B}$ , and each

such  $r$ , we choose such a band  $B$ , and write  $\mathcal{B}(r)$  for the set of all bands that arise in this way for a fixed  $r \in \mathbb{N}$ .

Given a subsurface,  $\Phi$ , of  $\Sigma$ , write  $\mathcal{B}_\Phi(r) \subseteq \mathcal{B}(r)$  for those bands in  $\mathcal{B}(r)$  whose base surface is a proper subsurface of  $\Phi$ .

Let  $\mathcal{D}$  be the brick decomposition of  $\Psi$  described earlier.

**Definition.** Given a subset  $Q \subseteq \Psi$  we define the *size* of  $Q$ , denoted  $\text{size}(Q)$ , to be the number of bricks of  $\mathcal{D}$  whose interiors meet the interior of  $Q$ .

We view  $\text{size}(Q)$  as a combinatorial measure of the volume of  $Q$ .

**Lemma 2.4.5.** *Given  $h, r, \xi$ , there is some  $\nu = \nu(h, r, \xi)$  such that if  $B$  is a band with  $H(B) \leq h$  and base surface  $\Phi$ , then*

$$\text{size}(B \setminus \bigcup \mathcal{B}_\Phi(r)) \leq \nu(h, r, \xi(\Phi)).$$

Here we are allowing the case where  $\Phi = \Sigma$ , in which case,  $\mathcal{B}_\Phi(r) = \mathcal{B}(r)$ .

*Proof.* We proceed by induction on  $\xi(\Phi)$ . If  $\xi(\Phi) = 1$ , then we see explicitly that  $\text{size}(B) \leq 3H(B) + 2$ .

Now suppose that  $\xi(\Phi) \geq 2$ . We can cut  $B$  into a set of  $H(B) + 1$  parallel bands each of height  $r$ . It is thus sufficient to deal with the case where  $H(B) = 0$ , in other words, some vertical annulus of  $\mathcal{W}$  cuts through  $B$ . Now the set of all such annuli that cut through  $B$  cut  $B$  into set of bands of lower complexity. The number (possibly just 1) of such bands is bounded by  $\xi(\Phi)$ . Let  $A \subseteq B$  be such a band, and let  $\Phi' = \pi_\Sigma A \subseteq \Phi$ . Note that  $\mathcal{B}_{\Phi'}(r) \subseteq \mathcal{B}_\Phi(r)$ . If  $H(A) \leq 2r$ , then  $\text{size}(A \setminus \bigcup \mathcal{B}_\Phi(r))$  is bounded (by  $\nu(2r, r, \xi(\Phi) - 1)$ ). If  $H(A) \geq 2r + 1$ , then  $H(M(A)) \geq 2r + 1$ , and so there is some  $C \in \mathcal{B}(r)$  with base surface  $\Phi'$ , so that each of the collars  $M(A) \setminus C$  has height  $r$ . Now  $C \in \mathcal{B}_\Phi(r)$  and  $A \setminus C$  consists of at most two bands each of height at most  $r$ . Applying the inductive hypothesis again, we see that  $\text{size}(A \setminus \bigcup \mathcal{B}_\Phi(r))$  is bounded (by  $2\nu(r, r, \xi(\Phi) - 1)$ ). Since there are at most  $\xi(\Phi)$  such bands  $A$ , this bounds  $\text{size}(B \setminus \bigcup \mathcal{B}_\Phi(r))$ , and the result follows by induction.  $\square$

## 2.5. THE COMBINATORIAL STRUCTURE OF THE MODEL.

We first describe the topology of the model, and its decomposition into bricks and tubes (Subsection 2.5.1). We equip this with a riemannian metric (Subsection 2.5.2) and describe some of its properties (Subsection 2.5.3). We will later need to describe how to extend the metric over Margulis tubes (see Subsection 2.8.1).

### 2.5.1. The topological construction.

Given an annulus system,  $\mathcal{W}$ , we can define  $\Lambda = \Lambda(\mathcal{W})$  as the metric completion of  $\Psi \setminus \mathcal{W}$  in its induced path metric. (This is different from the model metric



described in the next subsection.) In this way, we have a toroidal boundary component,  $\Delta(\Omega)$ , of  $\Lambda$ , associated to each  $\Omega \in \mathcal{W}$ . Indeed,  $\partial\Lambda = \partial\Psi \cup \bigcup_{\Omega \in \mathcal{W}} \Delta(\Omega)$ . There is natural projection,  $\pi_\Psi : \Lambda \rightarrow \Psi$  that is injective on  $\text{int } \Lambda$ . On each  $\Delta(\Omega)$  it is injective on  $\pi_\Psi^{-1}\partial_H\Omega$  and two-to-one elsewhere. As in Section 2.3,  $\Delta(\Omega)$  comes equipped with a free homotopy class of longitude, denoted  $l(\Omega)$  and meridian denoted  $m(\Omega)$ . (A longitude maps to a horizontal curve in  $\Omega$ , and a meridian to a vertical arc in  $\Omega$  connecting its boundary components.) We refer to this procedure as “opening out” the annuli of  $\mathcal{W}$ . We can also fill them back in again.

Given any subset,  $\mathcal{W}_0 \subseteq \mathcal{W}$  let  $\Lambda(\mathcal{W}, \mathcal{W}_0)$  be the manifold obtained from  $\Lambda(\mathcal{W})$  by gluing in a solid torus,  $T(\Omega)$  to  $\Delta(\Omega)$  for each  $\Omega \in \mathcal{W}_0$ . so that the meridian bounds a disc. We will write  $\Upsilon = \Upsilon(\mathcal{W}) = \Lambda(\mathcal{W}, \mathcal{W})$ . Thus  $\Upsilon$  is homeomorphic to  $\Psi$ . From a purely topological point of view this is a rather fruitless exercise. However, we will want to view these spaces as having different structures. We will be regarding  $\Psi$  together with  $\mathcal{W}$  as an essentially combinatorial object, whereas  $\Upsilon$  will be given a geometric structure.

Suppose that  $\mathcal{W}$  is a complete annulus system. We obtain a **brick decomposition** of  $\Lambda = \Lambda(\mathcal{W})$  by lifting each brick in  $\Psi$  to a **brick** in  $\Lambda$ . Note that two vertical boundary components of such a (lifted) brick may become identified under the projection map,  $\pi_\Psi$ , but the projection is otherwise injective on bricks. By abuse of notation we will also denote this lifted brick decomposition by  $\mathcal{D}$ . (Figure 2.11.) In this case, if two bricks meet one is of type 0 and the other of

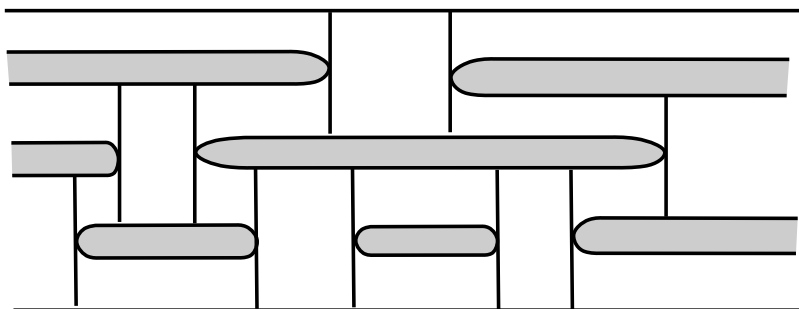


FIGURE 2.11. A brick decomposition,  $\mathcal{D}$ , of  $\Lambda$ . The glued-in solid tori are shaded in grey.

type 1. They meet along a horizontal 3HS.

To each  $\Omega \in \mathcal{W}$ , we have associated a toroidal boundary component,  $\Delta(\Omega)$ , of  $\Lambda(\mathcal{W})$  as described above. Let  $\mathcal{D}(\Omega)$  be the set of components of  $D \cap \Delta(\Omega)$  which are annuli, as  $D$  runs over the set of bricks,  $\mathcal{D}$ . Thus  $\Delta(\Omega) = \bigcup \mathcal{D}(\Omega)$  is a decomposition of this torus. We can view  $|\mathcal{D}(\Omega)|$  as a combinatorial measure of its length.

Similarly suppose  $P$  is a non-compact boundary component of  $\Lambda(\mathcal{W})$ . This must be a bi-infinite cylinder, identified with a boundary component of  $\Psi$ . We

get a decomposition of  $P$  into a collection  $\mathcal{D}(P)$  of compact annuli, by taking the intersection with bricks.

We observed that we can recover  $\Psi$  up to homeomorphism by gluing a solid torus  $T(\Omega)$  to each  $\Delta(\Omega)$ , so as to obtain the space  $\Upsilon$ . We can describe this more explicitly as follows. Given  $\Omega \in \mathcal{W}$ , we choose an explicit homeomorphism of  $T(\Omega) \setminus \partial_H \Omega$  with  $S^1 \times [0, 1] \times (0, 1)$ , and foliate  $T(\Omega) \setminus \partial_H \Omega$  with annuli of the form  $S^1 \times [0, 1] \times \{t\}$  for  $t \in (0, 1)$ . We set up the homeomorphism so that the two circles  $S^1 \times \{0\} \times \{t\}$  and  $S^1 \times \{1\} \times \{t\}$  are horizontal in  $\Delta(\Omega)$  and get identified with the same horizontal circle in  $\Omega$ , under the projection of  $\Lambda(\Omega)$  to  $\Psi$ . We add in two (“top” and “bottom”) degenerate circular leaves,  $\partial_- \Omega$  and  $\partial_+ \Omega$ , to complete the foliation of  $T(\Omega)$ .

Now suppose that  $S$  is a horizontal fibre in  $\Psi$ . Taking preimages in  $\Lambda(\mathcal{W})$  we get a disjoint union,  $\pi_\Psi^{-1}S \subseteq \Lambda(\mathcal{W})$ , of “horizontal” surfaces. We can now use the foliations on the tori  $T(\Omega)$  to complete this to a fibre of  $\Upsilon$ . These fibres collectively foliate  $\Upsilon$ . We refer to them as **horizontal fibres** of  $\Upsilon$ . We denote by  $S(x) \subseteq \Upsilon$  the fibre containing  $x \in \Upsilon$ . There is a natural projection of  $\Upsilon$  to  $\Psi$  collapsing each torus  $T(\Omega)$  to  $\Omega$ , so that the fibres of  $T(\Omega)$  are preimages of horizontal curves. The horizontal fibres of  $\Upsilon$  are preimages of horizontal fibres of  $\Psi$ . By a **band** in  $\Upsilon$  we mean the preimage of a band in  $\Psi$ .

Suppose now that there is some  $L \geq 0$  and a partition  $\mathcal{W} = \mathcal{W}_0 \sqcup \mathcal{W}_1$  of  $\mathcal{W}$  such that  $|\mathcal{D}(\Omega)| \leq L$  for all  $\Omega \in \mathcal{W}_1$ . (Such a situation will arise in Section 2.8 — see Theorem 2.8.2.) We write

$$\begin{aligned} \mathcal{T} &= \{T(\Omega) \mid \Omega \in \mathcal{W}_0\}, \\ \mathcal{T}_1 &= \{T(\Omega) \mid \Omega \in \mathcal{W}_1\}. \end{aligned}$$

Let

$$\Theta = \Lambda(\mathcal{W}, \mathcal{W}_1) = \Lambda(\mathcal{W}) \cup \bigcup \mathcal{T}_1.$$

(In other words,  $\Theta$  is the result of gluing in the solid tori corresponding to  $\mathcal{W}_1$ , and as such is homeomorphic to  $\Lambda(\mathcal{W}_0)$ .)

Note that  $\Theta$  is made out of a collection,  $\mathcal{D}$ , of bricks and “tubes”  $\mathcal{T}_1$ . We refer to the elements of  $\mathcal{D} \cup \mathcal{T}_1$  collectively as the **building blocks** of  $\Theta$ . Similarly, if  $R$  is a boundary component of  $\Theta$  (either a cylinder or a torus) we refer to the elements of  $\mathcal{D}(R)$  as the **building blocks** of  $R$ . If  $\beta$  is a path in  $\Theta$ , or in  $R$ , we define the **combinatorial length** to be equal to the number of building blocks that it meets (counting multiplicities).

Note that (since we are assuming that  $|\mathcal{D}(\Omega)| \leq L$  for all  $\Omega \in \mathcal{W}_1$ ), each building block of  $\Theta$  meets boundedly many others (in fact at most  $\max\{8, L\}$ ). It follows that given any  $x \in \Theta$  and any  $r \in \mathbb{N}$ , there is bound, depending on  $r$ , on the number of building blocks that can be connected to  $x$  by a path of combinatorial length at most  $r$ .

We want to make a couple of observations concerning the embedding of the boundary components of  $\Theta$  into  $\Theta$ . These will eventually be used to show that

boundary components (in the corresponding lifts) will be quasi-isometrically embedded in the model space (see Lemma 2.5.3).

We begin with the non-compact components, since the description is somewhat simpler. The geometrical interpretation of these statements is made more apparent by Lemma 2.5.3, which will eventually be used in Section 2.13 (see Lemma 2.13.7).

**Lemma 2.5.1.** *Suppose that  $\Pi$  is a non-compact boundary component of  $\Theta$ . Suppose that  $\beta$  is a path of combinatorial length  $n$  in  $\Theta$  connecting two points,  $x, y \in \Pi$ , and homotopic into  $\Pi$ , relative to  $\{x, y\}$ . Then  $x$  and  $y$  are connected by a path in  $\Pi$  whose combinatorial length is bounded above by some uniform linear function of  $n$ .*

*Proof.* Let  $C_x, C_y \in \mathcal{D}(\Pi)$  be annular blocks containing  $x$  and  $y$  respectively, and let  $\mathcal{D}_{x,y} \subseteq \mathcal{D}(\Pi)$  be the set of annular blocks between  $C_x$  and  $C_y$ . We want to bound  $|\mathcal{D}_{x,y}|$  linearly in terms of  $n$ .

Given  $z \in \Pi$ , recall that  $S(z)$  is the horizontal fibre of  $\Upsilon$  containing  $z$ . Let  $F(z)$  be the component of  $S(z) \cap \Theta$  containing  $z$ . There is a bound, say  $l_0$ , on the number of blocks that  $F(z)$  can meet, depending only on  $\xi(\Sigma)$ . Now if  $z \in \bigcup \mathcal{D}_{x,y}$  we see that  $F(z)$  must meet  $\beta$  (from the assumption that  $\beta$  is homotopic into  $\Pi$ ). It follows that  $z$  is connected to  $\beta$  by a path in  $\Theta$  of combinatorial length at most  $l_0$ . By the earlier observation (on the uniform local finiteness of our system of building blocks) we see that this gives some bound on  $|\mathcal{D}_{x,y}|$  in terms of  $n$ .

To make this a linear bound, let us fix our favourite positive integer, say 10, and let  $l_1$  be the bound when  $n$  is at most  $10 + 2l_0$ . This means that if  $z, w \in \Pi$  are separated by at least  $l_1$  blocks, then if  $\gamma$  is any path from  $F(z)$  to  $F(w)$  in  $\Theta$ , which can be homotoped into  $\Pi$  by sliding its endpoints along  $F(z)$  and  $F(w)$  respectively, then  $l(\gamma) \geq 10$ .

For the general case, we now choose a sequence of points  $x = z_0, z_1, \dots, z_p = y$  in  $\bigcup \mathcal{D}_{x,y}$  so that  $z_i$  and  $z_{i+1}$  are separated by at least  $l_1$  annular blocks in  $\mathcal{D}(\Omega)$ , and with  $|\mathcal{D}_{x,y}|$  bounded above by a fixed linear function of  $p$ . Now the path  $\beta$  must cross each of the surfaces  $F(z_i)$  and so  $p$  is in turn bounded above by a linear function of  $l(\beta)$ .  $\square$

We need a version of this where  $\Pi$  is replaced by a toroidal boundary component,  $\Delta = \Delta(\Omega)$  for some  $\Omega \in \mathcal{W}_0$ .

**Lemma 2.5.2.** *Suppose that  $\Delta$  is a compact boundary component of  $\Theta$ . Suppose that  $\beta$  is a path of combinatorial length  $n$  in  $\Theta$  connecting two points,  $x, y \in \Delta$ . If  $\beta$  is homotopic to a path in  $\Delta$ , relative to  $\{x, y\}$ , then we can find such a path in  $\Delta$  in the same relative homotopy class whose combinatorial length is bounded above by some uniform linear function of  $n$ .*

*Proof.* The argument is a slight refinement of that used for Lemma 2.5.1. If  $z \in \Delta$ , we can define the surface  $F(z)$  exactly as in Lemma 2.5.1.

Note that  $\partial_H \Omega$  cuts  $\Delta = \Delta(\Omega)$  into two annuli,  $A$  and  $A'$ , say. Suppose first that  $x, y \in A$  and that  $\beta$  is homotopic into  $A$  relative to  $x, y$ . Essentially the same argument as before gives a bound on the number of building blocks separating  $x$  and  $y$  in  $A$ . We can therefore construct surfaces  $F(z_i)$  as before so as to obtain a linear bound in terms of  $n$ .

The general case is complicated by the fact that  $\beta$  might wrap around  $\Delta$  many times in the vertical direction (that of a meridian curve). However, we can construct surfaces,  $F(z_i)$ , on both sides of  $\Delta$  (the annuli  $A$  and  $A'$ ), and we note that  $\beta$  must cross all of these surfaces in sequence (counting multiplicities). We should note that nothing we have said excludes the possibility that the total vertical length,  $\Delta(\Omega)$ , is small, (maybe smaller than  $l_0$ , for example) so we need to take at least one such surface.  $\square$

### 2.5.2. *The geometry of $\Lambda(\mathcal{W})$ .*

We next want to translate some of these combinatorial observations into more geometrical terms. To this end, we shall put a riemannian metric on  $\Upsilon$ . We write  $d_\Upsilon$  of the induced path metric. The construction of  $d_\Upsilon$  will be explained more carefully in Section 2.7, where we construct the model space. For the purposes of this section, we only care about the metric restricted to  $\Theta$ . We write  $d = d_\Theta$  for the induced path-metric on  $\Theta$ . The key points (which can be taken as hypotheses for the moment) are as follows. We will assume:

(\*\*): The local geometry of the decomposition of  $(\Theta, d)$  is bounded.

In particular, there is a uniform positive lower bound on the injectivity radius of  $(\Theta, d)$ . Each building block of  $\Theta$  (in  $\mathcal{D} \cup \mathcal{T}_1$ ) has bounded diameter. Moreover, there is a positive lower bound on the  $d$ -distance between any two disjoint building blocks. We can also assume that each of the building blocks of any boundary component of  $\Theta$  is a fixed isometry class of annulus, say  $S^1 \times [0, 1]$ . If  $x \in \Theta$ , the fibre  $S(x)$  meets each block of  $\Theta$  in a surface of bounded diameter. In particular, the diameter of each component of  $S(x) \cap \Theta$  is bounded.

Note that an immediate consequence is that, at least up to homotopy, the combinatorial length of a path in  $\Theta$  is bounded above by a linear function of its  $d$ -length. Since the building blocks are not simply connected, we do not have a converse statement keeping control of homotopy. However, distances between points, and diameters of sets, in the metric  $d$  are bounded above by a linear function of their combinatorial counterparts.

We can immediately translate Lemmas 2.5.1 and 2.5.2 into geometrical terms and express them in a unified fashion as follows.

Suppose that  $R$  is a boundary component of  $\Theta$ , and let  $H \subseteq \pi_1(\Theta)$  be the subgroup generated by a horizontal longitude. (Thus  $H \equiv \pi_1(R)$  if  $R$  is a bi-infinite cylinder.) Let  $\hat{\Theta}$  be the cover corresponding to  $H$ . Thus  $R$  lifts to a bi-infinite cylinder,  $\hat{R} \subseteq \hat{\Theta}$  (so that  $\hat{R} \equiv R$  in the non-compact case).

**Lemma 2.5.3.** *If  $R$  is a boundary component of  $\Theta$ , and  $\hat{R} \subseteq \hat{\Theta}$  constructed as above, then  $\hat{R}$  is quasi-isometrically embedded in  $\hat{\Theta}$ .*

For applications, we will need another riemannian metric,  $\rho$ , on  $\Upsilon$ , and its restriction to  $\Theta$ . This can be taken to be equal to  $d$  on  $\Theta$  and equal to 0 on each tube  $T \in \mathcal{T}$ . In other words we force each element of  $T$  to have diameter 0. (This is the electric pseudometric, referred to in Section 2.1.) As stated, we just get a pseudometric, and it will be discontinuous at the toroidal boundary components. If we want, we can smooth it out in a small neighbourhood of these boundaries. The only important requirement is that each element of  $\mathcal{T}$  should have bounded diameter with respect to  $\rho$ .

### 2.5.3. Properties of the model.

In summary, at this point, we have a manifold,  $\Upsilon$  diffeomorphic to  $\Sigma \times \mathbb{R}$ , a collection,  $\mathcal{T}$ , of unlinked solid tori in  $\Upsilon$ , the “thick part”,  $\Theta = \Upsilon \setminus \text{int } \mathcal{T}$ . We have an electric pseudometric,  $\rho = \rho_{\Upsilon}$ , in  $\Upsilon$ , which is identically zero restricted to each  $T \in \mathcal{T}$ . This restricts to an electric pseudometric,  $\rho = \rho_{\Theta}$ , on  $\Theta$ .

Recall the definition of “fibres” from Section 2.3.

We can list some geometric properties of  $\Theta$  as follows:

**(W1)** Every point  $x \in \Theta$  lies in a fibre  $S(x)$  of  $\Upsilon$  with uniformly bounded  $\rho$ -diameter.

We can simply take  $S(x)$  to be the horizontal fibre as described above. In this way,  $S(x)$ , will vary continuously in  $x$ . (This will be used in Section 2.6.) Alternatively, we can push the fibre off each torus of  $\mathcal{T}$  so as to give us a surface,  $S(x)$ , in  $\Theta$ , while retaining a bound on its  $\rho$ -diameter in  $\Theta$ . (Here we are referring to the extrinsic diameter in  $\Theta$ , and not the induced path-metric in  $\Theta$ , which may be arbitrarily large.) In this case, however, we can no longer assume that  $S(x)$  varies continuously in  $x$ . (This alternative construction will be useful in Section 2.10.)

**(W2)** Each  $x \in \Theta$  is contained in a loop  $\gamma_x \subseteq \Theta$  of bounded  $d$ -length, and homotopic to a curve,  $[\gamma_x] \in \mathbf{C}(\Sigma)$ . If  $x$  lies in a component,  $R$ , of  $\partial\Theta \setminus \partial\Psi$ , then we can take  $\gamma_x$  to be the horizontal curve in  $R$  containing  $x$ .

In fact, if  $x \in D \in \mathcal{D}$ , we take  $\gamma_x$  to be freely homotopic into one of the vertical boundary components of  $D$ . Thus,  $\gamma_x$  is freely homotopic into an annulus

$\Omega_x \in \mathcal{W}$ . (In all cases, we can assume that  $\gamma_x \subseteq S(x)$ .) The following then follows from the combinatorial structure.

**(W3)** There is some  $\eta > 0$  such that if  $x, y \in \Theta$ , with  $d(x, y) \leq \eta$ , then  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  is bounded above.

Here we can take  $\eta > 0$  to be the lower bound on injectivity radius of  $\Theta$ , but any positive constant would do.

**(W4)** If  $x, y \in \Theta$ , then  $\rho(x, y)$  is bounded above by a uniform linear function of  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$ .

This is our geometric interpretation of the tautness condition (Theorem 2.4.3(P3)) in the case where the base surface is  $\Sigma$ . Note that  $\gamma_x$  and  $\gamma_y$  are homotopic to  $\Omega_x$  and  $\Omega_y$  and a bounded  $d$ -distance from the corresponding tubes  $T(\Omega_x)$  and  $T(\Omega_y)$  (these tubes might lie in either  $\mathcal{T}$  or  $\mathcal{T}_1$ ). By tautness, there is a ladder,  $\Omega_x = \Omega_0, \Omega_1, \dots, \Omega_n = \Omega_y$  in  $\mathcal{W}$ , with  $n$  bounded above by a linear function of  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  (in fact,  $n \leq d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y]) + c$ , where  $c$  is the tautness constant). Two consecutive  $\Omega_i$  and  $\Omega_{i+1}$  meet a horizontal fibre which has bounded  $\rho$ -diameter. We see that  $\rho(T(\Omega_i), T(\Omega_{i+1}))$  is bounded above, and so  $\rho(x, y)$  is linearly bounded in terms of  $n$  and hence in terms of  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  as claimed. (One needs to rephrase this slightly if  $\Sigma$  is 4HS or 1HT, but the argument is essentially the same — consecutive tubes are a bounded distance apart, since they meet a common building block. Here we use the metric on the modified curve graph  $\mathcal{G}(\Sigma)$ .)

All these statements have analogues for the case of a band  $B \subseteq \Upsilon$ . We are only really interested in the case where all vertical boundary components of  $B$  lie in  $\mathcal{W}_0$ . In this case,  $\partial_H B$  is the relative boundary of  $B$  in  $\Theta$ . Let  $\Phi$  be the base surface of  $B$ .

Let  $\mathcal{T}_B^0$  be the set of tubes in  $\mathcal{T}$  whose interiors meet  $B$ . We let  $d_B$  be the riemannian metric on  $B$  induced from  $d$ , and let  $\rho_B$  be the metric obtained from  $d_B$  by forcing each set  $T \cap B$  for  $T \in \mathcal{T}_B^0$  to have diameter 0 (the intrinsic electric pseudometric in  $B$ .) We can now perform the above constructions inside  $B$ . We get:

**(W5)** If  $x \in B \cap \Theta$ , then  $x$  lies in a fibre,  $F(x) \subseteq B$  of  $B$ , of uniformly bounded  $\rho_B$ -diameter.

**(W6)** If  $x \in B \cap \Theta$ , then  $x$  lies in a loop  $\gamma_x^B \subseteq B \cap \Theta$  of bounded  $d_B$ -length, with  $[\gamma_x^B] \subseteq \mathbf{C}(\Phi) \subseteq \mathbf{C}(\Sigma)$ . If  $x \in B \cap \partial\Theta$  we can take  $\gamma_x^B$  to be a horizontal curve of that boundary component.

(W7) If  $x, y \in B \cap \Theta$ , and  $d_B(x, y) \leq \eta$ , then  $d_{\mathcal{G}(\Phi)}([\gamma_x^B], [\gamma_y^B])$  is bounded above.

(W8) If  $x, y \in B \cap \Theta$ , then  $\rho_B(x, y)$  is bounded above by a uniform linear function of  $d_{\mathcal{G}(\Phi)}([\gamma_x^B], [\gamma_y^B])$ .

Finally we need to be able to recognise when a point of  $\Theta$  does not lie in a given maximal band. We can define a **maximal band** in  $\Theta$  to be the preimage of a maximal band in  $\Psi$ .

(W9) Let  $B \subseteq \Theta$  be a maximal band, with base surface,  $\Phi \subseteq \Sigma$ . If  $x \in \Theta \setminus B$ , then  $x$  lies in a loop  $\delta_x^B \subseteq B \setminus \Theta$  of bounded  $d_B$ -length such that either  $[\delta_x^B]$  is homotopic to a torus  $T \in \mathcal{T}$  not lying entirely inside  $B$ , or else  $[\delta_x^B]$  is not homotopic into  $\Phi$ .

To see this, let  $S(x)$  be the horizontal fibre through  $x$ , and let  $F$  be the component of  $S(x) \cap \Theta$  containing  $x$ . This has bounded  $d$ -diameter. If  $\pi_\Sigma F$  is not a subsurface of  $\Phi$ , then we can choose  $\delta_x^B \subseteq F$ , not homotopic into  $\Phi$ . If  $\pi_\Sigma F \subseteq \Phi$ , then by the maximality of  $B$ , at least one of the boundary components of  $F$  must be homotopic to a torus  $T \in \mathcal{T}$ , and this is not contained in  $B$ . We can take  $\delta_x^B \subseteq F$ , freely homotopic to this boundary component.

## 2.6. MARGULIS TUBES.

The constructions described in Section 2.4 are the basis of the model of the “thick part”. To complete the picture we will need some description of the “thin part”. There may be parabolic cusps, but the main thing we have to worry about is the existence of Margulis tubes.

Recall, from Section 1.3 that a **quasi-isometry** between two geodesic spaces,  $(X, d)$  and  $(X', d')$  is a map  $f : X \rightarrow X'$ , for which there exist constants  $k_1 > 0, k_2, k_3, k_4, k_5$  such that  $k_1 d(x, y) - k_2 \leq d'(f(x), f(y)) \leq k_3 d(x, y) + k_4$  for all  $x, y \in X$ , and with  $X' \subseteq N(f(X), k_5)$ . Here we shall make the following stronger definition:

**Definition.** A **sesquilipschitz** map is a surjective lipschitz quasi-isometry.

In other words, in the definition of quasi-isometry we put  $k_4 = k_5 = 0$ .

**Definition.** A **universally sesquilipschitz** map between two spaces is a homotopy equivalence whose lift to the universal covers is sesquilipschitz.

One can easily check that a universally sesquilipschitz map is indeed sesquilipschitz.

Throughout this section our results refer to implicitly assumed constants. We will take it as implied that the constants outputted are explicit functions of the

constants inputted, though we will not bother to calculate these function explicitly. (In this section, they are all computable.) We shall use the adjective “uniform” if we want to stress this point.

We shall begin our discussion in dimension 1. It easily seen that any quasi-isometry of the real line  $\mathbb{R}$  is a bounded distance from a bilipschitz homeomorphism. We note the following variation:

**Lemma 2.6.1.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  is sesquilipschitz then there is a sesquilipschitz homotopy to a bilipschitz map.*

In other words there is a sesquilipschitz map  $F : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  with  $f = [x \mapsto F(x, 0)]$  and with  $g := [x \mapsto F(x, 1)]$  bilipschitz. Note that it follows that  $f$  is a bounded distance from  $g$ .

*Proof.* We can assume that  $f$  is end-preserving. We fix some sufficiently large, but bounded, constant,  $k \geq 0$ , so that  $f(x + k) > f(x) + 1$  for all  $x \in \mathbb{R}$ . We set  $g|k\mathbb{Z} = f|k\mathbb{Z}$ , and interpolate linearly. We then take a linear homotopy between  $f$  and  $g$ .  $\square$

A similar argument can be carried out equivariantly. We write  $S(r) = \mathbb{R}/r\mathbb{Z}$  for the circle of length  $r$ . We obtain:

**Lemma 2.6.2.** *Suppose that  $r, s > 0$ , and that  $f : S(r) \rightarrow S(s)$  is a universally sesquilipschitz map. Then there is a universally sesquilipschitz homotopy from  $f$  to a bilipschitz map from  $S(r)$  to  $S(s)$ .*

In particular, the ratios  $s/r$  and  $r/s$  are bounded. Here, all constants depend on those of those of  $f$ .

More will be said about 1-dimensional quasi-isometries in Section 2.10, but this will do for the moment. We move on to 2 dimensions.

Let  $\Delta$  be a euclidean torus equipped with a preferred basis,  $(l, m_0)$  for the integral first homology. We refer to  $l$  as the **longitude** of  $\Delta$  and to  $m_0$  as the **standard meridian**. More generally a **meridian** will be a curve of the form  $m_0 + nl$  for some  $n \in \mathbb{Z}$ . In situations of interest to us, the length of the longitude will be bounded both above and below, and so it is often convenient to normalise so that its length is 1. In this case the structure on  $\Delta$  is determined by a complex **modulus**  $\lambda \in \mathbb{C}$  with imaginary part  $\Im(\lambda) > 0$ , so that  $\Delta = \Delta(\lambda) = \mathbb{C}/\langle [z \mapsto z + 1], [z \mapsto z + \lambda] \rangle$ , with  $[z \mapsto z + \lambda]$  giving us the standard meridian. Note that the shortest meridian in  $\Delta$  has length between  $\Im(\lambda)$  and  $\Im(\lambda) + \frac{1}{2}$ .

We refer to a geodesic longitude as being **horizontal**: it is the projection of a line parallel to the real axis. These foliate  $\Delta$  and we write  $S(\Delta)$  for the leaf space obtained by collapsing each leaf to a point. It is a circle of length  $\Im(\lambda)$ .

In most cases of interest, the injectivity radius of  $\Delta(\lambda)$  will be bounded below by some positive constant. One can see that this is equivalent to putting a lower bound on  $\Im(\lambda)$ . Moreover if there is an equivariant quasi-isometry between the



universal covers of two such tori, a lower bound on the injectivity radius of one gives a lower bound for the other.

Let  $\Delta = \Delta(\lambda)$  and  $\Delta' = \Delta(\lambda')$ .

**Definition.** A map  $f : \Delta \rightarrow \Delta'$  is *horizontally straight* if it sends each horizontal longitude of  $\Delta$  isometrically to a horizontal longitude of  $\Delta'$ .

In formulae, this means that, writing  $\tilde{\Delta} = \mathbb{R}^2 = \tilde{\Delta}'$ , we have  $\tilde{f}(x, y) = (x + \tilde{f}_H(y), \tilde{f}_V(y))$ , where  $f_H, f_V : \mathbb{R} \rightarrow \mathbb{R}$  satisfy  $\tilde{f}_H(y + \mathfrak{S}(\lambda)) = \tilde{f}_H(y) + \mathfrak{R}(\lambda') - \mathfrak{R}(\lambda)$  and  $\tilde{f}_V(y + \mathfrak{S}(\lambda)) = \tilde{f}_V(y) + \mathfrak{S}(\lambda')$ .

Note that such a map induces a map,  $S(f) : S(\Delta) \rightarrow S(\Delta')$  (lifting to  $f_V : \mathbb{R} \rightarrow \mathbb{R}$ ). One can easily check that if  $f$  is lipschitz (respectively, sesquilipschitz, universally sesquilipschitz) then so is  $S(f)$ .

**Lemma 2.6.3.** *Suppose  $\mathfrak{S}(\lambda) \geq \epsilon > 0$ . Suppose  $f : \Delta \rightarrow \Delta'$  is a lipschitz map sending the longitude (homotopically) to the longitude. Then there is a lipschitz homotopy of  $f$  to a (lipschitz) horizontally straight map. Moreover if  $f$  is (universally) sesquilipschitz, we can take the homotopy to be (universally) sesquilipschitz. Here the constants only depend on  $\epsilon$  and the initial (sesqui)lipschitz constants.*

*Proof.* Let  $m$  be a shortest meridian on  $\Delta$ . The lower bound on  $\mathfrak{S}(\lambda)$  means that there is a lower bound on its slope with respect to any horizontal longitude. We now define  $g : \Delta \rightarrow \Delta'$  by taking  $g|m = f|m$ , and extending in the unique way to a horizontally straight map. We now take a linear homotopy between  $f$  and  $g$ . The above properties are easily verified.  $\square$

**Lemma 2.6.4.** *Suppose that  $f : \Delta \rightarrow \Delta'$  is a universally sesquilipschitz horizontally straight map. Then there is a universally sesquilipschitz homotopy from  $f$  to a horizontally straight bilipschitz homeomorphism.*

*Proof.* By Lemma 2.6.2, there is a universally sesquilipschitz homotopy  $F$  of  $S(f)$  to a bilipschitz map. Lifting to  $\mathbb{R}$  gives us a homotopy  $\tilde{F} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  from  $f_V : \mathbb{R} \rightarrow \mathbb{R}$  to a bilipschitz map  $h$ . Now define  $\tilde{G} : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^2$  by  $\tilde{G}(x, y, t) = (x + \tilde{f}_H(y), \tilde{F}(y, t))$ . Projecting back down gives us a bilipschitz map,  $g : \Delta \rightarrow \Delta'$ , with  $g_H = f_H$  and  $g_V = h$ .  $\square$

We have assumed that  $f$  lifts to a quasi-isometry of universal covers  $\tilde{f} : \tilde{\Delta} \rightarrow \tilde{\Delta}'$ . However, we only really require that it is a quasi-isometry for the covers corresponding to the longitudes of  $\Delta$  and  $\Delta'$ , which are bi-infinite cylinders. This is enough to show that  $\tilde{f}_V$  is a quasi-isometry. If we assume that  $f$  is lipschitz, then the same conclusion holds.

We remark that the existence of a  $k$ -bilipschitz map from  $\Delta(\lambda)$  to  $\Delta(\lambda')$  implies both that  $k^{-1}\mathfrak{S}(\lambda) \leq \mathfrak{S}(\lambda') \leq k\mathfrak{S}(\lambda)$  and  $k^{-1}|\lambda| \leq |\lambda'| \leq k|\lambda|$ .

We need also to consider lipschitz maps to the circle. Suppose that  $f : \Delta \rightarrow S(1)$  is  $k$ -lipschitz. Let  $m$  be a shortest meridian on  $\Delta$ . It has length at most

$\mathfrak{S}(\lambda) + \frac{1}{2}$ , and so its image has length at most  $k(\mathfrak{S}(\lambda) + \frac{1}{2})$ . Thus the degree of  $f|m$  is at most  $k(\mathfrak{S}(\lambda) + \frac{1}{2})$  in absolute value.

**Lemma 2.6.5.** *Suppose the torus  $\Delta(\lambda)$  admits a  $k$ -lipschitz map to  $S(1)$  which has degree 1 on the longitude and degree 0 on the standard meridian. Then  $|\lambda| \leq (k+1)(\mathfrak{S}(\lambda) + \frac{1}{2})$ .*

*Proof.* Let  $l$  be the longitude, and  $m_0$  and  $m$  be the standard and shortest meridians respectively. Thus,  $m = m_0 + pl$  for some  $p \in \mathbb{Z}$ , and so  $\deg(f|m) = \deg(f|m_0) + p \deg(f|l) = p$ , where  $f$  is the  $k$ -lipschitz map. By the above observation,  $|p| \leq k(\mathfrak{S}(\lambda) + \frac{1}{2})$ . Now

$$\begin{aligned} |\lambda| &= \text{length}(m_0) \\ &\leq \text{length}(m) + |p| \\ &\leq (\mathfrak{S}(\lambda) + \frac{1}{2}) + k(\mathfrak{S}(\lambda) + \frac{1}{2}) \\ &= (k+1)(\mathfrak{S}(\lambda) + \frac{1}{2}). \end{aligned}$$

□

**Lemma 2.6.6.** *Given  $c, k > 0$ , there is some  $h > 0$  such that if a map  $f : \Delta(\lambda) \rightarrow S(1)$  is  $k$ -lipschitz and degree 1 on the longitude and degree 0 on the meridian, and if  $\mathfrak{S}(\lambda) \leq c$ , then there is an  $h$ -lipschitz homotopy of  $f$  to a  $k$ -lipschitz map,  $g$ , sending every geodesic standard meridian to a point.*

*Proof.* Let  $l$  be some horizontal longitude. There is a unique map  $g : \Delta(\lambda) \rightarrow S(1)$  so that  $g|l = f|l$  and sending every standard meridian to a point. This is also  $k$ -lipschitz. Clearly  $f$  and  $g$  are homotopic, and we take a linear homotopy between them. To bound its lipschitz constant, it is enough to note that every geodesic standard meridian of  $\Delta(\lambda)$  gets mapped under  $f$  to a curve of length at most  $k|\lambda| \leq k(k+1)(c + \frac{1}{2})$  by Lemma 2.6.5. □

We now move into 3 dimensions to consider Margulis tubes. For the purposes of this section, we use the term ‘‘Margulis tube’’ simply to mean a particular kind of hyperbolic structure on the solid torus. This can be described as follows.

Let  $r \geq 0$  and  $R = 2\pi \sinh r$ . Given  $t \geq 0$ , set  $a_r(t) = \cosh(rt) / \cosh r$  and set  $b_r(t) = \sinh(rt)$ . Define a riemannian metric on  $\mathbb{R} \times S^1 \times (0, 1]$  by  $ds^2 = a_r(t)^2 dx^2 + b_r(t)^2 dy^2 + r^2 dt^2$ , where  $(x, y, t)$  are the local coordinates. Let  $N$  be the metric completion of this space.

The space we have defined is isometric to the  $r$ -neighbourhood,  $N = N(\tilde{\alpha})$  of a bi-infinite geodesic,  $\tilde{\alpha}$  in  $\mathbb{H}^3$ . Its boundary,  $\partial N = \mathbb{R} \times S^1 \times \{1\}$  is isometric to  $\mathbb{R} \times S(R)$ , by an isometry that is the identity on the first co-ordinate.

A loxodromic isometry,  $g$ , with axis  $\tilde{\alpha}$  acts by translating the  $x$ -coordinate and rotating the  $y$ -coordinate. We refer to the quotient,  $T = N/\langle g \rangle$  as an (abstract) **Margulis tube**. Thus,  $\partial T$  is a euclidean torus. The quotient,  $\alpha = \tilde{\alpha}/\langle g \rangle$  is

the **core** of  $T$ . We refer to  $r = d(\alpha, \partial T)$  as the **depth** of  $T$ . We define the **standard meridian**,  $m_0$ , of  $\partial T$ , to be homotopically trivial in  $T$ . It has length  $R$ . We deem another curve,  $l$ , in  $\partial T$ , homotopic in  $T$  to the core curve,  $\alpha$ , to be the **longitude** of  $\partial T$ . (Its homotopy class gives us some additional structure to  $T$ .) Note that the length of the core curve is equal to  $\text{area}(\partial T)/R \cosh r = 2\pi \text{area}(\partial T)/R\sqrt{R^2 + 4\pi^2}$ .

The following seems well known:

**Lemma 2.6.7.** *Given any euclidean torus  $\Delta$  with preferred longitude and standard meridian, there is a unique (up to isometry) Margulis tube,  $T$ , with  $\partial T = \Delta$ .*

*Proof.* The cover of  $\Delta$  corresponding to the standard meridian is isometric to  $\mathbb{R} \times S(R)$  for some  $R > 0$ . Set  $r = \sinh^{-1}(R/2\pi)$  and construct  $N$  as above. The action of the longitude on  $\partial N = \mathbb{R} \times S(R)$  extends to a loxodromic on  $N$ , and we take the quotient.

Uniqueness is easily established (see the remark after Lemma 2.6.8 below).  $\square$

**Lemma 2.6.8.** *Suppose  $T, T'$  are Margulis tubes and  $f : \partial T \rightarrow \partial T'$  is a  $k$ -bilipschitz map sending the standard meridian of  $\partial T$  (homotopically) to the standard meridian of  $\partial T'$ . Then  $f$  extends to a  $k'$ -bilipschitz map,  $f : T \rightarrow T'$ , where  $k'$  depends only on  $k$ .*

*Proof.* Let  $r, s$  be the depths of  $T, T'$  respectively. The lengths of the standard meridians are  $2\pi \sinh r$  and  $2\pi \sinh s$ . Their ratios are bounded by  $k$ . This also gives bounds on the ratios of  $r$  to  $s$  and of  $\cosh r$  to  $\cosh s$ . For  $t \in (0, 1]$ ,  $a_r(t)/a_s(t)$  varies between  $\cosh s/\cosh r$  and 1 and  $b_r(t)/b_s(t)$  varies between  $r/s$  and  $\sinh r/\sinh s$ . We can thus define a bilipschitz map  $[(x, t) \mapsto (f(x), t)] : \partial T \times (0, 1] \rightarrow \partial T' \times (0, 1]$  and extend over the completions.  $\square$

We note that in the case where  $k = 1$ , we can take  $k' = 1$ , giving the uniqueness part of Lemma 2.6.7.

In the cases of interest to us, the length of the longitude will be bounded above and below, and so, up to bilipschitz equivalence, we can normalise so that it has length 1. (This is really just for notational convenience.) In this case, we can identify  $\partial T$  with  $\Delta(\lambda)$  for some modulus  $\lambda \in \mathbb{C}$ . Note that  $|\lambda| = R$  and that  $\text{area} \partial T = \Im(\lambda)$ . We see that the length of the core curve is  $L(\lambda) := 2\pi \Im(\lambda)/|\lambda| \sqrt{|\lambda|^2 + 4\pi^2}$ .

We earlier defined the leaf space,  $S(\partial T)$  of  $\partial T$  by collapsing each geodesic longitude to a point. Its length is  $\Im(\lambda)$ . We can define another leaf space,  $S_0(\partial T)$  by collapsing each geodesic standard meridian to a point. It has length  $\text{area}(\partial T)/R = \Im(\lambda)/|\lambda|$ . There is a natural linear homeomorphism from  $S_0(\partial T)$  to the core curve,  $\alpha$ , given by orthogonal projection. It contracts distances by a factor  $L(\lambda)/(\Im(\lambda)/|\lambda|) = |\lambda|L(\lambda)/\Im(\lambda) = 2\pi/\sqrt{|\lambda|^2 + 4\pi^2}$ . By precomposing the inverse of this projection with the projection of  $T$  to the core curve, we see

that the projection  $\partial T \rightarrow S_0(\Delta)$  extends to a  $(\sqrt{|\lambda|^2 + 4\pi^2}/2\pi)$ -lipschitz map  $T \rightarrow S_0(\Delta)$ .

**Lemma 2.6.9.** *Suppose that  $T$  is a Margulis tube where the longitude of  $\partial T$  has length 1. Suppose that  $\partial T$  has area at most  $c$ . Let  $f : \partial T \rightarrow S(1)$  be a  $k$ -lipschitz map which has degree 1 on the longitude and degree 0 on the standard meridian. Then  $f$  extends to a  $k'$ -lipschitz map  $f : T \rightarrow S(1)$ , where  $k'$  depends only on  $k$  and  $c$ .*

*Proof.* Using Lemma 2.6.6, we can reduce to the case where  $f$  sends each geodesic meridian to a point — since the lipschitz homotopy given by Lemma 2.6.6 can be carried out in a uniformly small neighbourhood of  $\partial T$  in  $T$ .

We can thus assume we have a  $k$ -lipschitz map  $f : \partial T \rightarrow S(1)$  which factors through the projection  $\partial T \rightarrow S_0(\partial T)$ . But the latter projection extends to a  $(\sqrt{|\lambda|^2 + 4\pi^2}/2\pi)$ -lipschitz map  $T \rightarrow S_0(\partial T)$ . Composing, this gives us a  $k(\sqrt{|\lambda|^2 + 4\pi^2}/2\pi)$ -lipschitz map  $T \rightarrow S(1)$ . Finally, we note that, by Lemma 2.6.5, we have  $|\lambda| \leq (k+1)(\mathfrak{S}(\lambda) + \frac{1}{2}) \leq (k+1)(c + \frac{1}{2})$ .  $\square$

## 2.7. SYSTEMS OF CONVEX SETS.

The purpose of this section is to describe some constructions of lipschitz maps which we will apply in Section 2.8 to get a lipschitz map from the thick part of our model space into the thick part of our 3-manifold. Since the actual set-up in which we are interested is somewhat complicated to describe, we will present most of it in the fairly general setting of systems of convex sets, only adding assumptions as we need them. The main application we have in mind here is described by Lemma 2.7.6, whose explicit hypotheses are laid out before its statement. In practice all the convex sets we deal with will be either horoballs or uniform neighbourhoods of geodesics. These will be lifts of closed geodesics or Margulis regions in our 3-manifold. The domain for our map will be a locally finite polyhedral complex, which in applications, arises out of the combinatorial construction of the model space. We note that any two sensible path metrics on such a model space will be bilipschitz equivalent, and so the actual choice doesn't much matter to us. We will describe a specific metric for definiteness. Much of the argument would apply in any dimension, though for simplicity we restrict our attention to 3.

Let  $\Pi$  be a 3-dimensional simplicial complex with vertex set  $\Pi^0$ . We write  $\Pi^i$  for the set of  $i$ -simplices. We assume  $\Pi^i$  to be locally finite away from  $\Pi^0$ . We write  $|\Pi|$  for its realisation. We are really interested in a **truncated realisation** of  $\Pi$ , denoted  $R(\Pi)$ , built out of truncated simplices. We can construct a truncated simplex by taking a regular euclidean simplex of side length 3, and removing a regular simplex of side length 1 about each vertex. The resulting polyhedron has all side lengths 1. (In dimension 2, for example, we get a regular hexagon.) Gluing these together we get a locally finite polyhedral complex,  $R(\Pi)$ , which

we can view as a closed subset of  $|\Pi|$ . Associated to each  $x \in \Pi^0$ , we have a polyhedral subset  $D(x) \subseteq R(\Pi)$  — the boundary of a neighbourhood of  $x$  in  $|\Pi|$ . This is a 2-dimensional simplicial complex. (Figure 2.12.)

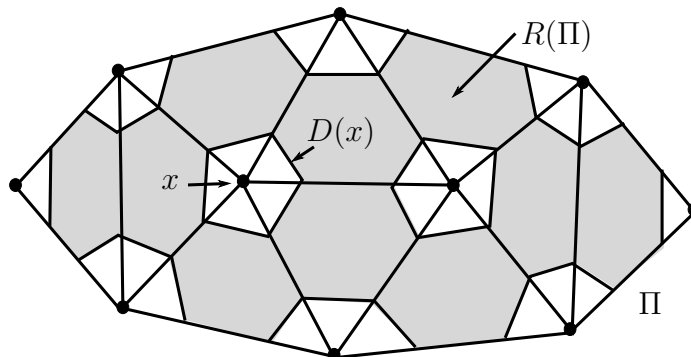


FIGURE 2.12. A schematic picture of  $R(\Pi)$  (one dimension down)

Given two convex subsets  $P, Q \subseteq \mathbb{H}^3$  we write  $\text{par}(P, Q) = \text{diam}(N(P, 1) \cap N(Q, 1))$ . (This is 0 if the intersection is empty.) We view this as a convenient measure of the extent to which  $P$  and  $Q$  remain close (or “parallel”). An upper bound in  $\text{par}(P, Q)$  means that they must diverge uniformly. (More precisely, for all  $t \geq 0$ , there is some set  $R$ , whose diameter is bounded in terms of  $t$  and  $\text{par}(P, Q)$  such that  $d(P \setminus R, Q \setminus R) \geq t$ .)

We start with a fairly simple construction that will be refined later. Suppose to each  $x \in \Pi^0$  we associate a closed convex set  $Q(x) \subseteq \mathbb{H}^3$ . We assume:

**(A1):** If  $x, y \in \Pi^0$  are distinct, then  $Q(x) \cap Q(y)$  consists of at most one point (a common boundary point), and  $\text{par}(Q(x), Q(y))$  is bounded above.

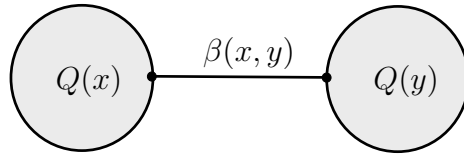
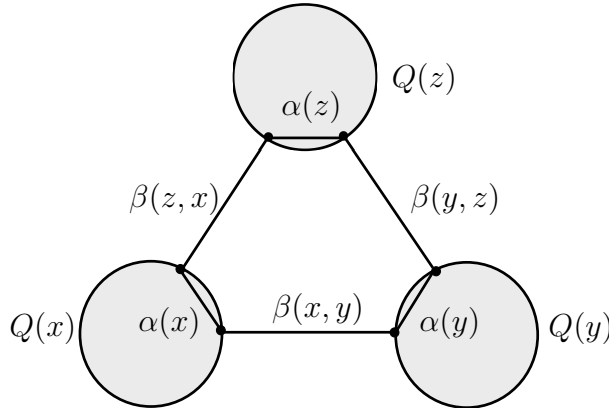
**(B1):** If  $xy \in \Pi^1$ , then  $d(Q(x), Q(y))$  is bounded above.

We will later consider other hypotheses that imply (B1). Implicit in these statements are constants which give the respective bounds. The constant of (B1) will eventually depend on the hypotheses (A2)–(A7) described later (see Lemma 2.7.7).

Given  $xy \in \Pi^1$ , let  $\beta(xy)$  be the shortest geodesic (possibly degenerate) from  $Q(x)$  to  $Q(y)$ . (Figure 2.13.) A key observation that is a simple exercise in hyperbolic geometry is the following:

**Lemma 2.7.1.** *If  $xyz \in \Pi^2$ , then  $\text{diam}(\beta(xy) \cup \beta(yz) \cup \beta(zx))$  is bounded in terms of the constant of (A1).*

*Proof.* Consider the hexagonal path  $\beta(xy) \cup \alpha(y) \cup \beta(yz) \cup \alpha(z) \cup \beta(zx) \cup \alpha(x)$  with  $\alpha(x) \subseteq Q(x)$  etc. all geodesics. (Figure 2.14.) The lengths of the  $\beta$ -paths

FIGURE 2.13. The geodesic segment  $\beta(x, y)$ FIGURE 2.14. The hexagon  $\beta(xy) \cup \alpha(y) \cup \beta(yz) \cup \alpha(z) \cup \beta(zx) \cup \alpha(x)$ 

are bounded. So, if the hexagon were very long, two of the  $\alpha$  paths would have to run close together over a long distance, contradicting (A1).  $\square$

**Lemma 2.7.2.** *With the above hypotheses ((A1) and (B1)) there is a canonical uniformly lipschitz map  $\psi : R(\Pi) \rightarrow \mathbb{H}^3$  such that  $\psi(D(x)) \subseteq Q(x)$  for all  $x \in \Pi^0$ , and such that if  $a \in Q(x) \cap \psi(R(\Pi))$  then  $d(a, \partial Q)$  is uniformly bounded above.*

Here the lipschitz constant and bound depend on the bounds assumed in (A1) and (B1).

*Proof of Lemma 2.7.2 :* For the proof, we will define the ‘‘centre’’ of a finite diameter subset,  $B \subseteq \mathbb{H}^3$ , as the unique point  $c \in \mathbb{H}^3$  such that  $B \subseteq N(c, r)$  with  $r$  minimal.

Our construction of  $\psi$  is as follows. For any  $xy \in \Pi^1$ ,  $\text{length } \beta(xy) = d(Q(x), Q(y))$  is bounded. Let  $p(x, y) = \beta(xy) \cap Q(x) \in \partial Q(x)$  be the nearest point in  $Q(x)$  to  $Q(y)$ . We map the corresponding edge of  $R(\Pi)$  linearly to  $\beta(xy)$ . By this process, we will map the vertex set of each  $D(x)$  into  $\partial Q(x)$ . We now extend  $\phi$  to the 1-skeleton of  $D(x)$  again mapping the remaining edges linearly to geodesics in  $\mathbb{H}^3$ . By Lemma 2.7.2, this is uniformly lipschitz. Finally we extend over 2-simplices of  $D(x)$ . We can do this, for example, by coning over the centre of its vertex set. In

this way, each simplex gets mapped to totally geodesic simplex in  $\mathbb{H}^3$ . The map is still uniformly lipschitz. By convexity,  $\psi(D(x)) \subseteq Q(x)$ .

Given  $xyz \in \Pi^2$ , write  $c(xyz)$  for the centre of  $\beta(xy) \cup \beta(yz) \cup \beta(zx)$ . Associated to  $xyz$ , we have a hexagonal 2-cell in  $R(\Pi)$ , and we have already defined  $\psi$  on its boundary. We now extend over the interior by sending its centre to  $c(xyz)$  and coning linearly over the boundary. (Figure 2.15.)

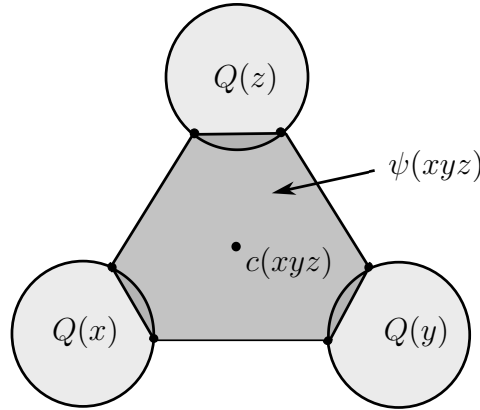


FIGURE 2.15. Extending  $\phi$  over a 2-cell

Similarly, given  $xyzw \in \Pi^3$ , we let  $c(xyzw)$  be the centre of  $\beta(xy) \cup \beta(yz) \cup \beta(zx) \cup \beta(xw) \cap \beta(yw) \cup \beta(zw)$ . We have already defined  $\psi$  on the boundary of the associated 3-cell of  $R(\Pi)$  and now cone linearly over the centre  $c(xyzw)$ .

This gives us our lipschitz map  $\psi$ , proving Lemma 2.7.2.  $\square$

Note that every point of  $\psi(R(\Pi))$  lies a bounded distance from two distinct sets  $Q(x)$ , from which it follows that this image can only boundedly enter any such convex set. Explicitly we note:

(\*) There is some constant  $K \geq 0$  depending only on the constants of (A1) and (B1) such that for all  $x \in \Pi^0$ , if  $a \in Q(x) \cap \psi(R(\Pi))$  then  $d(a, \partial Q(x)) \leq K$ .

We want to refine the above construction to push  $\psi$  off the interiors of convex sets. For this we need some additional assumptions.

Suppose that  $A \subseteq \mathbb{H}^3$  is convex, and that  $Q = N(A, t)$  for some  $t \geq 0$ . Given any  $r \in (0, t)$  write  $Q_r = N(A, t - r)$ . Thus  $Q_0 = Q$ . We can define an outward projection  $\pi : Q_r \setminus A \rightarrow \partial Q$  so that each  $a \in Q_r$  lies on the shortest geodesic from  $\pi(a)$  to  $A$ . This projection is  $(\sinh t / \sinh(t - r))$ -lipschitz.

We can refine Lemma 2.7.2 as follows. Suppose that to each  $x \in \Pi^0$  we have associated some convex set,  $A(x)$  and some  $t(x) \geq 0$ . Let  $Q(x) = N(A(x), t(x))$ . We suppose that the collection  $(Q(x))_{x \in \Pi^0}$  satisfies the assumptions (A1) and (B1). Suppose that  $t_0 \geq K + 1$ , where  $K$  is the constant of (\*) above, and suppose that  $t_1 \geq t_0$ .

Now suppose that we decompose  $\Pi^0$  into two subsets,  $\Pi_0^0 \sqcup \Pi_1^0$  satisfying:

For all  $x \in \Pi_0^0$ ,  $t(x) \geq t_0$ , and

for all  $x \in \Pi_1^0$ ,  $t(x) \leq t_1$ .

**Lemma 2.7.3.** *With the above hypotheses, we can find a canonical uniformly lipschitz map,  $\phi : R(\Pi) \rightarrow \mathbb{H}^3$  such that*

- (1) *if  $x \in \Pi_1^0$ , then  $\phi(D(x)) \subseteq A(x)$ ,*
- (2) *if  $x \in \Pi_0^0$ , then  $\phi(D(x)) \subseteq \partial Q(x)$ , and*
- (3) *if  $x \in \Pi_0^0$ , then  $Q(x) \cap \phi(R(\Pi)) \subseteq \partial Q(x)$ .*

(In our application,  $A(x)$  will be the axis of a loxodromic, or a horoball fixed by a parabolic, and  $Q(x)$  will be a Margulis region or an axis: see Lemma 2.7.6.)

*Proof.* We start with a map  $\psi : R(\Pi) \rightarrow \mathbb{H}^3$  as given by Lemma 2.7.2. If  $x \in \Pi_0^0$ , then  $Q(x) \cap \phi(R(\Pi)) \subseteq Q_K(x)$ . By composing with the outward projection  $\pi : Q_{t_0}(x) \rightarrow \partial Q(x)$  described above, we can push the image off the interior of  $Q(x)$ , while maintaining a control on the lipschitz constant.

If  $x \in \Pi_1^0$ , we have  $\phi(D(x)) \subseteq Q(x) \subseteq N(A(x), t_1)$ . We can now project  $\phi(D(x))$  to  $A(x)$  by nearest point projection, and extending by linear homotopy carried out in a uniformly small neighbourhood of  $D(x)$  in  $R(\Pi)$ . In this way we can arrange that  $\phi(D(x)) \subseteq A(x)$ , again maintaining control over the lipschitz constant.  $\square$

In applying these results, we will start from slightly different hypotheses. We suppose we have convex sets,  $(Q(x))_x$  for  $x \in \Pi^0$ , satisfying (A1), but we do not a-priori assume (B1). This we will need to deduce.

Let  $\Gamma$  be a group acting both simplicially on  $\Pi$  and isometrically on  $\mathbb{H}^3$ .

We begin by assuming:

**(A2):**  $(\forall x \in \Pi^0)(\forall g \in \Gamma)(Q(gx) = gQ(x))$ .

**(A3):** The setwise stabiliser of each element of  $\Pi^1$  and of  $\Pi^2$  is trivial.

(Note that (A3) implies that  $\Gamma$  acts freely and isometrically on  $R(\Pi)$ .)

We write  $\Gamma(x)$  for the stabiliser of  $x$  in  $\Gamma$ .

**(A4):** If  $g \in \Gamma(x) \setminus \{1\}$ , then for all  $a \in \partial Q(x)$ ,  $d(a, ga) \geq \epsilon$  for some fixed constant  $\epsilon > 0$ .

Note that an immediate consequence of this is that if  $y \in \Pi^0$  with  $xy \in \Pi^1$ , and  $g \in \Gamma(x) \setminus \{1\}$ , then the geodesic segments  $\beta(xy)$  and  $g\beta(xy)$  diverge uniformly. More precisely, given any  $t > 0$ , they can remain  $t$ -close only over a distance bounded above in terms of  $t$  and  $\epsilon$ .



We also suppose we have a  $\Gamma$ -equivariant subset  $\Pi_0^2 \subseteq \Pi^2$  satisfying:

**(A5):** If  $xyz \in \Pi_0^2$ , then there are non-trivial elements  $g(x, y, z) \in \Gamma(x)$ ,  $g(y, z, x) \in \Gamma(y)$  and  $g(z, x, y) \in \Gamma(z)$  with  $g(x, y, z)g(y, z, x)g(z, x, y) = 1$  in  $\Gamma$ .

**Lemma 2.7.4.** *We assume (A1)–(A5). If  $xyz \in \Pi_0^2$ , then the lengths of  $\beta(xy)$ ,  $\beta(yz)$  and  $\beta(zx)$  are all bounded above in terms of the constants of (A1) and (A4).*

*Proof.* By symmetry, it is enough to bound  $\beta(xy)$ . Let  $g = g(x, y, z) \in \Gamma(x)$  and  $h = g(y, z, x) \in \Gamma(y)$ , so that  $gh = g(z, x, y)^{-1} \in \Gamma(z)$ . Let  $w = g^{-1}z = hz$ . Since  $\Gamma(x) \cap \Gamma(z)$  is trivial,  $w \neq z$ .

There are geodesics segments  $\alpha(x), \alpha(y), \alpha(z), \alpha(w)$  in  $\mathbb{H}^3$ , respectively contained in  $Q(x), Q(y), Q(z), Q(w)$ , so that

$$\alpha(x) \cup \beta(xz) \cup \alpha(z) \cup \beta(zy) \cup \alpha(y) \cup \beta(yw) \cup \alpha(w) \cup \beta(wx)$$

forms a closed path — an “octagon”. (Figure 2.16.) Consecutive edges of the

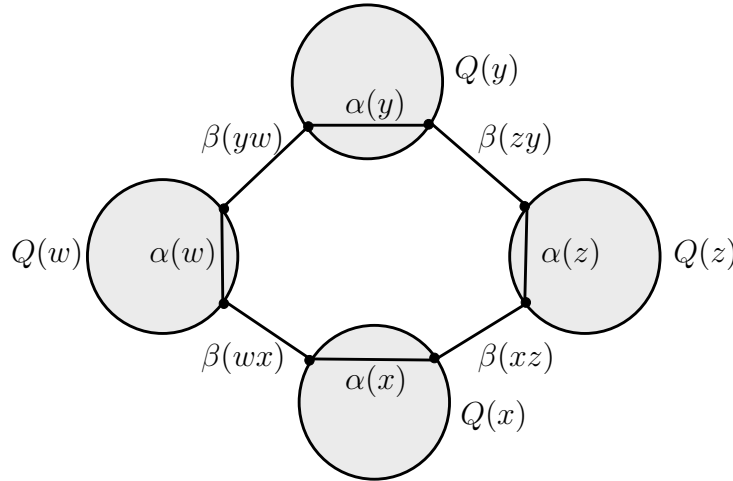


FIGURE 2.16. An octagon

octagon meet at an angle at least  $\pi/2$ , and can remain close only over a bounded distance. Moreover,  $\beta(xz)$  and  $\beta(xw)$  have the same length, and as mentioned above, (A4) implies that they can and remain close only over a bounded distance. The same also applies to  $\beta(yz)$  and  $\beta(yw)$ .

Now a simple exercise in hyperbolic geometry shows that if  $\alpha(x)$  and  $\alpha(y)$  are far apart, then  $\alpha(z)$  and  $\alpha(w)$  remain close over a large distance. (Consider the projections of  $\alpha(z)$  and  $\alpha(w)$  to  $\beta(xy)$ .) In particular, if  $\beta(xy)$  is very long, then  $\text{par}(Q(z), Q(w))$  is large, contradicting (A1).

This shows that  $\beta(xy)$  has bounded length, as claimed. □

Next we assume:

**(A6):** If  $xyz \in \Pi^2$ , at least two of the edges  $xy, yz, zx$  lie in simplices of  $\Pi_0^2$ .

**(A7):** Each edge of  $\Pi^1$  lies in at least two simplices of  $\Pi^2$ .

**Lemma 2.7.5.** *We assume (A1)–(A7). If  $xy \in \Pi^1$ , then the length of  $\beta(xy)$  is bounded above in terms of the constants of (A1) and (A4).*

*Proof.* By (A7) there are distinct  $z, w \in \Pi^0$  with  $xyz, xyw \in \Pi^2$ . Now if  $\beta(xy)$  is very long, then by Lemma 2.7.4,  $xy$  cannot lie in any simplex of  $\Pi_0^2$ . Thus, by (A7),  $xy, yz, xw, yw$  must all lie in simplices in  $\Pi_0^2$ . By Lemma 2.7.4 again, the lengths of each of  $\beta(xy), \beta(yz), \beta(xw)$  and  $\beta(yw)$  are bounded.

As in the proof of Lemma 2.7.4 we consider the octagon

$$\alpha(x) \cup \beta(xz) \cup \alpha(z) \cup \beta(zx) \cup \alpha(y) \cup \beta(yw) \cup \alpha(w) \cup \beta(wx).$$

This time, we note that all the  $\beta$ -edges have bounded length, and that  $\alpha(x)$  and  $\alpha(y)$  are far apart. Again a simple exercise in hyperbolic geometry shows that  $\alpha(z)$  and  $\alpha(w)$  remain close over a large distance, a contradicting (A1) as before.  $\square$

In other words, we have shown (A1)–(A7) imply (B1). In particular, in Lemma 2.7.3, we can substitute hypothesis (B1) with (A2)–(A7).

Since the construction was canonical, the map  $\phi$  we get is  $\Gamma$ -equivariant.

We now finally get to the specific application we have in mind.

To be clear about our hypotheses, we go back to the beginning. Let us suppose that  $\Pi$  is a simplicial complex, and let  $R(\Pi)$  and  $(D(x))_{x \in \Pi^0}$  be as constructed earlier. We assume that  $R(\Pi)$  is locally finite. Let  $\Gamma$  be a group acting simplicially on  $\Pi$ , and so we get an induced isometric action on  $R(\Pi)$ . Given  $x \in \Pi^0$ , we write  $\Gamma(x)$  for the stabiliser of  $x$  in  $\Gamma$ . Let  $\Pi_0^2$  be a  $\Gamma$ -invariant subset of  $\Pi^2$ . We now suppose:

**(C1):** Every edge of  $\Pi$  is contained in at least two simplices in  $\Pi^2$ .

**(C2):** If  $x \in \Pi^0$ , then  $\Gamma(x)$  is infinite cyclic.

**(C3):** If  $x, y \in \Pi^0$  are distinct, then  $\Gamma(x) \cap \Gamma(y)$  is trivial.

**(C4):** If  $xyz \in \Pi_0^2$ , we can choose the generators,  $g(x, y, z), g(y, z, x), g(z, x, y)$  respectively for  $\Gamma(x), \Gamma(y), \Gamma(z)$  so that  $g(x, y, z)g(y, z, x)g(z, x, y) = 1$ .

**(C5):** If  $xyz \in \Pi^2$ , then at least two of its edges,  $xy, yz, zx$ , are also edges of some element of  $\Pi_0^2$  (not necessarily the same element).

We now suppose that  $\Gamma$  also acts freely and properly discontinuously on hyperbolic 3-space,  $\mathbb{H}^3$ . Given  $x \in \Pi^0$ , we let  $l(x)$  for the infimum translation distance of  $g(x)$  on  $\mathbb{H}^3$ . We assume:

**(C6):** There is some  $L \geq 0$  such that for all  $x \in \Pi^0$  we have  $l(x) \leq L$ .

We note that this “translation bound” constant,  $L$ , is the only constant we are inputting into the proceedings. All other constants arising can be chosen dependent on that (though we will be free to exercise some choice).

Let  $\Pi_P^0 = \{x \in \Pi^0 \mid l(x) = 0\}$ . Thus if  $x \in \Pi_P^0$ , then  $g(x)$  is parabolic with fixed point in  $\partial\mathbb{H}^3$ . If  $x \in \Pi^0 \setminus \Pi_P^0$ , then  $g(x)$  is loxodromic and translates some axis,  $\alpha(x)$ , a distance  $l(x)$ .

Now fix some  $\epsilon_0$  less than the 3-dimensional Margulis constant, and sufficiently small in relation to  $L$  as we describe shortly. Suppose  $x \in \Pi^0$  with  $l(x) < \epsilon_0$ . Let  $P_0(x) \subseteq \mathbb{H}^3$  be the set of points,  $y$ , such that there exists a non-trivial  $h \in \Gamma(x)$  with  $d(y, hy) \leq \epsilon_0$ . (This of course, need not be a generator.) Thus,  $P_0(x)$  is the **Margulis region** corresponding to  $\Gamma(x)$ . (We refer to it as the “ $\epsilon_0$ -Margulis region” if we need to specify the constant.) If  $x \notin \Pi_P^0$  then  $P_0(x) = N(\alpha(x), r(x))$  for some  $r(x) > 0$ . We refer to  $r(x)$  as the **depth** of  $P_0(x)$ . If  $x \in \Pi_P^0$  then  $P_0(x)$  is a horoball centred at the fixed point. We set  $r(x) = \infty$ . The Margulis lemma tells us that distinct Margulis regions are disjoint. Indeed we can assume the distance between them to be bounded below. (For the moment we are not excluding the possibility that there may be other points of  $\mathbb{H}^3$  translated a very small distance by some element of  $\Gamma \setminus \{1\}$  outside the regions we have described.) In the above situation ( $l(x) < \epsilon_0$ ), we set  $Q(x) = P_0(x)$ . If  $l(x) \geq \epsilon_0$ , we set  $Q(x) = \alpha(x)$ .

Our choice of  $\epsilon_0$  depends on the following standard fact of hyperbolic geometry. Given any  $L > 0$ , there is some  $\epsilon(L) > 0$  such that if  $g, h$  are hyperbolic isometries generating a discrete group with  $d(x, gx) \leq L$  and  $d(x, hx) \leq \epsilon(L)$ , then  $g$  and  $h$  generate an elementary (i.e. virtually abelian) group. In our case this will be cyclic. Thus, if we choose  $\epsilon_0 \leq \epsilon(L)$ , then no axis  $\alpha(x)$  can enter any Margulis region  $P_0(y)$ . It follows that for all distinct  $x, y \in \Pi^0$ ,  $Q(x) \cap Q(y)$  is at most one point.

The following construction is most conveniently described now, even though its logical place in the argument comes a bit later. We fix some constant  $r_0$  sufficiently large (to be specified later). Let  $\Pi_0^0$  be the set of  $x \in \Pi^0$  such that  $l(x) < \epsilon_0$  and  $r(x) \geq r_0$ , and let  $\Pi_1^0 = \Pi \setminus \Pi_0^0$ . Note that  $\Pi_P^0 \subseteq \Pi_0^0$ . If  $x \in \Pi_P^0$ , let  $A(x)$  be the horoball about the fixed point such that so that  $Q(x) = N(A(x), r_0)$ . If  $x \in \Pi_0^0 \setminus \Pi_P^0$ , we let  $A(x) = N(\alpha(x), r(x) - r_0)$ , so that again  $Q(x) = N(A(x), r_0)$ . If  $x \in \Pi_1^0$ , then we set  $A(x) = \alpha(x)$ . In this case, we have  $Q(x) = N(A(x), r)$  for some  $r \leq r_0$ .

Suppose now that  $x, y \in \Pi^0$  are distinct. It now follows that  $\text{par}(Q(x), Q(y))$  is bounded above. If these are both loxodromic axes, this is a standard fact following from the upper bound on translation lengths (C6) and the discreteness of  $\Gamma$ . Otherwise it is a standard fact about the geometry of Margulis regions: they cannot remain parallel over large distance, nor can they remain parallel to any loxodromic axis of bounded translation length. We have thus verified property (A1).

Now suppose that  $xyz \in \Pi^2$ . We obtain (A5) in the same way as (C4).

We are now in a set-up applicable to Lemma 2.7.5. Note that this makes reference only to the sets  $Q(x)$ , and so our construction of the sets  $A(x)$  is irrelevant for the moment. In particular, Lemma 2.7.5 gives us an upper bound on  $d(Q(x), Q(y))$  for all  $xy \in \Pi^1$ . This ultimately depends only on  $L$ . We are now free to choose  $r_0$  so that  $r_0 - 1$  is greater than the constant given by Lemma 2.7.2. We are now in a position to apply Lemma 2.7.3 with  $t_0 = t_1 = r_0$ .

We finally note that for all  $x \in \Pi_1^0$ , we have  $l(x) > \epsilon_1$  for some  $\epsilon_1$  depending only on  $\epsilon_0$  and  $r_0$ , and thus ultimately only on  $r_0$ .

Let us summarise what we have shown:

**Lemma 2.7.6.** *Let  $\Gamma$  be a group acting on  $\Pi$  and on  $\mathbb{H}^3$  in the manner described above, in particular satisfying (C1)–(C6). Then there are positive constants,  $k, \epsilon_0, \epsilon_1$ , depending only on the translation bound of property (C6), such that we can write  $\Pi^0$  as an  $\Gamma$ -invariant disjoint union  $\Pi^0 = \Pi_0^0 \sqcup \Pi_1^0$  such that there exists an equivariant  $k$ -lipschitz map,  $\phi : R(\Pi) \rightarrow \mathbb{H}^3$  satisfying:*

(1) *If  $x \in \Pi_1^0$ , then the generator of  $\Gamma(x)$  translates an axis  $\alpha(x)$  a distance at least  $\epsilon_1$ , and  $\phi(D(x)) \subseteq \alpha(x)$ .*

(2) *If  $x \in \Pi_0^0$ , the  $\epsilon_0$ -Margulis region,  $P_0(x)$ , corresponding to  $\Gamma(x)$  is non-empty and  $\phi(D(x)) \subseteq \partial P_0(x)$ .*

(3) *For all  $x \in \Pi_0^0$ ,  $P_0(x) \cap \phi(R(\Pi)) \subseteq \partial P_0(x)$ .*

This map projects to a map,  $f : R(\Pi)/\Gamma \rightarrow \mathbb{H}^3/\Gamma$ . We write  $\tilde{\Theta} = \mathbb{H}^3 \setminus \bigcup_{x \in \Pi_0^0} \text{int } P_0(x)$ , and let  $\Theta = \tilde{\Theta}/\Gamma$ . Thus by (3), we have  $f(R(\Pi)/\Gamma) \subseteq \Theta$ .

We note:

**Lemma 2.7.7.** *There is some  $\epsilon_2 > 0$  depending only on the bound,  $L$ , of (C6), such that if  $f(R(\Pi)/\Gamma) = \Theta$ , then the systole of  $\Theta$  is at least  $\epsilon_2$ .*

*Proof.* In this case, every point of  $\Theta$  is a bounded distance from some set of the form  $\phi(D(x))/\Gamma(x)$  for some  $x \in \Pi^0$ . This is either a closed geodesic whose length is bounded below by  $\epsilon_1$  and above by  $L$ , or else the boundary of an  $\epsilon_0$ -Margulis region. This places an upper bound on the depth of any Margulis region contained in  $\Theta$  and hence a positive lower bound on the systole.  $\square$

In fact, this argument shows in general that the systole is bounded below in the image of  $f$ .

## 2.8. THE MODEL SPACE.

In this section, we give a description of the model space for a doubly degenerate manifold. Building on the results of Section 2.5, we will show how the results of Section 2.4 can be used to construct a lipschitz map into such a hyperbolic 3-manifold. For the purposes of exposition, we will mostly deal with the doubly degenerate case. We give some discussion of other cases at the end of the section.

2.8.1. *Construction of the model.*

Let  $W = \bigcup \mathcal{W}$  be a complete annulus system in  $\Psi = \Sigma \times \mathbb{R}$ . Let  $\Lambda = \Lambda(\mathcal{W})$  be the completion of  $\Psi \setminus W$ . In Section 2.5, we described the associated “brick decomposition”,  $\mathcal{D} = \mathcal{D}(\mathcal{W})$  of  $\Lambda$ . Each element  $B \in \mathcal{D}$  has the form  $\Phi \times [0, 1]$ , where  $\Phi$  is a 3HS (“type 0”) or a 4HS or 1HT (“type 1”). Suppose  $B$  is of type 1. There is a curve  $\gamma_+ \subseteq \partial_+ B$  that cuts  $\partial_+ B$  into one or two 3HS components, each the lower boundary of an adjacent type 0 brick. We have a similar curve,  $\gamma_- \subseteq \partial_- B$ . By construction, the intersection number  $\iota(\gamma_-, \gamma_+)$  is minimal (1 for a 1HT and 2 for a 4HS). Thus, if we forget about the marking (the map to  $\Sigma$ ), then the local combinatorics of  $\mathcal{D}$  is bounded.

We want to put a path-metric on  $\Lambda$  using our combinatorial structure. Since we are only interested in the metric up to bilipschitz equivalence, it doesn’t much matter how we do this, but a fairly specific procedure is as follows. We fix the unique hyperbolic metric on the 3HS so that every boundary component has length 1. This will be our *standard* 3HS. For a type 0 brick, we just take a product with the unit interval. Suppose  $B$  is a type 1 brick. We put hyperbolic structures on  $\partial_{\pm} B$  so that each component of  $\partial_{\pm} B \setminus \gamma_{\pm}$  is a standard 3HS and so that there is no “twisting” (the topological symmetries give geometric symmetries). We now choose (once and for all) our favourite path between these structures in the space of pointwise smooth riemannian metrics for which the boundary components are always geodesic of length 1. This gives us a riemannian metric on  $B = \Phi \times [0, 1]$ . (It would be natural to do this in such a way that the topological symmetries of  $B$  give geometric symmetries, though this doesn’t really matter to us.)

We can now glue all the bricks back together to give us a riemannian metric on  $\Lambda$ . Each boundary component,  $\Delta(\Omega)$  is a locally geodesic euclidean torus of the form  $\Delta(\lambda)$  with respect to the longitude and standard meridian (in the notation of Section 2.6). Note that  $\text{area}(\Delta(\lambda)) = \mathfrak{S}(\lambda)$  is the same as the “combinatorial length” of  $\Delta(W)$  as defined in Section 2.4.

Recall that  $\Upsilon = \Lambda(\mathcal{W}) \cup \bigcup_{\Omega \in \mathcal{W}} T(\Omega)$  is obtained by gluing in a solid torus,  $T(\Omega)$  to each  $\Delta(\Omega)$  so that the standard meridian is trivial in  $T(\Omega)$ . The standard meridian was defined in such a way that  $\Upsilon$  gives us back  $\Psi$  up to homeomorphism. In particular, there is a projection map  $\pi_{\Sigma} : \Upsilon \rightarrow \Sigma$ , well defined up to homotopy.

Now Lemma 2.6.7 gives us a riemannian metric on  $T(\Omega)$  isometric to a standard Margulis tube. In this way, we get a riemannian metric on all of  $\Upsilon$ . (It is comforting to observe that by Lemma 2.6.8, if we chose a different metric on  $\Lambda(\mathcal{W})$  with euclidean boundary and in the same bilipschitz class we would get a bilipschitz equivalent metric on  $\Omega$ , so the construction is quite “robust”. However, once we have constructed our model space, we don’t formally need to know this.)

Finally to arrive at our model space,  $P = P(\mathcal{W}) = \text{int } \Sigma \times \mathbb{R}$ , we glue in a Margulis cusp to each boundary component of  $\Upsilon$ . Any such boundary component is a bi-infinite cylinder,  $S(1) \times \mathbb{R}$ , and the Margulis cusp is the quotient of a

horoball by a  $\mathbb{Z}$ -action. We can regard  $\Upsilon$  as a subset of  $P$ , which we shall denote by  $\Psi(P) \cong \Psi(\mathcal{W})$  and refer to it as the *non-cuspidal* part of  $P$ .

Another way to describe the metric up to bilipschitz equivalence is given in the next subsection.

### 2.8.2. The model as a simplicial complex.

Subsection 2.8.1 tells us all we need to know about the model space to understand the statements of the main results of this section, notably Theorem 2.8.2. For proofs, however, we need more combinatorial constructions, in order to apply the results of Section 2.7. In particular, we want to view the model, up to bilipschitz equivalence, as derived from a simplicial complex satisfying properties (C1)–(C6) of Section 2.7.

We will first need to cut up  $\Lambda(\mathcal{W})$  into truncated simplices. This is done in a number of steps.

First, we replace our brick decomposition with a “block decomposition” — which is the same combinatorial structure as the “block decomposition” of Minsky [Mi4]. This is a fairly trivial adjustment which gets rid of the 3HS bricks. For each type 0 brick,  $B \equiv F \times [0, 1]$ , we take the horizontal 3HS,  $F_B = F \times \{\frac{1}{2}\} \subseteq B$ . The union of all these surfaces,  $\bigcup_B F_B$  as  $B$  ranges over all type 0 bricks, cuts  $\Lambda(\mathcal{W})$  into a collection of compact **blocks**, each of which is a type 1 brick with either two or four type 0 half bricks attached to it, depending on whether the base surface,  $\Phi$ , is a 1HT or a 4HS. (Figure 2.17.) Such a block,  $C$ , is homeomorphic to  $\Phi \times [0, 1]$ .

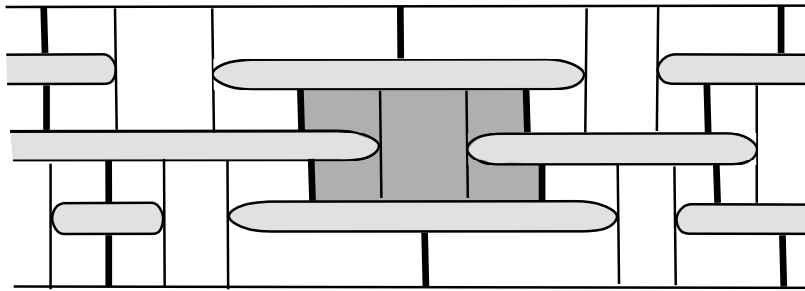


FIGURE 2.17. A decomposition of  $\Lambda$  into blocks, with one block highlighted

We write  $\partial_V C = \partial\Phi \times [0, 1]$ ,  $\partial_- C = \Phi \times \{0\}$  and  $\partial_+ C = \Phi \times \{1\}$ . We can choose the homeomorphism in such a way that  $C \cap \partial\Lambda = \partial_V C \sqcup A_- \sqcup A_+$ , where  $A_\pm$  is an annulus in  $\partial_\pm C$ , with core curves  $\gamma_\pm$ , say. These core curves correspond to components of  $W$  (meeting the original type 1 brick). By the tautness assumption on the type 1 band, these must have minimal intersection in the surface,  $\Phi$ .

Before proceeding to the second step, we make the following observation regarding a 3HS,  $F$ , which we can take to have the standard hyperbolic structure. If  $\alpha \subseteq \partial F$  is a boundary component, write  $\sigma(\alpha)$  for the shortest geodesic from

$\alpha$  to itself that separates the other two boundary components of  $F$ . If  $\beta$  is another boundary component, write  $\sigma(\alpha, \beta)$  for the shortest geodesic from  $\alpha$  to  $\beta$ . (Figure 2.18.) We can cut  $F$  into two right-angled hexagons in four different

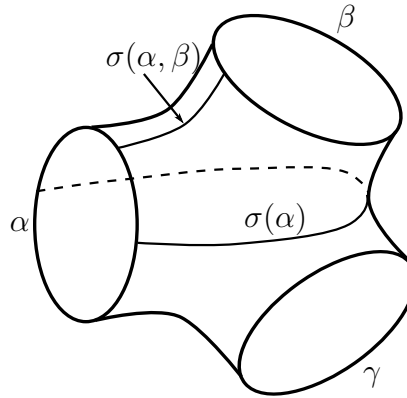


FIGURE 2.18. The arcs  $\sigma(\alpha)$  and  $\sigma(\alpha, \beta)$  on a 3HS

ways. We can cut it along  $\sigma(\alpha, \beta) \cup \sigma(\beta, \gamma) \cup \sigma(\gamma, \alpha)$ , or we can cut it along  $\sigma(\alpha) \cup \sigma(\alpha, \beta) \cup \sigma(\alpha, \gamma)$  for any boundary curve  $\alpha$ . We say these decompositions are of “type  $D_0$ ” or “type  $D_\alpha$ ” respectively.

The second step is to cut each block into truncated octahedra. The process is more conveniently described in reverse. Let  $O$  be a truncated octahedron — it has six square and eight hexagonal faces. We label the edges 1, 2, 3 so that the edges of each square face are alternately labelled 2 and 3, and the edges of each hexagonal face are either labelled alternately 1 and 2 or alternately 1 and 3. Thus all three labels appear at each vertex. Any two hexagons meet, if at all, in a 1-edge. There are four 12-hexagons and four 13-hexagons arranged alternately. (Figure 2.19.)

To describe a 4HS block, we take two copies of  $O$ , and identify the corresponding pairs of 13-hexagons. This gives us a genus-3 handlebody,  $H$ . The square faces turn into a set of six disjoint annuli embedded in  $\partial H$  (each bounded by two curves labelled 2). Each component of the complement of these annuli in  $\partial H$  is a 3HS and is cut into two hexagons by three 1-arcs. (Figure 2.20.) This decomposition is of type  $D_0$ . To identify  $H$  as a 4HS block, we select four annuli which cut  $\partial H$  into two 4HS’s, and deem them to be vertical. The other two annuli give us our non-vertical annuli. (They correspond to a pair of opposite squares in each copy of  $O$ .)

To describe a 1HT block, take one copy of  $O$  and partition the 13-hexagons into two pairs. Thus, the two hexagons in a pair meet a common square face. Now identify the two hexagons of a pair in such a way that their common adjacent square turns into an annulus. This gives us a genus-2 handlebody,  $H$ , and two annuli in  $\partial H$ . The other four square faces of  $O$  get strung together to form a

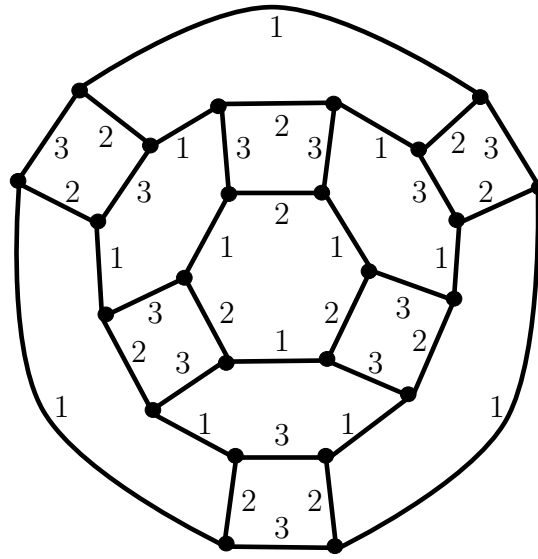


FIGURE 2.19. A truncated octahedron, with edge labels, 1, 2, 3

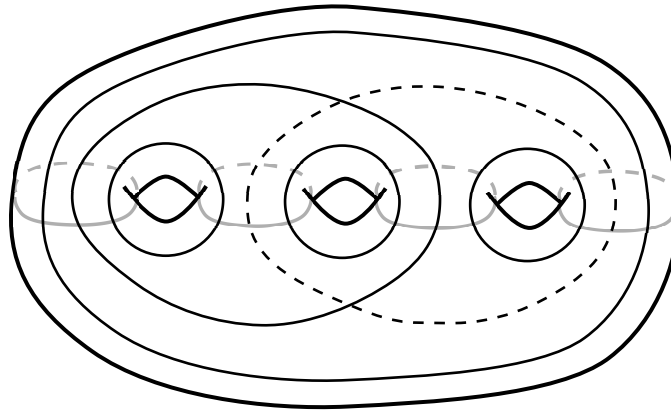


FIGURE 2.20. Pattern of curves on genus-3 handbody: Grey curves are images of 13-hexagons (as they would appear after collapsing each annulus to its core curve). Black curves are the cores of the annuli arising from squares.

third annulus, which we deem to be vertical. (Figure 2.21.) It separates  $\partial H$  into two 1HT's each containing a non-vertical annulus. This gives us a 1HT block. Each of the four 3HS components of the complement of these annuli is cut into two hexagons by three 1-arcs, as before. This time, these decompositions are of type  $D_\alpha$ , where  $\alpha$  is the boundary component in the vertical annulus.



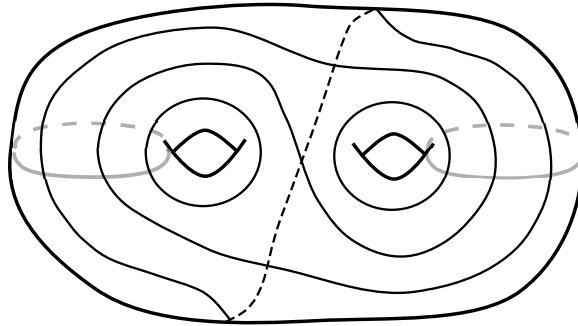


FIGURE 2.21. Pattern of curves on genus-2 handlebody: Grey curves represent images of 13-hexagons. Black curves are the cores of the annuli arising from squares.

We note that, in fact, the decomposition of a block as one or two octahedra in the manner described above is combinatorially canonical. We can thus reverse to process to cut each block of our decomposition up in this way.

The third step arises from the complication that the two decompositions of a horizontal 3HS into two hexagons (arising from the blocks on either side) might not match up. For example if a 4HS block meets a 1HT block along a horizontal 3HS  $F$ , one decomposition will be of type  $D_0$  and the other of type  $D_\alpha$ . We can fix this by replacing  $F$  by a truncated simplex. Writing  $\partial F = \alpha \cup \beta \cup \gamma$ , we can think of one pair of opposite edges of this simplex as corresponding to  $\sigma(\alpha)$  and  $\sigma(\beta, \gamma)$ . Another pair of opposite edges corresponding to  $\sigma(\alpha, \beta)$  get identified, and a third pair, corresponding to  $\sigma(\alpha, \gamma)$  also get identified. It is also possible to get two decomposition of type  $D_\alpha$  and  $D_\beta$  with  $\alpha \neq \beta$ , arising from two 1HT blocks. In this case we replace  $F$  by two truncated simplices via an intermediate  $D_0$  decomposition by applying the above construction.

This gives a polyhedral decomposition of  $\Lambda$  into truncated simplices and truncated octahedra. Since our discussion in Section 2.7 only considered truncated simplices, we should apply a fourth step. Each octahedron can be cut into four 3-simplices by connecting two opposite vertices by an edge and adding in four 2-simplices. After truncating, this cuts a truncated octahedron into four truncated simplices. There are choices involved, but the manner in which we do it is not important.

To relate this to the discussion of Section 2.7, we need to pass to covers. Let  $\Gamma = \pi_1(\Sigma)$ , and write  $\Psi(P) = \Upsilon$  for the non-cuspidal part of our model space. We have  $\Lambda(\mathcal{W}) \subseteq \Psi(P)$ , and we let  $R$  be the preimage of this in the universal cover of  $\Psi(P)$ . Thus  $\Gamma$  acts on  $R$  with quotient  $\Lambda(\mathcal{W})$ . We can lift the polyhedral decomposition of  $\Lambda(\mathcal{W})$  we just constructed to a polyhedral decomposition of  $R$ . This has the form  $R = R(\Pi)$ , where  $\Pi$  is the simplicial complex obtained by shrinking each boundary component of the 3-manifold  $R$  to a point. These points become the vertices,  $\Pi^0$ , of  $\Pi$ . The higher dimensional truncated simplices turn

into simplices of  $\Pi$ . Thus, for each  $x \in \Pi^0$ , the complex  $D(x)$  described in Section 2.7 is a boundary component of  $R(\Pi)$ . We let  $\Pi_P^0$  be the set of  $x \in \Pi^0$  such that  $D(x)$  is homeomorphic to  $\mathbb{R}^2$ . In this case,  $D(x)/\Gamma(x)$  is a bi-infinite cylinder, in fact a boundary component of  $\Psi(P)$ . If  $x \in \Pi^0 \setminus \Pi_P^0$ , then  $D(x)$  is a bi-infinite cylinder, and  $D(x)/\Gamma(x)$  is a torus of the form  $\Delta(\Omega) = \partial T(\Omega)$  for some  $\Omega \in \mathcal{W}$ . In all cases,  $\Gamma(x)$  is infinite cyclic (Property (C2)).

In Section 2.7, we defined a polyhedral metric on  $R(\Pi)$ , which also gives us a polyhedral metric on  $R(\Pi)/\Gamma = \Lambda(\mathcal{W})$ . Provided we carry out the subdivision of  $\Lambda(\mathcal{W})$  in a geometrically sensible way, this will be bilipschitz equivalent to the model metric on  $\Lambda(\mathcal{W})$  we described above. (Note that we are carrying out very explicit, locally bounded, combinatorial operations.)

We set  $\Pi_0^2$  to be the set of 2-simplices that arose from hexagons in horizontal 3HS's. In other words, these are 12-hexagons in the truncated octahedra constructed by the end of the second step, together with the hexagons introduced in the truncated simplices of the third step. (There will be other simplices arising from 13-hexagons in octahedra as well as those arising in the fourth step of the construction.) Note that every 2-simplex of  $\Pi$  has at least two edges in 2-simplices in  $\Pi_0^2$ . This is property (C5). Clearly every edge lies in a 2-simplex, and so (C1) holds.

If  $xyz \in \Pi_0^2$ , then we can choose generators,  $g(x), g(y), g(z)$ , of  $\Gamma(x), \Gamma(y), \Gamma(z)$  with  $g(x)g(y)g(z) = 1$ . This is just an observation about the boundary curves in the fundamental group of a 3HS. This is property (C4).

For property (C3), we need another assumption on  $\mathcal{W}$ , namely that no two annuli are parallel, i.e. if  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$  then  $\Omega = \Omega'$ . In this case, distinct boundary components of  $\Lambda(\mathcal{W})$  project to distinct curves in  $\Sigma$  (allowing peripheral curves for the cusp boundaries).

In summary we have shown:

**Lemma 2.8.1.** *The complex  $\Pi$  constructed above satisfies the hypotheses (C1)–(C5) of Section 2.7 (as used in Lemma 2.7.6).*

### 2.8.3. Lipschitz maps to the model space.

It remains to verify the final hypothesis, (C6). For this, we need finally to introduce group actions on  $\mathbb{H}^3$ .

Suppose that  $M = \mathbb{H}^3/\Gamma$  is a complete hyperbolic 3-manifold, with a strictly type-preserving homotopy equivalence  $\pi_\Sigma^M : M \rightarrow \Sigma$ . Now every curve  $\alpha \in \mathbf{C}(\Sigma)$  can be realised as a closed geodesic  $\alpha^* = \alpha_M^*$  in  $M$ . We will abuse notation and write  $\alpha^* \subseteq M$ , even if it is not embedded. We write  $l_M(\alpha)$  for the length of  $\alpha^*$ . Given  $r \geq 0$ , we write  $\mathbf{C}(M, r) = \{\alpha \in \mathbf{C}(\Sigma) \mid l_M(\alpha) \leq r\}$ .

Suppose that  $\mathcal{W}$  is a complete annulus system such that no two annuli are parallel. Recall the notation  $\mathbf{C}(\mathcal{W}) = \{\pi_\Sigma \Omega \mid \Omega \in \mathcal{W}\}$ . We shall make the following ‘‘a-priori bounds’’ assumption:

**(APB)**: There is some constant  $L \geq 0$ , such that  $\mathbf{C}(\mathcal{W}) \subseteq \mathbf{C}(M, L)$ .

In other words, the curves corresponding to the annuli of  $W$  have bounded length when realised in  $M$ . (This (APB) condition will be justified in the case of interest by Proposition 2.13.4.)

The constant,  $L$ , now gives us another constant,  $\epsilon_0$ , arising from Lemma 2.7.6. This is less than the Margulis constant. We write  $\Psi(M)$  for the non-cuspidal part of  $M$  with respect to this constant, in other words  $M$  minus the  $\epsilon_0$ -cusps. By tameness [Bon] this is homeomorphic to  $\Psi = \Sigma \times \mathbb{R}$ . Given  $\alpha \in \mathbf{C}(M, \epsilon_0)$  we write  $T_0(\alpha^*)$  for the  $\epsilon_0$ -Margulis tube about  $\alpha$ .

Let  $P = P(\mathcal{W})$  be the model space constructed above, and let  $\Psi(P) = \Psi(P)$  be its non-cuspidal part. The closures of components of  $P \setminus \Psi(P)$  we shall refer to as **cusps**. Given  $\Omega \in \mathcal{W}$ , write  $\Omega^* = \pi_\Sigma(\Omega)^*$  for the corresponding closed geodesic in  $M$ .

**Theorem 2.8.2.** *Let  $\mathcal{W}$  and  $M$  be as above, in particular, satisfying (APB). Let  $P = P(\mathcal{W})$  be the model space constructed above, and let  $\Psi(P) \subseteq P$  be its non-cuspidal part. Then there is a proper map  $f : P \rightarrow M$  such that  $\pi_\Sigma^M \circ f$  is homotopic to  $\pi_\Sigma$  with the following properties. Each cusp of  $P$  gets sent to an  $\epsilon_0$ -cusp of  $M$ , and  $f(\Psi(P)) \subseteq \Psi(M)$ . Moreover, we can write  $\mathcal{W} = \mathcal{W}_0 \sqcup \mathcal{W}_1$  so that if  $\Omega \in \mathcal{W}_1$ , then  $f(T(\Omega)) = \Omega^*$  and  $\Omega^*$  has length at least  $\epsilon_1 > 0$ . If  $\Omega \in \mathcal{W}_0$ , then  $\Omega^*$  has length less than  $\epsilon_0$  and  $T(\Omega) = f^{-1}T_0(\Omega^*)$ . The map  $f$  is  $k$ -lipschitz on the complement of  $\bigcup_{\Omega \in \mathcal{W}_0} \text{int } T(\Omega)$ . Here, the constants,  $\epsilon_0 \geq \epsilon_1 > 0$  and  $k \geq 0$  depend only on the constant  $L$  of the (APB) hypothesis.*

*Proof.* We first pass to the the covers corresponding to  $\Gamma = \pi_1(\Sigma)$ , and construct the polyhedral complex,  $R(\Pi)$  as above. This satisfies (C1)–(C5), and (APB) gives us (C6). Thus, Lemma 2.7.6 gives us an equivariant map  $\tilde{f} : R(\Pi) \rightarrow \mathbb{H}^3$ , which projects to a map  $f : \Lambda(\mathcal{W}) \rightarrow \Psi(M)$ . This is lipschitz with respect to the polyhedral metric on  $R(\Pi)/\Gamma = \Lambda(\mathcal{W})$ , and hence also with respect to the model metric on  $\Lambda(\mathcal{W})$ .

The  $\Gamma$ -invariant partition  $\Pi^0 \setminus \Pi_P^0 = (\Pi_0^0 \setminus \Pi_P^0) \sqcup \Pi_1^0$  gives us a partition of  $\mathcal{W}$  as  $\mathcal{W}_0 \sqcup \mathcal{W}_1$ . If  $\Omega \in \mathcal{W}_0$ , then, by construction  $f(\partial T(\Omega)) \subseteq \partial T_0(\Omega^*)$ , and no other part of  $\Lambda(\mathcal{W})$ , nor indeed the cusps, can enter  $\text{int}(T_0(\Omega^*))$ . We can now extend  $f$  topologically over  $T(\Omega)$ . By slight adjustment in a neighbourhood of the boundary torus, we can assume that  $f^{-1}(T_0(\Omega^*)) = T(\Omega)$ . (This property will not be affected by our remaining construction.)

Now suppose that  $\Omega \in \mathcal{W}_1$ . Now  $\partial T(\Omega) = \Delta(\Omega)$  is a euclidean torus of the form  $\Delta(\lambda)$  for some modulus  $\lambda$  (in the notation of Section 2.6). Moreover,  $f|_{\partial T}$  is a lipschitz map to a circle,  $\Omega^* \subseteq M$ , whose length is bounded above (by  $L$ ), and below by the constant  $\epsilon_1 > 0$  of Lemma 2.7.6. Thus, if we can place an upper bound on  $\Im(\lambda)$  we can apply Lemma 2.6.9 to extend it to a lipschitz map of  $T(\Omega)$  to  $\Omega^*$ .

To bound  $\mathfrak{S}(\lambda)$  we need again the assumption that no two annuli in  $\Omega$  are parallel. From this it is a simple matter to construct a set of  $n$  curves in  $\Lambda(\mathcal{W})$ , all a bounded distance from  $\Delta(\Omega)$  and of bounded length, which correspond to distinct elements of  $\mathbf{C}(\Sigma)$  and such that  $\mathfrak{S}(\lambda)$  is bounded above by fixed (linear) function of  $n$ . For example we can take these curves to be boundary curves of type 0 bricks meeting  $\Delta(\Omega)$  and deleting repetitions.

Now the images of these curves in  $M$  also have bounded length, and are a bounded distance from  $\Omega^*$ . Since  $\Omega^*$  has length at most  $L$ , these curves all lie in a subset of  $M$  of bounded diameter. Moreover, they are all homotopically distinct in  $\Sigma$  and hence in  $M$ . But a set of bounded diameter in any hyperbolic 3-manifold contains boundedly many distinct curves of bounded length (unless it includes multiples of a very short geodesics, which cannot arise here). Thus,  $n$  is bounded, and so is  $\mathfrak{S}(\lambda)$  as required. This allows us to extend  $f$  over all tubes, and gives us a map  $f : \Psi(P) \rightarrow \Psi(M)$ .

Extending over a cusp,  $C$ , of  $\Psi(P)$  is a fairly trivial operation. Note that we can write  $C = \partial C \times [0, \infty)$ , where  $\{x\} \times [0, \infty)$  is a (riemannian) geodesic ray in  $C$ . Now,  $f$  sends  $\partial C \subseteq \partial \Psi$  to a horosphere in  $M$  bounding a cusp. We extend  $f$  by mapping  $\{x\} \times [0, \infty)$  linearly to the ray in the cusp of  $M$ . This process does not change the lipschitz constant.

We still need to show that  $f$  is proper. First, to see that  $f|\Lambda(\mathcal{W})$  is proper, we can use a variation on the above argument. Any bounded set of  $M$  can meet only finitely many toroidal boundaries, and hence only finitely many sets of the form  $f(\Delta(\Omega))$  for  $\Omega \in \mathcal{W}$ . Since every point of  $\Lambda(\mathcal{W})$  is a bounded distance from some  $\Delta(\Omega)$  the properness of  $f|\Lambda(\mathcal{W})$  follows. The fact that  $f$  is proper on all of  $P(\mathcal{W})$  now follows easily from the manner in which we have extended over tubes and cusps.  $\square$

Now let  $\Theta(M) = \Psi(M) \setminus \bigcup_{\Omega \in \mathcal{W}_0} \text{int } T(\Omega^*)$  and  $\Theta(P) = \Theta_M(P) = \Lambda(\mathcal{W}, \mathcal{W}_1) = \Psi(\mathcal{W}) \setminus \bigcup_{\Omega \in \mathcal{W}_0} \text{int } T(\Omega)$ . Thus,  $\Theta(P) \subseteq f^{-1}(\Theta(M))$ . Note that the definition of  $\Theta(P)$  uses the partition of  $\mathcal{W}$  as  $\mathcal{W}_0 \sqcup \mathcal{W}_1$  coming from Theorem 2.8.2, and so (at least a-priori) may depend on  $M$ . (Since it did not feature in the construction of  $P$  or  $f$ , this will not matter.)

Now the map  $f : \Psi(P) \rightarrow \Psi(M)$  is proper, and both spaces are homeomorphic to  $\Psi = \Sigma \times \mathbb{R}$ . It thus sends each end of  $\Psi(P)$  to an end of  $\Psi(M)$ . We make the following ‘‘end consistency’’ assumption:

(EC) Distinct ends of  $\Psi(P)$  get sent to distinct ends of  $\Psi(M)$ .

Note that, in this case  $f|\Psi(P)$  has degree 1 to  $\Psi(M)$ , and in particular is surjective. It also follows that  $f : P \rightarrow M$  has degree 1 and is surjective. In this case, the manifold  $M$  must be doubly degenerate.

**Lemma 2.8.3.** *If (EC) is satisfied, then the set of Margulis tubes  $T_0(\Omega^*)$  for  $\Omega \in \mathcal{W}_0$  is unlinked in  $\Psi(M)$ .*

*Proof.* The preimage of this set under  $f$  is a set of Margulis tubes in  $\Psi(P)$ , which is unlinked by construction. The statement now follows from Proposition 2.2.1.  $\square$

**Proposition 2.8.4.** *If (EC) is satisfied, then the injectivity radius of  $\Theta(M)$  is bounded below by some constant  $\epsilon_2$  depending only in  $L$ .*

*Proof.* This is an immediate consequence of Lemma 2.7.7.  $\square$

**Proposition 2.8.5.** *The map  $f|\Theta(P) : \Theta(P) \rightarrow \Theta(M)$  is homotopic to a homeomorphism.*

*Proof.* By Lemma 2.8.3, or by the result of Otal [Ot3], the set of Margulis tubes  $T_0(\Omega^*)$  for  $\Omega \in \mathcal{W}_0$  is unlinked in  $\Psi(M)$ . We can thus apply the results of Section 2.3, in particular, Proposition 2.3.2.  $\square$

Thus, applying Corollary 2.3.9, we get:

**Lemma 2.8.6.** *Suppose  $F$  is a properly embedded  $\pi_1$ -injective surface in  $\Theta(P)$ . Let  $U$  be any neighbourhood of  $f(F)$  in  $\Theta(M)$ . Then there is a proper embedding  $g : F \rightarrow U$  such that  $f|F$  is homotopic to  $g$  in  $\Theta(M)$  relative to  $\partial M$ .*

We shall write

$$\mathcal{T}(P) = \{T(\Omega) \mid \Omega \in \mathcal{W}_0\}$$

and write

$$\mathcal{T}(M) = \{f(T) \mid T \in \mathcal{T}(P)\}$$

for the corresponding set of Margulis tubes in  $M$ . Thus  $\Theta(P) = \Psi(P) \setminus \text{int} \bigcup \mathcal{T}(P)$  and  $\Theta(M) = \Psi(M) \setminus \text{int} \bigcup \mathcal{T}(M)$ .

#### 2.8.4. Singly degenerate groups.

Before leaving this section, we describe how the constructions can be adapted to finite or semi-infinite model spaces. This will be used in Section 2.10 to prove some of the statements of Section 2.1 (though they are incidental to the proof of the Ending Lamination Theorem). It is also relevant to the construction of the model in the general case (Chapter 3).

We start with  $\Sigma \times I$ , where  $I \subseteq \mathbb{R}$  is any interval. Let  $W = \bigcup \mathcal{W} \subseteq \Sigma \times I$  be an annulus system. We assume that  $\partial_V \Omega \neq I$  for all  $\Omega \in \mathcal{W}$  (i.e. no annulus crosses  $\Sigma \times I$ ). Let  $\mathcal{W}_\partial = \{\Omega \in \mathcal{W} \mid \partial_V(\Omega) \cap \partial I \neq \emptyset\}$  be the set of “boundary annuli” in  $W$ . Let  $\Lambda(\mathcal{W})$  be the space obtained from  $\Sigma \times I$  by opening up each annulus as before, and let  $p : \Lambda(\mathcal{W}) \rightarrow \Sigma \times I$  be the natural quotient map. Given  $\Omega \in \mathcal{W}$  let  $\Delta(\Omega) = p^{-1}(\Omega) \subseteq \partial \Lambda(\mathcal{W})$ . If  $\Omega \in \mathcal{W} \setminus \mathcal{W}_\partial$ , then  $\Delta(\Omega)$  is a torus, and we glue in a solid torus as before. If  $\Omega \in \mathcal{W}_\partial$ , then  $\Delta(\Omega)$  is an annulus, and we leave it alone. This gives us a model space,  $\Psi(P) := \Psi(P(\mathcal{W}))$ . (We have no need to construct “ $P$ ” at this point, but use this notation in order to

maintain consistency.) Note that  $\Psi(P)$  is homeomorphic to  $\Sigma \times I$ , and we can write  $\partial\Psi(P) = \partial_V\Psi(P) \cup \partial_H\Psi(P)$ , where the vertical and horizontal boundaries correspond to  $\partial\Sigma \times I$  and  $\Sigma \times \partial I$  respectively. The annuli  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}_\partial$  give us a disjoint collection of annuli in  $\partial_H\Psi(P)$ . As before,  $\Psi(P)$  has a riemannian metric,  $d = d_P$ .

Suppose that  $M$  is a complete hyperbolic 3-manifold with  $\Psi(M) \cong \Sigma \times \mathbb{R}$ . Suppose that  $M$  satisfies the a-priori bound condition (APB) with constant  $L$ . In other words,  $\text{length}(\Omega^*) \leq L$  for all  $\Omega \in \mathcal{W}$ . Thus, as before, we construct a lipschitz homotopy equivalence,  $f : \Psi(P) \rightarrow \Psi(M)$ , with  $f^{-1}(\Psi(M)) = \partial_V\Psi(P)$ . Each margulis tube of  $\Psi(P)$ , as well as each boundary annulus  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}_\partial$ , gets sent either to a Margulis tube in  $\Psi(M)$ , or to a closed geodesic (of length at most  $L$ ) in  $\Psi(M)$ .

As an example, suppose that  $\alpha, \beta \in \mathbf{C}(M, l)$  and that  $d_G(\alpha, \beta) \geq 3$  (i.e.,  $\alpha \cup \beta$  fills  $\Sigma$ , and the corresponding geodesics,  $\alpha_M^*$  and  $\beta_M^*$  in  $M$  have length at most  $L$ ). Applying Lemma 2.4.1, we can construct an annulus system  $W = \bigcup \mathcal{W}$  in  $\Sigma \times [0, 1]$ , with  $\alpha$  and  $\beta$  homotopic to boundary annuli in the horizontal boundary components. All the curves in this construction lie in a hierarchy built out of tight geodesics. As a result, it satisfies the (APB) condition, where the bound,  $L$ , depends only in  $\xi(\Sigma)$  and  $l$ . (See Theorem 1.6.11.) We thus get a lipschitz homotopy equivalence,  $f : \Psi(P) \rightarrow \Psi(M)$ , where the lipschitz constant depends only on  $\xi(\Sigma)$  and on  $l$ . This will be used at the end of Section 2.10, to give a proof of Theorem 2.1.4.

## 2.9. BOUNDED GEOMETRY.

In this section, we describe some properties of “bounded geometry” manifolds. In Sections 2.10 and 2.12, these will be applied to the “thick parts”,  $\Theta(P)$  and  $\Theta(M)$  of our model space and hyperbolic 3-manifolds respectively. Much of the discussion is quite general. The arguments here introduce non-computable constants into the proceedings. It would be interesting to find ways to render these arguments more effective.

We remark that another way of viewing bounded geometry is in terms of triangulations. This will be used in Chapter 4, but is not needed here.

Let  $\Theta$  be a riemannian  $n$ -manifold (for us  $n = 3$ ) with boundary  $\partial\Theta$ . Here, for simplicity, we shall assume that everything is smooth.

Write  $B(\mathbb{R}^n) = \{x \mid \|x\| \leq 1\}$  for the unit ball in  $\mathbb{R}^n$ , and write  $B^+(\mathbb{R}^n) = B(\mathbb{R}^n) \cap \{x \mid x_n \geq 0\}$  for the unit half-ball.

**Definition.** We say that  $\Theta$  has *bounded geometry* if there is some  $\mu > 0$  such that every  $x \in \Theta$  has a neighbourhood  $N \ni x$ , with a smooth  $\mu$ -bilipschitz homeomorphism to either  $B(\mathbb{R}^n)$  or  $B^+(\mathbb{R}^n)$  taking  $x$  to a point a distance at most  $\frac{1}{2}$  from the origin.

(If we allow ourselves to modify  $\mu$ , we can equivalently replace  $\frac{1}{2}$  by any constant strictly between 0 and 1.)

One can draw a few immediate conclusions. The neighbourhood  $N$  contains and is contained in a ball of uniform positive radius about  $x$ . In particular, the systole,  $\text{sys}(\Theta)$ , is bounded below by some positive constant. We fix some positive constant,  $\eta_0 = \eta_0(\theta) < \text{sys}(M)$  depending only on  $\mu$ . If  $x, y \in \Theta$  with  $d(x, y) \leq \eta_0$ , we write  $[x, y]$  for the unique geodesic between them.

There are increasing functions,  $V_{\pm} : [0, \eta_0) \rightarrow [0, \infty)$  with  $V_{\pm}(0) = 0$  such that for all  $x \in \Theta$  and  $r \in [0, \infty)$ , we have  $V_-(r) \leq \text{vol}(N(x, r)) \leq V_+(r)$ .

**Definition.** Given  $\epsilon > 0$ , a subset  $P \subseteq \Theta$  is  $\epsilon$ -separated if  $d(x, y) > \epsilon$  for all distinct  $x, y \in P$ .

We see that  $|P| \leq \text{vol}(N(P, \epsilon/2))/V_-(\epsilon/2)$ . If  $\text{diam}(P) \leq r$ , then  $|P| \leq V_+(r + \epsilon/2)/V_-(\epsilon/2)$ .

**Definition.** If  $P \subseteq Q \subseteq M$ , we say that  $P$  is  $\epsilon$ -**dense** in  $Q$  if  $Q \subseteq N(P, \epsilon)$ .

**Definition.**  $P$  is an  $\epsilon$ -**net** in  $Q$  if it is  $(\epsilon/2)$ -separated and  $\epsilon$ -dense.

(This is all with respect to the metric  $d_{\Theta}$ .)

Note that any maximal  $(\epsilon/2)$ -separated subset of  $Q$  is an  $\epsilon$ -net in  $Q$ . We shall be taking  $\epsilon < \eta_0$ . The cardinality of any  $\epsilon$ -net is thus bounded above in terms of  $\epsilon$  and  $\text{vol}(N(Q, \eta_0))$ . We will use the following technical lemma in Section 2.12.

**Lemma 2.9.1.** *Suppose  $\Theta, \Theta'$  are bounded geometry manifolds, and  $f : \Theta \rightarrow \Theta'$  is  $\lambda$ -lipschitz. If  $Q \subseteq \Theta$ , then  $\text{vol}(N(f(Q), \eta_0))$  is bounded above in terms of  $\text{vol}(N(Q, \eta_0))$ ,  $\lambda$  and the bounded-geometry constants.*

(Here we are taking the same constant  $\eta_0$  for  $\Theta$  and  $\Theta'$ .)

*Proof.* Let  $P \subseteq Q$  be an  $\eta_0$ -net. Thus  $|P|$  is bounded in terms of  $\text{vol}(N(Q, \eta_0))$ . Now  $f(Q) \subseteq N(f(P), \lambda\eta_0)$ , so  $N(Q, \eta_0)$  lies inside  $N(f(P), \lambda\eta_0 + \eta_0)$  whose volume is bounded above in terms of  $|f(P)| \leq |P|$  and  $\lambda, \eta_0$ .  $\square$

The following “nerve” construction will serve as a substitute for certain “geometric limit” arguments given elsewhere (for example, in [BroCCM]). (Indeed it is the basis of many precompactness results in bounded geometry, cf. [Gr2].)

Given  $P \subseteq \Theta$  and  $\epsilon > 0$ , let  $K = K_{\epsilon}(P)$  be the simplicial 2-complex with vertex set  $V(K) = P$  and with  $A \subseteq P$  deemed to be a simplex in  $K$  if  $\text{diam}(A) \leq 3\epsilon$ . For us,  $P$  will be discrete, and so  $K$  will be locally finite. We write  $K^1$  for its 1-skeleton. If  $\epsilon < \eta_0$ , then the inclusion of  $P$  into  $M$  extends to a map  $\theta : K \rightarrow \Theta$ . This can be taken to send each edge of  $K$  linearly to a geodesic segment. We then extend over each 2-simplex by coning over a vertex. (The latter construction may entail putting some order on the vertices, and so may not be canonical.) We easily verify that  $\theta(K) \subseteq N(P, 3\epsilon)$ .

**Definition.** We say that two paths  $\alpha$  and  $\beta$  in  $\Theta$  are  $\eta$ -**close** if we can parameterise them so that  $d(\alpha(t), \beta(t)) \leq \eta$  for all parameter values,  $t$ .

Note that if  $\alpha$  and  $\beta$  have the same endpoints, and  $\eta \leq \eta_0$ , then this implies that  $\alpha$  and  $\beta$  are homotopic relative to their endpoints.

Suppose now that  $Q \subseteq \Theta$  and that  $P$  is  $\epsilon$ -dense in  $Q$ , with  $3\epsilon \leq \eta_0$ . If  $\alpha$  is a path in  $Q$  with endpoints in  $P$ , then we can find a path  $\bar{\alpha}$  in  $K^1$  with the same endpoints of combinatorial length at most  $3((\text{length}(\alpha)/\epsilon) + 1)$ , and such that  $\theta \circ \bar{\alpha}$  is  $3\epsilon$ -close to  $\alpha$ . In particular, if  $Q$  is connected, then so is  $K$ , and the image of  $\pi_1(Q)$  in  $\pi_1(M)$  is contained in  $\theta_*(\pi_1(K))$ .

If  $\pi_1(Q)$  injects into  $\pi_1(M)$ , it would be nice to say that  $\pi_1(Q)$  were isomorphic to  $\pi_1(K)$ , but this is complicated by the fact that  $Q$  may have wiggly boundary. To help us cope with this problem, we make the following definition:

**Definition.** We say that  $Q$  is  *$r$ -convex* if given any  $x, y \in Q$  with  $d(x, y) \leq \eta_0$ , there is some arc  $\alpha$  in  $Q$  from  $x$  to  $y$  so that  $\alpha \cup [x, y]$  bounds a (singular) disc of diameter at most  $r$  in  $\Theta$ .

Note that an immediate consequence is that if  $P \subseteq Q$  is  $\epsilon$ -dense for some  $\epsilon \leq \eta_0$ , then if  $\beta$  is any path in  $K^1$  then  $\theta \circ \beta$  is homotopic (in  $\Theta$ ) relative to its endpoints into  $Q$ . In particular, given our previous observation, we see that the image of  $\pi_1(Q)$  in  $\pi_1(M)$  must equal  $\theta_*(\pi_1(K))$ . The problem remains that  $\pi_1(K)$  may have lots of non-trivial loops “near the boundary”.

Suppose then that  $Q$  is  $r$ -convex, and let  $Q' = N(Q, r)$ . Let  $P \subseteq Q$  be an  $\epsilon$ -net in  $P$  and extend to an  $\epsilon$ -net  $P' \subseteq Q'$ . We thus have an inclusion of  $K$  in  $K'$ . Write  $\Gamma(K, K')$  for the image of  $\pi_1(K)$  in  $\pi_1(K')$ . Note that  $\theta$  induces a natural map of  $\Gamma(K, K')$  into  $\pi_1(M)$ .

**Lemma 2.9.2.** *Suppose that  $Q$  is  $r$ -convex and suppose that  $\pi_1(Q)$  injects into  $\pi_1(\Theta)$ . Let  $K$  and  $K'$  be as constructed above. Then the natural map of  $\Gamma(K, K')$  into  $\pi_1(\Theta)$  is injective, and its image equals the image of  $\pi_1(Q)$ . (In particular,  $\Gamma(K, K')$  is isomorphic to  $\pi_1(Q)$ .)*

*Proof.* We have already observed that the image of  $\pi_1(K)$  and hence of  $\Gamma(K, K')$  in  $\pi_1(\Theta)$  equals the image of  $\pi_1(Q)$ . We thus need to show that the map of  $\Gamma(K, K')$  into  $\pi_1(\Theta)$  is injective. If  $\beta$  is any closed curve in  $K^1$ , then  $\theta \circ \beta$  consists of a sequence of geodesics arcs of length at most  $3\epsilon \leq \eta_0$  connecting points of  $P \subseteq Q$ . By  $r$ -convexity,  $\theta \circ \beta$  can be homotoped into  $Q$  inside  $Q' = N(Q, r)$ . If  $\theta \circ \beta$  is trivial in  $\pi_1(\Theta)$ , then the homotoped curve is also trivial in  $\pi_1(Q)$ . It thus follows that  $\theta \circ \beta$  bounds a disc in  $Q'$ . We can now pull back this disc to  $K'$  showing that  $\beta$  is trivial in  $\pi_1(K')$ , and hence in  $\Gamma(K, K')$  as required.  $\square$

As an application, we have the following lemma (to be used in Section 2.10). For the purposes of this lemma, we can define a “band” in a 3-manifold,  $\Theta$ , to be a closed subset,  $B \subseteq \Theta$ , homeomorphic to  $\Sigma \times [0, 1]$ , where  $\Sigma$  is compact surface such that  $B \cap \partial\Theta = \partial_V B$ , where  $\partial_V B = \partial\Sigma \times [0, 1]$  is the “vertical boundary”. Note that the relative boundary of  $B$  in  $\Theta$  is the “horizontal boundary”  $\partial_H B = \Sigma \times \{0, 1\}$ . We regard the decomposition  $\partial B = \partial_H B \cup \partial_V B$  as part of the structure of the band.



**Lemma 2.9.3.** *Let  $\Theta$  be a bounded geometry 3-manifold, with  $\eta_0 \leq \frac{1}{2} \text{inj}(\Theta)$  as before. Suppose that  $B \subseteq \Theta$  is a band with  $\pi_1(B)$  injecting into  $\pi_1(\Theta)$  and that  $B$  is  $r$ -convex. Suppose there is a constant  $s \geq 0$  such that each component of  $\partial_V B$  is homotopic to a curve of length at most  $s$ . Suppose that  $\alpha_B, \beta_B$  are curves in  $B$  of length at most  $t$  for some other constant  $t \geq 0$ , and such that  $\alpha$  and  $\beta$  are homotopic in  $B \cong \Sigma \times [0, 1]$  to curves  $\alpha, \beta$  in  $\mathbf{C}(\Sigma)$ . Then the intersection number,  $\iota(\alpha, \beta)$ , of  $\alpha$  and  $\beta$  in  $\Sigma$  is bounded above in terms of  $r, s, t, \text{diam}(B)$  the constant of bounded geometry (including  $\eta_0$ ), and the complexity,  $\xi(\Sigma)$ , of the surface  $\Sigma$ .*

Note that the intersection number is independent of the choice of homeomorphism of  $B$  with  $\Sigma \times [0, 1]$ , respecting the vertical and horizontal boundaries. The bound also places a bound on the distance,  $d_{\mathbf{C}}([\alpha], [\beta])$ , between  $\alpha$  and  $\beta$  in the curve graph (which is what we are really interested in).

The proof relies on the observation that the intersection number of two curves is a function of the pair of conjugacy classes in  $\pi_1(\Sigma)$  representing their free homotopy class. (We need not explicitly describe what this function is, though of course this is in principle computable.) In the case where  $\pi_1(\Sigma)$  has boundary, we need also to take into account the peripheral structure — the set of conjugacy classes of boundary curves.

In practice the “short” peripheral curves in  $B$  will just be core curves of the corresponding annuli.

*Proof.* Fix some  $\epsilon \leq \eta_0/6$ . Let  $P \subseteq B$  be an  $\epsilon$ -net and extend to an  $\epsilon$ -net  $P'$  of  $B' = N(B, r)$ , and construct  $K = K_\epsilon(P)$  and  $K' = K_\epsilon(P')$  as above. Note that the diameter of  $B'$  is bounded, and so  $|V(K')| = |P'|$  is bounded. By Lemma 2.9.2, there is a natural isomorphism of  $\pi_1(B) \cong \pi_1(\Sigma)$  with  $\Gamma(K, K')$ . Note that  $\alpha_B$  and  $\beta_B$  correspond to curves,  $\bar{\alpha}$  and  $\bar{\beta}$  of bounded length in the 1-skeleton of  $K$ . If  $\Omega_1, \dots, \Omega_n$  is the (possibly empty) set of vertical boundary components, then each  $\Omega_i$  is homotopic to a curve,  $\gamma_i$  in  $B$  of bounded length, and thus corresponds to some bounded length curve,  $\bar{\gamma}_i$ , in the 1-skeleton of  $K$ . We see that there are boundedly many combinatorial possibilities for  $K, K', \bar{\alpha}, \bar{\beta}, (\bar{\gamma}_i)_i$ . Among all such possibilities for which  $\Gamma(K, K')$  is isomorphic to  $\pi_1(\Sigma)$  with the  $\gamma_i$  peripheral, there is a maximal intersection number of  $\alpha$  and  $\beta$  which will serve as our bound.  $\square$

We remark that this argument does not give us a computable bound, since it involves sifting out those pairs,  $K, K'$  for which  $\Gamma(K, K')$  is a surface group, and this is not algorithmically testable. In principle, the above argument could be translated into a “geometric limit” argument.

Here is another application of this construction, to be used in Section 2.12 (see Proposition 2.12.9). (See Figure 2.22.)

**Lemma 2.9.4.** *Suppose that  $f : \Theta \rightarrow \Theta'$  is a surjective lipschitz homotopy equivalence between two bounded geometry manifolds  $\Theta$  and  $\Theta'$ . Suppose there is some positive  $\epsilon < \eta_0$  such that if  $x, y \in \Theta$  with  $d'(f(x), f(y)) \leq \epsilon$ , then there is*

a path  $\alpha$  from  $x$  to  $y$  in  $\Theta$  with  $\text{diam}(\alpha)$  bounded such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc of bounded diameter in  $\Theta'$ . Then  $f$  is universally sesquilipschitz.

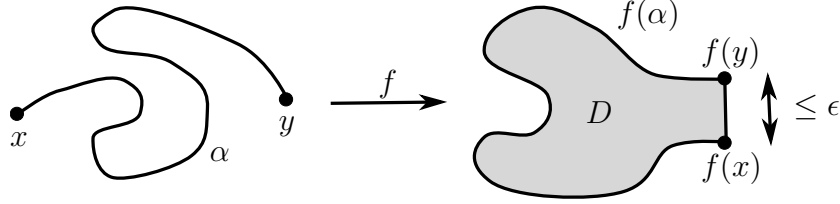


FIGURE 2.22. Hypotheses of Lemma 2.9.4

In other words, the lift,  $\tilde{f} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$  is a quasi-isometry. The constants of quasi-isometry depend only on the constants of the hypotheses (though, again, we do not show this dependence to be computable). Of course, the fact that  $f$  itself is sesquilipschitz (i.e. a quasi-isometry) is an immediate consequence of the hypotheses.

Note that, since  $f$  is a homotopy equivalence, the conclusion means that  $\alpha$  must lie in a particular homotopy class relative to its endpoints  $x$  and  $y$ . We refer to this as the “right homotopy class”. The hypotheses tell us that, in particular, there is path of bounded diameter in the right class, and we need to find one of bounded length.

*Proof.* For the purposes of the proof (rescaling the metric on  $\Theta$  or  $\Theta'$  if necessary, and modifying the bounded geometry constants) we can assume, for notational convenience, that  $f$  is 1-lipschitz. We can also take the same constant  $\eta_0$  for both  $\Theta$  and  $\Theta'$ . Fix some  $\epsilon \leq \eta_0/6$ .

Let  $R$  be the bound on the diameter of  $\alpha$  in  $\Theta$  and let  $R'$  be the bound on the diameter of the disc in  $\Theta'$ . We assume  $R \leq R'$ . Let  $Q = N(x, R) \subseteq \Theta$  and let  $Q' = N(f(x), R' + 6\epsilon) \subseteq \Theta'$ . Let  $P_0$  and  $P'_0$  be  $\epsilon$ -nets in  $Q$  and  $Q'$  respectively. Let  $P = P_0 \cup \{x, y\}$ , and let  $P' = P'_0 \cup f(P) \subseteq Q'$ . Thus,  $P$  and  $P'$  are  $\epsilon$ -dense in  $Q$  and  $Q'$  respectively. Note that since the diameters of  $Q$  and  $Q'$  are bounded,  $|P_0|$  and  $|P'_0|$  and hence  $|P|$  and  $|P'|$  are bounded.

Let  $K = K_\epsilon(P)$  and  $K' = K_\epsilon(P')$ , and write  $\theta : K \rightarrow \Theta$  and  $\theta' : K' \rightarrow \Theta'$  for the corresponding maps as constructed earlier. Note that the map  $f : V(K) \rightarrow V(K')$  extends to a simplicial map  $g : K \rightarrow K'$ . We see that  $f \circ \theta$  and  $\theta' \circ g$  are  $3\epsilon$ -close on the 1-skeleton of  $K$ .

Let  $\alpha$  be the path connecting  $x$  to  $y$  as given by the hypotheses. Now  $\alpha \subseteq Q$ , so there is a path  $\bar{\alpha}$  from  $x$  to  $y$  in  $K$  such that  $\theta \circ \bar{\alpha}$  is  $3\epsilon$ -close to  $\alpha$ . Thus  $f \circ \alpha$  is  $3\epsilon$ -close to  $f \circ \theta \circ \bar{\alpha}$  and hence  $6\epsilon$ -close to  $\theta' \circ g \circ \bar{\alpha}$ . Now  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc  $D \subseteq N(f(x), R')$ , and so  $\theta' \circ g \circ \bar{\alpha} \cup [f(x), f(y)]$  bound a disc in  $Q' = N(f(x), R' + 6\epsilon)$ . This pulls back to a disc in  $K'$  bounding  $(g \circ \bar{\alpha}) \cup e$ , where  $e$  is the edge connecting  $f(x)$  to  $f(y)$  in  $K'$ .

In summary, we have two simplicial 2-complexes,  $K, K'$  with  $|V(K)|$  and  $|V(K')|$  bounded, a simplicial map  $g : K \rightarrow K'$  and vertices  $x, y \in V(K)$ , with the property that  $f(x)$  and  $f(y)$  are connected by some edge  $e$  in  $K'$ , and such  $x$  and  $y$  are connected by some path whose image under  $f$  together with  $e$  bounds a disc in  $K'$ . Now there are boundedly many combinatorial possibilities for  $K, K', g, x, y$ . For each such  $K, K', g, x, y$ , we choose some path, say  $\beta$  from  $x$  to  $y$ , so that  $(g \circ \beta) \cup e$  bounds a disc. Since there are only finitely many cases, there is some upper bound for the length of any such  $\beta$  depending only on the bounds on  $|V(K)|$  and  $|V(K')|$ . Let  $\gamma = \theta \circ \beta$ . This has bounded length in  $\Theta$ , and since  $(g \circ \beta) \cup e$  bounds a disc in  $K$ , we see that  $(\theta' \circ g \circ \beta) \cup [f(x), f(y)]$  bounds a disc in  $\Theta'$  (of bounded diameter). Now  $\theta' \circ g \circ \beta$  is  $3\epsilon$ -close to  $f \circ \theta \circ \beta = f \circ \gamma$ , so  $(f \circ \gamma) \cup [f(x), f(y)]$  bounds a disc in  $\Theta'$ .

In other words, we can find a path from  $x$  to  $y$  of bounded length in  $\Theta$  in the right homotopy class (since  $f$  is a homotopy equivalence). The rest of the argument is now fairly standard as follows.

Reinterpreting in terms of universal covers, we have a lift  $\tilde{f} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$ . If  $a, b \in \tilde{\Theta}$  with  $d(\tilde{f}(a), \tilde{f}(b)) \leq \epsilon$ , then  $d(a, b)$  is bounded by some constant, say  $k$ . Given any  $a, b \in \Theta$ , we can connect  $a$  to  $b$  by a geodesic  $\sigma$  in  $\Theta'$  from  $\tilde{f}(a)$  to  $\tilde{f}(b)$ . Choose points  $\tilde{f}(a) = c_0, c_1, \dots, c_n = \tilde{f}(b)$  along  $\sigma$  so that  $d'(c_i, c_{i+1}) \leq \epsilon$  for all  $i$ , and so that  $n \leq d'(\tilde{f}(a), \tilde{f}(b))/\epsilon + 1$ . Since  $f$  is assumed surjective,  $\tilde{f}$  is also surjective, so we can find points  $a = a_0, a_1, \dots, a_n = b$  in  $\Theta$  with  $\tilde{f}(a_i) = c_i$ . Thus,  $d(a_i, a_{i+1}) \leq k$  for all  $i$ , so that  $d(a, b)$  is bounded above by a linear function of  $d'(\tilde{f}(a), \tilde{f}(b))$ .

Since  $f$  is lipschitz, so is  $\tilde{f}$ , and so  $\tilde{f}$  is a quasi-isometry as required.  $\square$

We finish this section with the observation that if a group  $\Gamma$  acts freely properly discontinuously on a riemannian manifold  $\Theta$ , then  $\Theta/\Gamma$  has bounded geometry if and only if  $\Theta$  has bounded geometry and the orbits of  $\Gamma$  are uniformly separated sets.

## 2.10. LOWER BOUNDS.

We have constructed, in Section 2.8, a lipschitz map between a model space and our 3-manifold. In this section, we begin the project of showing that there is also a linear lower bound on distortion of distances. We will do this in a series of steps. First we shall restrict our attention to the electric pseudometrics,  $\rho = \rho_P$  and  $\rho' = \rho_M$  as defined at the end of Section 2.4. We will need a result about 1-dimensional quasi-isometries. In what follows we will write  $[t, u] = [u, t]$  for the interval between  $t, u \in \mathbb{R}$ , regardless of the order of  $t$  and  $u$ .

Suppose that  $I \subseteq \mathbb{R}$  is an interval and that  $\sigma : I \rightarrow \mathbb{R}$  is a continuous map. We shall say that  $\sigma$  is **quasi-isometric** if it is a quasi-isometry to its range,  $\sigma(I)$ . Writing  $I = [\partial_- I, \partial_+ I]$ , it necessarily follows that  $\sigma(I)$  lies in a bounded

neighbourhood of  $[\sigma(\partial_- I), \sigma(\partial_+ I)]$ . We shall allow the possibility that  $\partial_- I = -\infty$  and  $\partial_+ I = \infty$ . In this case,  $\sigma$  is a self quasi-isometry of  $\mathbb{R}$ .

We list the following properties of a continuous map  $\sigma : I \rightarrow \mathbb{R}$  which together will imply that it is quasi-isometric. Let  $Q$  be a closed subset of  $I$ . We suppose:

- (Q1)  $(\forall k)(\exists K_1(k))$  if  $t, u \in I$  with  $|t - u| \leq k$  then  $|\sigma(t) - \sigma(u)| \leq K_1(k)$ .
- (Q2)  $(\forall k)(\exists K_2(k))$  if  $t, u \in Q$  and  $|\sigma(t) - \sigma(u)| \leq k$  then  $|t - u| \leq K_2(k)$ .
- (Q3)  $(\forall k)(\exists K_3(k))$  if  $t, u \in I$  and  $\text{diam}(\sigma[t, u]) \leq k$ , then  $|t - u| \leq K_3(k)$ .
- (Q4)  $(\exists k_4)(\forall k)(\exists K_4(k))$  if  $t, u \in I$  and  $N([\sigma(t), \sigma(u)], k_4) \cap \sigma(Q) = \emptyset$  and  $|\sigma(t) - \sigma(u)| \leq k$  then  $|t - u| \leq K_4(k)$ .
- (Q5)  $(\exists k_5)$  if  $t, u \in I$ ,  $[t, u] \cap Q = \emptyset$ , then  $[\sigma(t), \sigma(u)] \cap \sigma(Q) \subseteq N(\{\sigma(t), \sigma(u)\}, k_5)$ .

We can paraphrase the above conditions informally as follows. (Q1) gives an upper bound on distortion, and (Q2) gives a lower bound restricted to  $Q$ . (Q3) tells us that no long interval can get sent into a short interval. (Q4) gives a lower bound on distortion, so long as we stay away from  $\sigma(Q)$ . Finally (Q5) tells us that intervals in the complement of  $Q$  can not fold too deeply over  $Q$ . (We remark that we will always apply (Q5) to a subinterval  $[t', u'] \subseteq [t, u]$  with  $\sigma([t', u']) = [\sigma(t'), \sigma(u')] = \sigma([t, u])$ .)

**Lemma 2.10.1.** *Let  $\sigma : I \rightarrow \mathbb{R}$  be a continuous map, with  $Q \subseteq I$  satisfying (Q1)–(Q5) above. Then  $\sigma$  is a quasi-isometry, and the constants of quasi-isometry depend only on the constants of the hypotheses.*

*Proof.* Let  $Q' = \sigma(Q)$  and let  $R' = \{t \in \mathbb{R} \mid N(t, 2k_4) \cap Q' = \emptyset\}$ , and let  $R = \sigma^{-1}R'$ . Then  $R \subseteq I$  is closed. We first claim that each component of  $I \setminus (R \cup Q)$  has bounded length.

Note that each component of  $\mathbb{R} \cap (R' \cup Q')$  has length at most  $2k_4$ . Suppose that  $J \subseteq I$  is an interval with  $J \cap (R \cup Q) = \emptyset$ . Now  $\sigma(J) \cap R' = \emptyset$ , and property (Q5) bounds the extent to which  $\sigma(J)$  can cross  $Q'$ . In fact, we get  $\text{diam}(\sigma(J)) \leq 2k_5 \cup 2k_4$ . It now follows by (Q3) that  $\text{diam}(J)$  is bounded (by  $K_3(2k_5 + 2k_4)$ ). This proves the claim.

Now fix some  $k_0 < k_4$ , and suppose that  $t < u \in I$  with  $|\sigma(t) - \sigma(u)| \leq k_0$ . We now claim that  $u - t$  is bounded. If  $t \in R$ , then  $\sigma(u) \in N(R', k_0) \subseteq N(R', k_4)$  so  $N([\sigma(t), \sigma(u)], k_4) \cap Q' = \emptyset$ , and  $u - t$  is bounded by (Q4). Similarly this holds if  $u \in R$ . In order to prove this second claim, we can therefore assume that  $t, u \notin R$ .

Now if  $[t, u] \cap (R \cup Q) = \emptyset$ , then  $u - t$  bounded by our earlier claim. If not, let  $t_0$  and  $u_0$  be, respectively, the minimal and maximal points of  $[t, u] \cap (R \cup Q)$ . Again,  $u - u_0$  and  $t_0 - t$  are bounded. It in turn follows that  $|\sigma(t_0) - \sigma(u_0)|$  is bounded (by (Q1)). If  $t_0, u_0 \in Q$ , then  $u_0 - t_0$  is bounded by (Q2), and so  $u - t$  is bounded. If  $t_0 \in R$ , then let  $t_1$  be the maximal point of  $[t, u] \cap \sigma^{-1}\sigma(t_0)$ . Since  $\sigma(t_0) = \sigma(t_1) \in R'$ ,  $t_1 - t_0$  is bounded by (Q4), and so it is enough to

consider the interval  $[t_1, u_0]$ . Similarly, if  $u_0 \in R$ , let  $u_1$  be the minimal point of  $[t, u] \cap \sigma^{-1}\sigma(u_0)$ , we see that  $u_0 - u_1$  is bounded. But if  $\sigma|[t, u]$  enters any component of  $R'$  it must eventually leave by the same point. Thus the above observations allow us to reduce to the case where  $(t, u) \cap R = \emptyset$ , and so again we get  $u - t$  bounded, as before.

This proves the second claim. The fact that  $\sigma$  is a quasi-isometry is now elementary.  $\square$

We shall be applying this to spaces quasi-isometric to intervals, and we will need some general observations concerning such spaces.

Suppose that  $\Psi$  is a locally compact locally connected and connected space with two ends, deemed “positive” and “negative”. (In practice  $\Psi \cong \Sigma \times \mathbb{R}$ .) By an **end-separating** set  $Q$ , we mean a compact connected subset which separates the two ends of  $\Psi$ . We write  $C_+(Q), C_-(Q)$  for the components of  $\Psi \setminus Q$  containing the positive and negative ends of  $\Psi$  respectively, We write  $C_0(Q) = \Psi \setminus (C_-(Q) \cup C_+(Q))$ . i.e. we adjoin all relatively compact components of the complement. (Figure 2.23.) (Note that  $C_0(C_0(Q)) = C_0(Q)$ .) If  $Q'$  is another end-separating

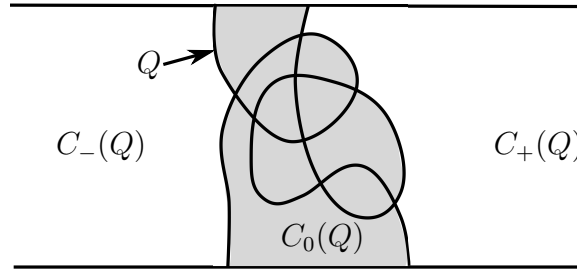


FIGURE 2.23. Definitions of  $C_0(Q), C_-(Q)$  and  $C_+(Q)$

set, we write  $Q < Q'$  to mean that  $Q \subseteq C_-(Q')$ . One can verify that this is equivalent to stating that  $Q' \subseteq C_+(Q)$ , and that  $<$  is a total order on any locally finite pairwise disjoint collection of end-separating sets. If  $Q < Q'$  we write  $[Q, Q'] = [Q', Q]$  for the closure of  $C_+(Q) \cap C_-(Q')$ . This is the compact region of  $\Psi$  between  $Q$  and  $Q'$ . We also note that if  $P \subseteq Q$  is end-separating, then  $C_-(Q) \subseteq C_-(P)$  and  $C_+(Q) \subseteq C_+(P)$ .

**Definition.** By a **quasiline** we mean a locally compact connected path-metric space  $(\Psi, \rho)$ , such that every point of  $\Psi$  lies in some end-separating set of  $\rho$ -diameter at most  $l$ , where  $l \geq 0$  is some constant.

Suppose now that  $\Psi$  is a quasiline. We choose such a set  $Q(x)$ . Note that if  $y \in C_0(Q(x))$ , then  $Q(y) \cap Q(x) \neq \emptyset$ , and so  $\text{diam } C_0(Q(x)) \leq 3l$ . In particular,  $C_0(Q(x))$  is another end-separating set of bounded diameter containing  $x$ .

Let  $\pi : \mathbb{R} \rightarrow \Psi$  be a bi-infinite end-respecting geodesic, i.e.  $\pi|(-\infty, 0]$  goes out the negative end, and  $\pi|[0, \infty)$  goes out the positive end of  $\Psi$ . (Such must

exist by a simple compactness argument.) Clearly any end-separating set must meet  $\pi(\mathbb{R})$  and so  $\Psi = N(\pi(\mathbb{R}), l)$ . In particular,  $\pi$  is a quasi-isometry. We see that any quasiline is quasi-isometric to  $\mathbb{R}$ . (Indeed we could equivalently define a quasiline as a locally compact path metric space quasi-isometric to  $\mathbb{R}$ .) We can see in fact, that any two geodesics in  $\Psi$  with the same endpoints must remain a bounded distance (in fact  $l$ ) apart. The same remains true if the respective endpoints are bounded distance apart. If  $x, y \in \Psi$  with  $Q(x) \cap Q(y) = \emptyset$ , then  $[Q(x), Q(y)]$  is a bounded Hausdorff distance from any geodesic from  $x$  to  $y$ . If  $t, u \in \mathbb{R}$  with  $u > t + 2l$ , then  $Q(\pi(t)) \cap Q(\pi(u)) = \emptyset$ , and  $[Q(\pi(t)), Q(\pi(u))]$  is a bounded Hausdorff distance from  $\pi([t, u])$ .

We shall be applying this in the case where  $\Psi = \Sigma \times \mathbb{R}$ , and every point of  $\Psi$  lies in the image of a homotopy equivalence,  $\psi$ , of  $\Sigma$  into  $\Psi$  of bounded diameter in  $\Psi$ . (Here all homotopy equivalences are assumed to be relative to the boundaries  $\partial\Sigma$  and  $\partial\Psi$ .) By the result of [FHS] (see Theorem 1.6.1 here) we can find an embedded surface,  $Z$ , in an arbitrarily small neighbourhood of  $\psi(\Sigma)$ . This is a ‘‘fibre’’ of  $\Psi$  in the sense discussed in Section 2.3. If we have two such  $\psi$  and  $\psi'$ , with the  $\psi(\Sigma) \cap \psi'(\Sigma) = \emptyset$  and  $Z, Z'$  are nearby fibres, then  $[Z, Z']$  is a band in  $\Psi$  (with base surface  $\Sigma$ ) and  $[Z, Z']$  is a bounded Hausdorff distance from  $[\psi(\Sigma), \psi'(\Sigma)]$ .

Suppose that  $(\Psi, \rho)$  and  $(\Psi', \rho')$  are two such product spaces, and  $f : \Psi \rightarrow \Psi'$  is a proper lipschitz end-preserving map. Let  $\pi : \mathbb{R} \rightarrow \Psi$  and  $\pi' : \mathbb{R} \rightarrow \Psi'$  be bi-infinite end-respecting geodesics. We can find a map  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  such that for all  $t \in \mathbb{R}$ ,  $\rho'(\pi'(\sigma(t)), f(\pi(t)))$  is uniformly bounded.

Let us now focus on the case of interest where  $f : \Psi(P) \rightarrow \Psi(M)$  is the map between the non-cuspidal parts of our model space  $P$  and hyperbolic 3-manifold  $M$ . These have riemannian metrics,  $d$  and  $d'$  respectively. Let  $\mathcal{T}(P)$  and  $\mathcal{T}(M)$  be the sets of Margulis tubes in  $\Psi(P)$  and  $\Psi(M)$  respectively, and let  $\Theta(P) = \Psi(P) \setminus \text{int} \bigcup \mathcal{T}(P)$  and  $\Theta(M) = \Psi(M) \setminus \text{int} \bigcup \mathcal{T}(M)$  be the respective thick parts. Now  $f$  maps  $\Theta(P)$  onto  $\Theta(M)$  and is lipschitz with respect to the metrics  $d$  and  $d'$ . We can define electric riemannian pseudometrics  $\rho$  and  $\rho'$  on  $\Psi(P)$  and  $\Psi(M)$  respectively, agreeing with  $d$  or  $d'$  on the thick parts, and equal to zero on each Margulis tube. The map  $f : (\Psi(P), \rho) \rightarrow (\Psi(M), \rho')$  is then lipschitz. In what follows, all distances and diameters etc. refer to the metrics  $\rho$  or  $\rho'$  unless otherwise specified. As observed in Section 2.5, we can foliate  $\Psi(P)$  with fibres of bounded diameter. We write  $S(x)$  for the fibre containing  $x$ , so that  $S(x)$  varies continuously in the Hausdorff topology. We are thus in the situation described above with  $\Psi = \Psi(P)$  and  $\Psi' = \Psi(M)$ . We have geodesics  $\pi$  and  $\pi'$  and a map  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ .

We claim:

**Lemma 2.10.2.** *The map  $\sigma$  arising as above (from the map  $f : \Psi(P) \rightarrow \Psi(M)$ ) is a self-quasi-isometry of  $\mathbb{R}$ .*

We set  $Q = \{t \in \mathbb{R} \mid S(\pi(t)) \cap \bigcup \mathcal{T}(P) \neq \emptyset\}$ . (Figure 2.24.) Thus  $Q$  is a closed subset of  $\mathbb{R}$ . Note that property (Q1) of Lemma 2.10.1, is an immediate

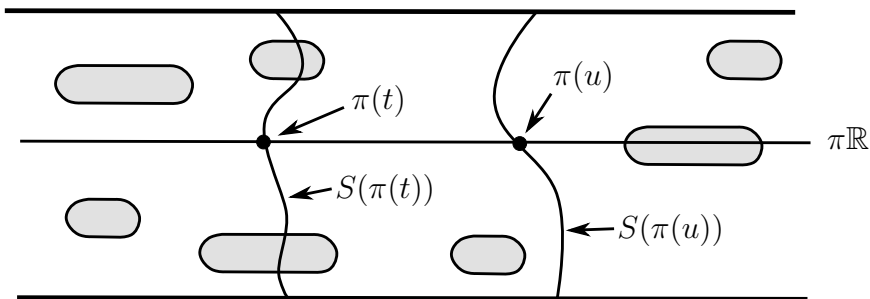


FIGURE 2.24. Definition of  $Q$ :  $t \in Q$ ,  $u \notin Q$ . The grey regions represent Margulis tubes in  $\Psi(P)$ .

consequence of the construction and the fact that  $f$  is lipschitz. We now set about verifying properties (Q2)–(Q5) of Lemma 2.10.1.

(Q1): This is an immediate consequence of the construction and the fact that  $f$  is lipschitz.

(Q2): This argument is based on a similar construction due to Bromberg, which is discussed in [Brocb]. (We are using images of fibres under  $f$ , in place of interpolations of pleated surfaces used by Bromberg.) Indeed the proof of (Q2) given here could be shortened by quoting Bromberg’s result (namely that two Margulis tubes a bounded distance apart in  $M$  are a bounded distance apart in the curve complex). However, we will need some variants of this construction later.

Suppose  $t, u \in Q$  with  $|\sigma(t) - \sigma(u)| \leq k$ . By the definition of  $Q$ ,  $S(\pi(t))$  and  $S(\pi(u))$  meet Margulis tubes  $T_0, T_1 \in \mathcal{T}(P)$  respectively. Let  $T'_i = f(T_i) \in \mathcal{T}(M)$ . We want to show that  $\rho(T_0, T_1)$  is bounded, since it then follows that  $\rho(\pi(t), \pi(u))$  is bounded, and so, since  $\pi$  is geodesic,  $|t - u|$  is bounded, as required. Clearly, we can assume that  $T_0 \neq T_1$ , or there is nothing to prove.

Now  $f(S(\pi(t)))$  meets  $T'_0$  and has bounded diameter. Also  $f(\pi(t)) \in f(S(\pi(t)))$  is a bounded distance from  $\pi'(\sigma(t))$  (by definition of  $\sigma$ ) and so  $\rho'(\pi'(\sigma(t)), T'_0)$  is bounded. Similarly,  $\rho'(\pi'(\sigma(u)), T'_1)$  is bounded. Also  $\rho'(\sigma(t), \sigma(u)) = |\sigma(t) - \sigma(u)| \leq k$ , and so  $\rho(T'_0, T'_1)$  is bounded (in terms of  $k$ ). In other words we can connect  $T'_0$  to  $T'_1$  by a path  $\beta$  of bounded  $\rho'$ -length. Indeed we can assume that  $\beta \cap \Theta(M)$  consists of a bounded number of paths  $\beta_1, \dots, \beta_n$  of bounded  $d$ -length connecting different Margulis tubes.

Since  $f : \Psi(P) \rightarrow \Psi(M)$  is a homotopy equivalence and  $f^{-1}(T'_0) = T_0$  and  $f^{-1}(T'_1) = T_1$ , there is a path  $\alpha \subseteq f^{-1}\beta$  connecting  $T_0$  to  $T_1$  in  $\Psi(P)$ . For each  $x \in \alpha$ , there is a loop  $\gamma_x$  in  $\Psi(P)$  based at  $x$  of bounded  $d$ -length, with the properties (W2)–(W4) described in Section 2.5. In particular,  $\gamma_x$  is freely homotopic to a curve,  $[\gamma_x] \in \mathbf{C}(\Sigma)$  (via the natural homotopy equivalence). We can also take  $\gamma_x$  to lie either in  $\Theta(P)$  or else inside a Margulis tube and hence freely homotopic to

the core of that tube. If  $x, y$  are sufficiently close then  $d_{\mathbf{C}}([\gamma_x], [\gamma_y])$  is bounded: less than  $2r \geq 0$  say. (“Sufficiently” can be taken to imply some uniform positive constant, but uniformity is not needed here.) Let  $Y = \{[\gamma_x] \mid x \in \alpha\} \subseteq \mathbf{C}(\Sigma)$ . Since  $\alpha$  is connected, by the above observation, the  $r$ -neighbourhood of  $Y$  in  $\mathcal{G}(\Sigma)$  is connected. We claim that  $|Y|$  is bounded. This will place a bound on the diameter of  $Y$  in  $\mathbf{C}(\Sigma)$ .

To see that  $|Y|$  is bounded, note that if  $x \in \alpha$ , then  $f(\gamma_x)$  is a loop in  $\Psi(M)$  based at  $f(x) \in \beta$ . Either  $f(x)$  lies in one of the segments  $\beta_i$  or else it is freely homotopic to the core of one of (the bounded number of) Margulis tubes passed through by  $\beta$ . But now for each  $\beta_i$ , there can be only boundedly many possibilities for the free homotopy classes of  $\beta_i$ . This follows from the standard fact of hyperbolic 3-manifolds that there are a bounded number of (based) homotopy classes of curves of bounded length based at any point of the 3-manifold (at least if we rule out multiples of a very short curve, which cannot happen here). Alternatively, this is a general statement about bounded geometry manifolds, given that we are working in the thick part. We see that there is bound on the number of possible  $[\gamma_x]$ . This bounds  $|Y|$  as required.

But now the core curves of  $T_0$  and  $T_1$  lie in  $Y$  and so we have bounded the distance between these curves in  $\mathcal{G}(\Sigma)$ . By the tautness assumption (see (W4) Section 2.5), it follows that  $\rho(T_0, T_1)$  is bounded as required, and so (Q2) follows.

(Q3): This is essentially a variation on the same argument. Suppose  $t, u \in \mathbb{R}$  with  $\text{diam} \sigma([t, u])$  bounded. Let  $R \subseteq \Psi$  be the region between  $S(\pi(t))$  and  $S(\pi(u))$ . Again, since  $f : \Psi(P) \rightarrow \Psi(M)$  is a homotopy equivalence, there is a path  $\alpha \subseteq R \cap f^{-1}(\pi'(\mathbb{R}))$  connecting  $S(\pi(t))$  to  $S(\pi(u))$ . Now  $\alpha$  will be a bounded Hausdorff distance from any geodesic also connecting  $S(\pi(t))$  to  $S(\pi(u))$  in  $\Psi(P)$ , in particular, the geodesic  $\pi([t, u])$ . Thus,  $f(\alpha)$  is a bounded Hausdorff distance from  $f(\pi([t, u]))$ , which, by the definition of  $\sigma$ , is in turn a bounded Hausdorff distance from  $\pi'(\sigma([t, u]))$  in  $\Psi(M)$ . By assumption,  $\text{diam}(\sigma([t, u]))$  is bounded. It therefore follows that  $\text{diam}(f(\alpha))$  is bounded. We see that  $f(\alpha) \cap \Theta(M)$  consists of a bounded number of segments  $\beta_1, \dots, \beta_n$ , each of bounded  $d'$ -length, and we proceed exactly as in (Q2) to show that  $|t - u|$  is bounded.

(Q4): Let us suppose that  $t, u \in \mathbb{R}$  and that  $N([\sigma(t), \sigma(u)], k_4) \cap \sigma(Q) = \emptyset$  for some constant  $k_4$  to be determined shortly, and suppose that  $|\sigma(t) - \sigma(u)| \leq k$ . Let  $x = \sigma(t)$  and  $y = \sigma(u)$ . We want to bound  $|t - u|$ , which is the same as bounding  $\rho(x, y)$ . Since  $t, u \notin Q$ , we have  $S(x), S(y) \subseteq \Theta(P)$ . In particular,  $x, y \in \Theta(P)$  and so the paths  $\gamma_x$  and  $\gamma_y$  lie in  $\Theta(P)$ . Since  $|\sigma(t) - \sigma(u)|$  is bounded,  $\rho'(f(x), f(y))$  is bounded. Also  $f(\gamma_x)$  and  $f(\gamma_y)$  have bounded  $d'$ -length. We want to apply Lemma 2.5.2. This means finding a band  $B \subseteq \Theta(M) \subseteq \Psi(M)$ , with base surface  $\Sigma$ , of bounded diameter, containing  $\gamma_x$  and  $\gamma_y$ , and which is  $r$ -convex for some uniform  $r > 0$ . (See the definition in Section 2.9.) For the latter, it is enough to find another band  $A \subseteq \Theta(M)$  containing a uniform neighbourhood of  $B$ .



Let  $h_1 > h_0 > 0$  be sufficiently large constants, to be determined shortly. Let  $t' = \sigma(t)$  and  $u' = \sigma(u)$ . We can suppose that  $t' \leq u'$ . Let  $t_0 = t' - h_0$ ,  $t_1 = t' - h_1$ ,  $u_0 = u' + h_0$ ,  $u_1 = u' + h_1$ . Thus  $t_1 < t_0 < t' < u' < u_0 < u_1$ . For  $i = 0, 1$ , let  $Z_i^-, Z_i^+$  be fibres in a small neighbourhood of  $f(S(\pi(t_i)))$  and  $f(S(\pi(u_i)))$  respectively. Now if  $h_0$  and  $h_1 - h_0$  are sufficiently large, then we have  $Z_1^- < Z_0^- < Z_0^+ < Z_1^+$ . Let  $B = [Z_0^-, Z_0^+]$  and  $A = [Z_1^-, Z_1^+]$ , so that  $B \subseteq A$ . Note that  $B$  and  $A$  are a bounded Hausdorff distance from  $\pi'([t_0, u_0])$  and  $\pi'([t_1, u_1])$  respectively, and that  $f(x)$  and  $f(y)$  are a bounded distance from  $\pi'(t')$  and  $\pi'(u')$  respectively. Thus, again by choosing  $h_0$  and  $h_1 - h_0$  sufficiently large, we can assume that  $f(\gamma_x), f(\gamma_y) \subseteq B$  and that  $N(B, r) \subseteq A$ , for some uniform  $r > 0$  (as usual, with respect to the metric  $\rho'$ ).

We claim that, provided  $k_4 - h_1$  is sufficiently large,  $A \subseteq \Theta(M)$ . Suppose that  $T \cap A \neq \emptyset$  for some  $T \in \mathcal{T}(M)$ . Let  $s \in I$  be such that the fibre  $S(\pi(s))$  meets the corresponding Margulis tube in  $P$ . By definition,  $s \in Q$ . Thus  $f(S(\pi(s))) \cap T \neq \emptyset$ . Now  $\pi'(\sigma(s))$  is a bounded distance from  $f(\pi(s))$  which is a bounded distance from  $T$  and hence from  $\pi'([t_1, u_1])$ . Thus  $\sigma(s)$  is a bounded distance from  $[t_1, u_1]$ . But  $\sigma(s) \in \sigma(Q)$ , so we get a contradiction by taking  $k_4 - h_1$  large enough. This shows that  $A \subseteq \Theta(M)$  as claimed.

But now  $\gamma_x, \gamma_y, B$  satisfy the hypotheses of Lemma 2.9.3, which means that  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  is bounded. By (W4) of Subection 2.5.3, it follows that  $\rho(x, y)$  is bounded, thereby giving a bound on  $|t - u|$  as required.

(Q5): Suppose  $t, u \in \mathbb{R}$  with  $[t, u] \cap Q = \emptyset$ . Let  $x = \pi(t)$  and  $y = \pi(u)$ . Let  $R = [S(x), S(y)]$  be the band bounded by  $S(x)$  and  $S(y)$ . Note that  $R \subseteq \bigcup_{v \in [t, u]} S(\pi(v))$ . Since  $[t, u] \cap Q = \emptyset$ , we have  $R \subseteq \Theta(P)$ . If  $f(S(x)) \cap f(S(y)) \neq \emptyset$ , then  $\rho'(f(x), f(y))$  is bounded, so  $\rho'(\pi'(\sigma(t)), \pi'(\sigma(u)))$  and hence  $|\sigma(t) - \sigma(u)|$  is bounded, and there is nothing to prove. If not, let  $R' = [f(S(x)), f(S(y))]$  be the compact region between  $f(S(x))$  and  $f(S(y))$ . Thus  $R'$  is a bounded Hausdorff distance from the geodesic segment  $\pi'[\sigma(t), \sigma(u)]$ . Now, for homological reasons,  $f|f^{-1}R'$  must have degree 1, and so  $R' \subseteq f(R)$ . Since  $f(\Theta(P)) = \Theta(M)$ , we see that  $R' \subseteq \Theta(M)$ . Suppose now that  $v \in [\sigma(t), \sigma(u)] \cap \sigma(Q)$ . Let  $v = \sigma(s)$  for  $s \in Q$ . Thus  $S(\pi(s))$  meets some Margulis tube  $T \in \mathcal{T}(P)$ . Now  $f(S(\pi(s)))$  has bounded diameter, and is a bounded distance from  $\pi'(v)$  and from  $f(T)$ . But  $R' \subseteq \Theta(M)$ , so  $f(T) \cap R' = \emptyset$ . It follows that  $\pi'(v)$  must be a bounded distance from an endpoint of the segment  $\pi'([\sigma(t), \sigma(u)])$ , and so  $v$  is a bounded distance from either  $\sigma(t)$  or  $\sigma(u)$  as required.

We have verified properties (Q1)–(Q5) for the map,  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ , and so by Lemma 2.10.1, it is a quasi-isometry, proving Lemma 2.10.2.

We note the following immediate consequence:

**Proposition 2.10.3.** *The map  $f : (\Psi(P), \rho) \longrightarrow (\Psi(M), \rho')$  is a quasi-isometry.*

The constants only depend on  $\xi(\Sigma)$  and the Margulis constant defining  $\Psi(M)$ .

We need a version of this result for bands. We can express this by passing to appropriate covers.

Let  $B \subseteq \Psi(P)$  be a band with base surface  $\Phi$ . Let  $\mathcal{T}_B(P) = \{T \in \mathcal{T}(P) \mid T \cap \partial_V B \neq \emptyset\}$ . Thus  $\partial_V B$  consists of a set of annuli in the boundaries of elements of  $\mathcal{T}_B(P)$  and components of  $\partial\Psi(P)$ . Let  $\Xi_B(P) = \Psi(P) \setminus \text{int} \bigcup \mathcal{T}_B(P)$ , and let  $\Psi_B(P)$  be the cover of  $\Xi_B(P)$  corresponding to  $B$ . Thus  $B$  lifts to a compact subset of  $\Psi_B(P)$ , which we also denote by  $B$ . We note that  $B$  cuts  $\Psi_B(P)$  into two non-compact components bounded by  $\partial_- B$  and  $\partial_+ B$  respectively. The inclusion of  $B$  into  $\Psi_B(P)$  is a homotopy equivalence. Indeed, if we remove those boundary components of  $\Psi_B(P)$  that do not meet  $B$ , then the result is homeomorphic to  $\Phi \times \mathbb{R}$ .

We can perform the same construction in  $\Psi(M)$ . We let  $\mathcal{T}_B(M) = \{f(T) \mid T \in \mathcal{T}_B(P)\}$ , and  $\Xi_B(M) = \Psi(M) \setminus \text{int} \bigcup \mathcal{T}_B(M)$ . By Lemma 2.3.8,  $f : \Xi_B(P) \longrightarrow \Xi_B(M)$  is a homotopy equivalence. We let  $\Psi_B(M)$  be the cover of  $\Xi_B(M)$  corresponding to  $\Psi_B(P)$ , so that  $f$  lifts to a homotopy equivalence  $\tilde{f} : \Psi_B(P) \longrightarrow \Psi_B(M)$ . Indeed,  $\Psi_B(M)$  is homeomorphic to  $\Phi \times \mathbb{R}$  after removing certain boundary components, so we are in the same topological situation as before (where the base surface was all of  $\Sigma$ ). We write  $g : B \longrightarrow \Psi_B(M)$  for its restriction to  $B$ .

We shall assume that  $B$  has positive height. Let  $\mathcal{T}_B^0(P) = \{T \in \Theta(P) \setminus \mathcal{T}_B(P) \mid T \cap B \neq \emptyset\}$ . If  $T \in \mathcal{T}_B^0(P)$ , then either  $T \subseteq B$ , or  $T \cap B$  is a half-torus bounded by an annulus in  $\partial_H B \cap T$ . We denote the lifted riemannian metric  $d$  by  $d_B$ , and write  $\rho_B$  for the ‘‘electric’’ metric obtained by setting the metric equal to zero on each  $T \in \mathcal{T}_B^0(P)$ . Every point  $x \in B$  lies in a fibre  $F(x) \subseteq B$ , of bounded  $\rho_B$ -diameter. If  $x \in \partial_\pm B$ , we can take  $F(x) = \partial_\pm B$ . We can assume that such fibres foliate  $B$ . We similarly define a metric  $\rho'_B$  on  $\Psi_B(M)$ . In what follows, all distances are measured with respect to  $\rho_B$  or  $\rho'_B$ , unless otherwise specified. Note that  $g : (B, \rho_B) \longrightarrow (\Psi_B(M), \rho'_B)$  is lipschitz.

Let  $\pi : [a, b] \longrightarrow B \subseteq \Psi_B(P)$  be a shortest geodesic from  $\partial_- B$  to  $\partial_+ B$ . Each fibre,  $F(x)$ , of  $B$  meets  $\pi([a, b])$  and we see that  $B$  lies in a uniform neighbourhood of  $\pi([a, b])$ . Let  $\pi' : \mathbb{R} \longrightarrow \Psi_B(M)$  be a bi-infinite geodesic, with  $\pi'|(-\infty, 0]$  and  $\pi'|[0, \infty)$  going out a negative and positive end of  $\Psi_B(M)$  respectively. If  $x \in B$ , then  $g(F(x))$  intersects  $\pi'(\mathbb{R})$ . This enables us to define a continuous map  $\sigma_B : [a, b] \longrightarrow \mathbb{R}$  such that  $\rho'_B(\pi'(\sigma_B(t)), g(\pi(t)))$  is uniformly bounded.

**Lemma 2.10.4.** *The map  $\sigma_B : [a, b] \longrightarrow \mathbb{R}$  is a uniform quasi-isometric embedding.*

*Proof.* We define  $Q = \{t \in [a, b] \mid F(\pi(t)) \cap \bigcup \mathcal{T}_B^0(P)\} \neq \emptyset$ . Property (Q1) is an immediate consequence of the fact that  $g$  is lipschitz. We need to verify (Q2)–(Q5). The argument is essentially the same as before. There are a few subtleties

we should comment on.

(Q1): Immediate from the construction.

(Q2): Here we use the loops  $\gamma_x^B \subseteq B$  instead of  $\gamma_x$ . This time property (W8) of Section 1.7 tells us that  $\rho_B(x, y)$  is bounded above in terms of the distance between  $[\gamma_x^B]$  and  $[\gamma_y^B]$  in the curve graph  $\mathcal{G}(\Phi)$ .

There is a slight complication in that  $g^{-1}\beta \subseteq B$  might not connect  $T_0$  to  $T_1$ . We may therefore need to allow the path  $\alpha$  to have up to three components, possibly connecting  $T_0$  or  $T_1$  to  $\partial_H B$ , and maybe also  $\partial_- B$  with  $\partial_+ B$ . But this makes no essential difference to the argument, since if  $x, y$  both lie in  $\partial_- B$  or both in  $\partial_+ B$ , then  $[\gamma_x^B]$  and  $[\gamma_y^B]$  are equal or adjacent in the curve graph. It follows that a metric  $r$ -neighbourhood of  $Y \subseteq \mathbf{C}(\Phi)$  is connected and the argument proceeds as before.

(Q3): As before.

(Q4): Here we apply [FHS] (Theorem 1.6.1 here) as before to find embedded surfaces in  $\Psi_B(M)$  close to the surfaces  $g(F(x))$ . These surfaces will be fibres in  $\Psi_B(M)$  and any two disjoint fibres bound a band.

There is, however, an added complication in that in order to find our surfaces  $Z_i^\pm$ , we will need that  $t_1$  and  $u_1$  lie in  $\sigma([a, b])$ . We therefore need that  $t$  and  $u$  are not too close to the boundary of  $\sigma([a, b])$ . We are saved by property (Q3) which have already proven.

Let  $\sigma([a, b]) = [a', b']$ , and suppose that  $t, u \in I$  with  $\rho'_B(\sigma(t), \sigma(u)) \leq k$  and that  $N([\sigma(t), \sigma(u)], 2k_4) \cap \sigma(Q) = \emptyset$ . Suppose that  $\sigma(t) \in N(a', k_4)$ , say. If  $\sigma([t, u]) \subseteq N(a', k_4)$ , then  $|t - u|$  is bounded using (Q3). If not, let  $s \in [t, u]$  be the first time that  $\sigma|[t, u]$  leaves  $N(a', k_4)$ . Now  $\sigma([t, s]) \subseteq N(a', k_4)$ , and so by (Q3)  $|s - t|$  is bounded. We can now replace  $t$  by  $s$  and continue the argument. We can do the same for the other end  $u$ . We are then reduced to the case where  $t, u \in [a' + k_4, b' - k_4]$ , and the argument proceeds as before. This time, the constant  $2k_4$  becomes our “new”  $k_4$ .

(Q5): As before. □

This shows that  $\sigma_B : [a, b] \rightarrow \mathbb{R}$  is quasi-isometric, and it follows that  $\sigma_B([a, b])$  lies in a bounded neighbourhood of  $[\sigma_B(a), \sigma_B(b)]$ . The constants depend only on the various constants inputted. To simplify notation in what follows, we will assume that  $\sigma(a) < \sigma(b)$ . If not, we could reinterpret everything by reversing the order on the range. (It will turn out, retrospectively, that if  $b - a$  is sufficiently large then this is necessarily the case, though we won't formally need to worry about this. The issue of vertical orientation of bands will eventually be taken care of automatically by the topology of the situation.)

If  $b - a$  is sufficiently large, then we can find embedded fibres close to  $f(\partial_- B)$  and  $f(\partial_+ B)$  in  $\Psi_B(M)$  which will together will bound a band. We would like to find an embedded band  $B'$  in  $\Psi(M)$ . It would be enough to show that the

projection to  $\Xi_B(M)$  is injective far enough away from  $\partial_H B'$ . For this, we use need the following lemma. For the statement, we can interpret the term “band” to be a 3-submanifold,  $A$  of  $\Psi$ , homeomorphic to  $\Phi \times [0, 1]$  with  $\partial_V A \equiv \partial\Phi \times [0, 1]$  and  $\partial_H A = \partial_- A \sqcup \partial_+ A = \Phi \times \{0, 1\}$ . As usual, a “subband” is a subset bounded by disjoint fibres. We shall assume that  $A$  carries a metric  $\rho$ . This need not be a path metric. (For our application it will be the restriction of an ambient path metric.)

**Lemma 2.10.5.** *Suppose that  $(A, \rho)$  is a band and that each point of  $A$  lies in a fibre of  $\rho$ -diameter at most  $k$ . Let  $\Xi$  be a complete non-compact orientable riemannian manifold, and suppose that  $\theta : A \rightarrow \Xi$  is a  $\pi_1$ -injective locally isometric map with  $\partial_V A = \theta^{-1}(\partial\Xi)$  (and so  $\theta$  is 1-lipschitz). Suppose that any fibre of  $A$  is homotopic to an embedded surface in  $\Xi$ . Then  $\theta|_{A \setminus N(\partial_H A, 13k)}$  is injective.*

*Proof.* First, we claim that if  $x \in A$  with  $\rho(x, \partial_H A) \geq 13k$ , then there is some subband,  $A' \subseteq A$  (with the same base surface), containing  $x$ , with  $\rho'(\theta(x), \theta(\partial_H A')) \geq 2k$ . To see this, we can argue as follows.

Let  $C_\pm$  be the set of  $p \in A$  such that there is an arc  $\tau_\pm(p)$  from  $p$  to  $\partial_\pm A$  such that  $\theta(\tau_\pm(p))$  is geodesic in  $\Xi$ . Clearly  $\partial_\pm A \subseteq C_\pm$  (set  $\tau_\pm(p) = \{p\}$ ). Also  $A = C_- \cup C_+$ , since if  $p \in A$ , we can find a geodesic ray  $\sigma$  in  $\Xi$  based at  $p$  in  $\Xi$  (since  $\Xi$  is non-compact), and some component of  $\theta^{-1}(\sigma)$  must connect  $p$  to  $\partial_H A$  in  $A$ , and so  $p \in C_- \cup C_+$ . Now  $A$  is connected, and  $C_-$  and  $C_+$  are closed, so there must be some  $p \in C_- \cap C_+$ . In other words, there are arcs,  $\tau_\pm$  from  $p$  to points  $a_\pm \in \partial_\pm A$  with  $\theta(\tau_\pm)$  geodesic in  $\Xi$ . Since the path  $\tau_+ \cup \tau_-$  connects  $\partial_- A$  to  $\partial_+ A$ , it meets every fiber, and so we have  $A = N(\tau_- \cup \tau_+, k)$ . Suppose now that  $x \in A$  with  $\rho(x, \partial_H A) \geq 13k$ . Let  $y \in \tau_- \cup \tau_+$  with  $\rho(x, y) \leq k$ , so  $\rho'(\theta(x), \theta(y)) \leq k$ . We can assume that  $y \in \tau_+$ . Since  $\theta$  is 1-lipschitz,  $\rho(y, a_+) \geq 12k$ . Since  $\theta(\tau_+)$  is geodesic,  $\rho'(\theta(y), \theta(a_+)) \geq 12k$ . If  $\rho'(\theta(y), \theta(p)) \leq 4k$ , let  $F$  be a fibre of diameter at most  $k$  through  $p$ . Note that  $\rho'(\theta(x), \theta(F)) \geq 4k - 2k = 2k$ , so we can set  $A'$  to be the band between  $F$  and  $\partial_+ A$ . On the other hand, suppose  $\rho'(\theta(y), \theta(p)) \geq 4k$ . The total length of  $\tau_-$  together with the segment of  $\tau_+$  from  $p$  to  $y$  is at least  $\rho(y, a_+) \geq 12k$ . Since  $\theta(\tau_-)$  and  $\theta(\tau_+)$  are both geodesic, it follows that  $\rho'(\theta(y), \theta(a_-)) \geq 12k - 8k = 4k$ . This time, we can take  $A' = A$ . This proves the claim.

Now if  $x \in A$  with  $\rho(x, \partial_H A) \geq 13k$ , we claim there is a fibre  $Z = Z(x)$  through  $x$ , with  $\theta|_Z$  injective. To see this, start with any fibre,  $F$ , through  $x$ . By hypothesis,  $\theta(F)$  is homotopic to an embedded surface in  $\Xi$ , and so by [FHS] (see Theorem 1.6.1 here) we can find such a surface  $S$ , in an arbitrarily small neighbourhood of  $\theta(F)$ . We shall assume that  $\theta(x) \in S$ . Since  $\text{diam}(S) < 2k$ , we have  $S \cap \theta(\partial_H A') = \emptyset$ . Now  $\theta|_{A'}$  is a local homeomorphism away from  $\partial_H A'$ . Let  $Z$  be the component of the preimage of  $S$  in  $A' \subseteq A$  with  $x \in Z$ . Therefore, the map from  $Z$  to  $S$  is a covering map, and so the inclusion of  $Z$  into  $A$  is  $\pi_1$ -injective. A simple degree argument shows that is also  $\pi_1$ -surjective, and it follows

that  $F$  must be a fibre of  $A$ . Thus, the map  $\theta|Z \rightarrow S$  is a homeomorphism. In particular,  $\theta|Z$  is injective as required.

Note also that, in the above construction, if  $\theta(x) = \theta(y)$  we could take the same surface  $S$  for both, and we get fibres  $Z(x)$  and  $Z(y)$  with  $\theta|Z(x)$  and  $\theta|Z(y)$  both homeomorphisms to  $S$ . If  $x \neq y$ , then we must have  $Z(x) \cap Z(y) = \emptyset$ .

Suppose finally for contradiction, that  $x, y \in A$  with  $x \neq y$ ,  $\rho(x, \partial_H A) \geq 13k$ ,  $\rho(y, \partial_H A) \geq 13k$  and  $\theta(x) = \theta(y)$ . Construct fibres  $Z(x)$  and  $Z(y)$  as above. Let  $C = [Z(x), Z(y)]$  be the band between  $Z(x)$  and  $Z(y)$ . We construct a closed manifold  $R$  by gluing together  $\partial_- C = Z(x)$  and  $\partial_+ C = Z(y)$  via the homeomorphism  $(\theta|Z(y))^{-1} \circ (\theta|Z(x))$ . Now  $\theta$  induces a map from  $R$  to  $\Xi$  which is a local homeomorphism away from  $Z(x) \equiv Z(y)$ , and hence, by orientation considerations, a local homeomorphism everywhere. It must therefore be a covering space, giving the contradiction that  $\Xi$  is compact.  $\square$

We now apply this in the situation of interest. We return to the set-up of Lemma 2.10.4. We will need to assume that  $b - a$  is sufficiently large. As described earlier, we will assume, for notational convenience that  $\sigma(a) < \sigma(b)$ . The first step is to note that  $\partial_- B = F(\pi(a))$  and so  $g(\partial_- B)$  is a bounded distance from  $\pi'(\sigma(a))$ . Similarly,  $\partial_+ B = F(\pi(b))$  is a bounded distance from  $\pi'(\sigma(b))$ . Thus if  $b - a$  and hence  $\sigma(b) - \sigma(a)$  is sufficiently large,  $g(\partial_- B) \cap g(\partial_+ B) = \emptyset$ . We can find disjoint fibres,  $Z_\pm$  in a small neighbourhood of  $g(\partial_\pm B)$ , and let  $A$  be the band between  $Z_-$  and  $Z_+$ . Let  $\theta : A \rightarrow \Xi_B(M)$  be the inclusion of  $A$  into  $\Psi_B(M)$  composed with the covering map  $\Psi_B(M) \rightarrow \Xi_B(M)$ . Now if  $l_0$ , and  $b - a - 2l_0$  are sufficiently large, then by Lemma 2.10.4, we can arrange that  $\sigma(a) < \sigma(a + l_0) < \sigma(b - l_0) < \sigma(b)$ . Moreover,  $g(F(\pi(a + l_0)))$  has bounded diameter and is a bounded distance from  $\pi(\sigma(a + l_0))$  and we can find a fibre,  $Z'_-$ , close to  $g(F(\pi(a + l_0)))$ . We similarly find  $Z'_+$  close to  $g(F(\pi(b - l_0)))$ . If  $l_0$  and  $b - a - 2l_0$  are sufficiently large, then we will have  $Z_- < Z'_- < Z'_+ < Z_+$ . Moreover, by Lemma 2.10.5, we can assume that  $\theta|B'$  is injective, where  $B' = [Z'_-, Z'_+]$ . Now, if  $l_1 > 0$  is sufficiently large, we can assume a uniform neighbourhood  $\sigma_B([a + l_0 + l_1, b - l_0 - l_1])$  lies inside  $[\sigma_B(a + l_0), \sigma_B(b - l_0)]$ . If this neighbourhood is large enough, then  $g(B_0) \subseteq B'$ , where  $B_0 = [F(a + l_0 + l_1), F(b - l_0 - l_1)]$ . Note that the depth of  $B_0$  in  $B$  (measured in the metric  $\rho_B$ ) is equal to  $l_0 + l_1$  up to an additive constant. Since  $B'$  embeds in  $\Psi_B(M)$ , we can project the whole picture to  $\Xi_B(P) \subseteq \Psi(P)$  and  $\Xi_B(M) \subseteq \Psi(M)$ .

In summary, we have shown:

**Lemma 2.10.6.** *There is some  $l > 0$  such that if  $B \subseteq \Psi(P)$  is a band, and  $B_0$  is a sub-band of depth at least  $l$ , then there is a band  $B' \subseteq \Psi(M)$  with the same base surface such that  $f(B_0) \subseteq B'$ .*

For the moment, we can just interpret this to mean that  $B' \cong \Phi \times [0, 1]$  with  $\partial_V B' = B' \cap \mathcal{T}_B(M)$ . (In Section 2.12, we will insist in addition that the horizontal boundaries of bands should lie in the thick part.) In fact, from our construction

of  $B'$  we see that every point of  $B'$  lies inside some fibre of bounded (extrinsic) diameter. By [FHS] (Theorem 1.6.1 here) we can take such fibres to be embedded (but we do not claim that such fibres foliate  $B'$ ). We can also refine Lemma 2.10.6 in various ways. Note, in particular, that if  $r \geq 0$ , then by choosing  $l = l(r)$  sufficiently large, we can assume that an  $r$ -neighbourhood of  $f(B_0)$  (with respect to  $\rho'$ ) lies inside  $B'$ .

At this point, it is still conceivable that  $f$  might also send a point far away from  $B$  into  $B'$ . To rule this out, we need to bring (W9) of Subsection 2.5.3 into play.

Suppose now that  $B$  is a maximal band. There is some  $r \geq 0$  such that if  $x \in \Psi(P) \setminus B$ , there there is some loop,  $\delta_x^B \ni x$ , of  $d$ -length at most  $r$  and such that either  $\delta_x^B$  is freely homotopic into a Margulis tube  $T$  not meeting  $B$ , or else  $[\delta_x^B]$  is not freely homotopic into the base surface,  $\Phi$ , in  $\Sigma$ . We can assume that  $B'$  does not meet any of the images of Margulis tubes of the above type, and so there is a bound on how deeply the image  $f(\delta_x^B)$  can enter into  $B'$ . Using the refinement of Lemma 2.10.6 mentioned above, we see:

**Lemma 2.10.7.** *There is some  $l' > 0$  such that if  $B \subseteq \Psi(P)$  is a maximal band and if  $B_0 \subseteq B$  is a parallel subband of depth at least  $l'$  in  $B$  (with respect to the metric  $\rho$ ) then there is a band,  $A \subseteq \Psi(M)$ , with the same base surface such that  $f(B_0) \subseteq A$  and  $f(\Psi(P) \setminus B) \cap A = \emptyset$ .*

To finish this section, we describe how the constructions can be adapted to the cases of finite or semi-infinite models. From this we can deduce Theorems 2.1.3 and 2.1.4 (though this is not directly relevant to the proof of the Ending Lamination Theorem).

Suppose that  $I \subseteq \mathbb{R}$  is any interval, and that  $W = \bigcup \mathcal{W}$  is an annulus system in  $\Sigma \times I$  with no annulus crossing  $\Sigma \times I$ . We assume that  $M$  satisfies the a-priori bounds condition (APB), that is the corresponding closed geodesics in  $M$  all have length at most  $L$ . We can construct a model  $\Psi(P)$  and lipschitz homotopy equivalence,  $f : \Psi(P) \rightarrow \Psi(M)$  as in Section 2.8.

We define an electric pseudometric,  $\rho = \rho_P$ , on  $\Psi(P)$  by forcing the preimage of each Margulis tube in  $M$  to have diameter 0 in  $(\Psi(P), \rho)$ . Such a preimage is either a Margulis tube in  $P$  or a boundary annulus, that is  $\Delta(\Omega)$  for some  $\Omega \in \mathcal{W}_\partial$  (though not all such sets arise in this way). We see that  $f : (\Psi(P), \rho_P) \rightarrow (\Psi(M), \rho_M)$  is also uniformly lipschitz. The argument of Lemma 2.10.3 now goes through as before to show that  $f$  is quasi-isometric.

*Proof of Theorem 2.1.4 :* Note that  $\rho(\alpha_M, \alpha_M^*)$  is bounded above in terms of  $l$ , so we can assume that  $\alpha_M = \alpha_M^*$ . Similarly, we can assume that  $\beta_M = \beta_M^*$ . Write  $\alpha = [\alpha_M]$  and  $\beta = [\beta_M]$ .

(1) We can reduce to the case where  $d_{\mathcal{G}}(\alpha, \beta) = 1$ . If  $\xi(\Sigma) = 1$ , then the statement follows directly from the construction of Section 2.8. If  $\xi(\Sigma) \geq 2$ , then  $\alpha$  and  $\beta$  are disjoint, and so we can extend  $\alpha_M$  and  $\beta_M$  to a pleating surface in  $M$ , which necessarily has bounded  $\rho_M$ -diameter. (Note that this gives explicit  $k_1$  and  $k_2$

computable in terms of  $\xi(\Sigma)$ .

(2) We can suppose that  $d_{\mathcal{G}}(\alpha, \beta) \geq 3$ . We construct a model,  $\Psi(P)$ , with  $\alpha, \beta$  as boundary curves, using Lemma 2.4.1 as described at the end of Section 2.8. Let  $f : \Psi(P) \rightarrow \Psi(M)$  be the map described above. As observed, we have  $d_{\mathcal{G}}(\alpha, \beta)$  linearly bounded above in terms of  $\rho_M(\alpha, \beta)$ . The linear function depends only on  $\xi(\Sigma)$  and  $l$ .  $\square$

We can now also prove Lemma 2.1.3. Note that we can choose a model  $\Psi(P)$  so that  $f(\Psi(P)) = \Psi(M) \cap \text{core}(M)$ . In the doubly degenerate case, this has already been done. For our purposes, it is enough that  $(\Psi(M) \cap \text{core}(M), \rho_M)$  lies a bounded distance from  $f(\Psi(P))$  in the electric pseudometric. For this, it is in turn enough that  $\partial_H \Psi(P)$  be sent a bounded electric distance from  $\Psi(M) \cap \text{core}(M)$ , as can be seen by a degree argument.

If  $M$  is singly degenerate, note that  $\partial \text{core}(M)$  is intrinsically a finite area hyperbolic surface. We choose any curve  $\alpha$  in  $\partial \text{core}(M)$  whose length is bounded in term of  $\xi(\Sigma)$ . We can now construct an annulus system in  $\Sigma \times [0, \infty)$ , with  $\alpha$  as a boundary curve, and with all curves in the hierarchy tending to the end invariant,  $a \in \partial \mathcal{G}(\Sigma)$ . In particular, all curves lie a bounded distance from a geodesic ray,  $\pi$ , in  $\mathcal{G}(\Sigma)$  tending to  $a$ . (This is based on a diagonal sequence argument — cf. Lemma 2.2.2.) In particular, (APB) is satisfied for some constant,  $L$ , depending only on  $\xi(\Sigma)$ , and we get a model,  $\Psi(P)$ , and a map  $f : \Psi(P) \rightarrow \Psi(M)$  as before. (This construction will be used again — see Lemma 2.14.3.)

Suppose that  $\alpha_M$  is a curve of length at most  $l$  in  $M$ . This lies a bounded  $\rho_M$  distance from  $f(\Psi(P))$ . In other words there is some  $x \in \Psi(P)$  with  $\rho_M(\alpha_M, f(x))$  bounded. By the construction of  $\Psi(P)$ ,  $x$  lies in some curve,  $\gamma$ , of bounded length in  $(\Psi(P), d_P)$  and with  $d_{\mathcal{G}}(\pi, \gamma)$  bounded (both in terms of  $\xi(\Sigma)$ ). Now  $\rho_M(\alpha_M, f(\gamma))$  is bounded, so it follows from Theorem 2.1.4 that  $d_{\mathcal{G}}(\alpha, \pi)$  is bounded in terms of  $\xi(\Sigma)$  and  $l$ . In particular, this shows that  $\mathbf{C}(M, l) \subseteq N(\pi, t)$  in  $\mathcal{G}(\Sigma)$ , where  $t$  depends only on  $\xi(\Sigma)$  and  $l$ . This proves Theorem 2.1.3 in this case.

The geometrically finite (quasifuchsian) case can be dealt with in a similar manner using a finite model, starting with curves,  $\alpha_M, \beta_M \in M$  of bounded length in each of the two boundary components of  $\text{core}(M)$ . In the case where  $d_{\mathcal{G}}(\psi, \beta) \leq 2$ , we need to observe that all the relevant subsets of  $\mathcal{G}(\Sigma)$  have bounded diameter.

## 2.11. FAMILIES OF PROPER SUBSURFACES.

In this section, we give some general topological results about maps between 3-manifolds which have controlled behaviour on certain families of proper subsurfaces. Under certain conditions, one can show that such a map is properly homotopic to a homeomorphism. Some of this will be needed in Section 2.12. The main result will be needed later when we consider promoting quasi-isometries to

bilipschitz maps (Chapter 4). Some of the constructions can be interpreted in terms of Bass-Serre theory, though we won't make that explicit here.

Let  $V$  be an orientable irreducible 3-manifold with (possibly empty) boundary  $\partial V$ . A **proper subsurface** in  $V$  is an embedded compact connected orientable  $\pi_1$ -injective surface,  $S$ , with  $S \cap \partial V = \partial S$ . Recall that two proper surfaces,  $S_1, S_2$  are **parallel** if they bound a product region,  $P \cong S_i \times [0, 1]$ , in  $V$ . We write  $S_1 \parallel S_2$ . By Waldhausen's theorem (given here as Theorem 1.6.2), this is equivalent to saying that they are homotopic in  $V$ , relative to  $\partial V$  (i.e. sliding their boundaries in  $\partial V$ ). Note that if  $V$  is not a circle bundle with fibre  $S_1$  (or equivalently  $S_2$ ) then  $P$  is uniquely determined by  $S_1$  and  $S_2$ .

Let  $\mathcal{S}$  be a locally finite family of disjoint proper subsurfaces.

By a **complementary region**,  $R$ , we mean the completion of a component of  $V \setminus \bigcup \mathcal{S}$ . In other words, we compactify a component of  $V \setminus \bigcup \mathcal{S}$  by adjoining finite collection of surfaces, to give a subsurface,  $\partial_0 R$ , of the boundary,  $\partial R$ . Here,  $\partial_0 R$  maps surjectively to  $\bigcup \mathcal{S}(R) \subseteq V$ , where  $\mathcal{S}(R)$  is the set of elements of  $\mathcal{S}$  which bound the component of  $V \setminus \bigcup \mathcal{S}$ . We allow two components of  $\partial_0 R$  to map to the same element,  $S$ , of  $\mathcal{S}(R)$ . This happens when there is a closed curve in  $V$  which meets  $\bigcup \mathcal{S}$  in a single point of  $S$ , where it crosses it transversely. Elsewhere map is injective from  $\partial_0 R$  to  $\bigcup \mathcal{S}(R)$ . We write  $\mathcal{R} = \mathcal{R}(\mathcal{S})$  for the set of such complementary regions.

We can give another description of this as follows. Let  $R \in \mathcal{R}$ , so that  $\text{int } R$  is a component of  $V \setminus \bigcup \mathcal{S}$ . Let  $V_R$  be the cover of  $V$  corresponding to  $\text{int } R$ . We can lift  $\text{int } R$  to an open subset of  $V_R$  and identify  $R$  with its closure. The inclusion  $R \hookrightarrow V_R$  is a homotopy equivalence. Under this identification, the covering map restricted to  $R$ , denoted  $\pi_R$ , is the same as the quotient map described above.

Suppose that  $V'$  is another orientable aspherical 3-manifold, and  $\mathcal{S}'$  is a collection of disjoint proper subsurfaces of  $V'$ . Write  $\mathcal{R}' = \mathcal{R}(\mathcal{S}')$ . We aim to show:

**Lemma 2.11.1.** *Let us suppose that no two elements of  $\mathcal{S}$  are parallel. Suppose that we have a bijection,  $[S \mapsto S'] : \mathcal{S} \rightarrow \mathcal{S}'$ , and a proper relative homotopy equivalence,  $f : V, \partial V \rightarrow V', \partial V'$ . Suppose that for all  $S \in \mathcal{S}$ ,  $f|_S$  is homotopic to the inclusion of  $S'$  into  $V'$ , relative to  $\partial V$  (precomposed with a homeomorphism of  $S$  to  $S'$ ). Then  $f$  is properly homotopic, relative to  $\partial V$ , to a map  $f_0 : V \rightarrow V'$ , such that for all  $S \in \mathcal{S}$ ,  $(f_0)^{-1}S' = S$  and  $f_0|_S : S \rightarrow S'$  is a homeomorphism.*

In fact,  $f$  also induces a bijection,  $[R \mapsto R'] : \mathcal{R} \rightarrow \mathcal{R}'$ , such that for all  $R$ , there is a proper homotopy equivalence,  $f_R : R \rightarrow R'$ , with  $f_R|_{\text{int } R} = f_0|_{\text{int } R}$ . Also if two components,  $S_1, S_2$ , of  $\partial_0 R$  get identified in  $V$ , then the corresponding components of  $\partial_0 R'$  get identified in  $V'$ , and  $f_R$  commutes with these identifications. (This can be deduced from the conclusion of Lemma 2.11.1, though it also will follow directly from the proof given below.)

Since  $f$  is a homotopy equivalence, we get an equivariant lift,  $\tilde{f}_0 : \tilde{V} \rightarrow \tilde{V}'$ , to the respective universal covers. Let  $\mathcal{S}'$  be the set of components of preimages in  $\tilde{V}$  of the surfaces in  $\mathcal{S}$ . We similarly define  $\tilde{\mathcal{R}}, \tilde{\mathcal{S}}'$  and  $\tilde{\mathcal{R}}'$ . There is an equivariant



bijection,  $[\tilde{S} \mapsto \tilde{S}'] : \tilde{\mathcal{S}} \longrightarrow \tilde{\mathcal{S}}'$ , such that  $\tilde{S}' = \tilde{f}_0(\tilde{S})$  for all  $\tilde{S} \in \tilde{\mathcal{S}}$ , where  $\tilde{f}_0 : \tilde{V} \longrightarrow \tilde{V}'$  is the corresponding lift of  $f_0$ . The above discussion gives us an equivariant bijection,  $[\tilde{R} \mapsto \tilde{R}'] : \tilde{\mathcal{R}} \longrightarrow \tilde{\mathcal{R}}'$ , so that  $\tilde{S} \subseteq \tilde{R}$  if and only if  $\tilde{S}' \subseteq \tilde{R}'$ . (This can be expressed by saying that there is an equivariant isomorphism of the dual Bass-Serre trees.) In particular, the families of surfaces  $\tilde{\mathcal{S}}$  and  $\tilde{\mathcal{S}}'$  have the same separation properties:

**Corollary 2.11.2.** *Let  $V, V', \mathcal{S}, \mathcal{S}', f$  be as in Lemma 2.11.1, and let  $[\tilde{S} \mapsto \tilde{S}'] : \tilde{\mathcal{S}} \longrightarrow \tilde{\mathcal{S}}'$  be the corresponding bijection in the universal covers as described above. Suppose that  $\tilde{S}_0, \tilde{S}_1, \tilde{S}_2 \in \tilde{\mathcal{S}}$ . Then  $\tilde{S}_0$  separates  $\tilde{S}_1$  from  $\tilde{S}_2$  in  $\tilde{V}$  if and only if  $\tilde{S}'_0$  separates  $\tilde{S}'_1$  from  $\tilde{S}'_2$  in  $\tilde{V}'$ .*

For the proof of Lemma 2.11.1, we will use the following characterisation of parallel surfaces.

**Lemma 2.11.3.** *Suppose that  $S_1, S_2 \subseteq V$  are disjoint proper subsurfaces and that  $S_1$  is homotopic into  $S_2$ , relative to  $\partial V$ . Then  $S_1$  and  $S_2$  are parallel.*

*Proof.* We can assume that  $V$  is not a circle bundle with fibre  $S_1$ . Let  $h : S_1 \times [0, 1] \longrightarrow V$  be a homotopy of  $S_1$  into  $S_2$ , with  $h^{-1}(\partial V) = \partial S_1 \times [0, 1]$ . Let  $P \subseteq V$  be the set of points to which  $h$  maps with degree 1 modulo 2. Then  $P$  has relative boundary,  $\partial_0 P = S_1 \sqcup S_2$ . If  $V$  is not a circle bundle, then it is not hard to see that  $h$  has degree 0 on  $V \setminus P$ , hence degree  $\pm 1$  on  $P$ . (In our application,  $V$  will be non-compact, and so this is clear.) It follows that the map  $[x \mapsto h(x, 1)] : S_1 \longrightarrow S_2$  also has degree 1. But this map is also  $\pi_1$ -injective, and so it is a homotopy equivalence (by a standard fact about compact surfaces). The statement now follows by Waldhausen's Cobordism Theorem (stated as Theorem 1.6.2 here).  $\square$

Let  $f : V \longrightarrow V'$  be as in the hypotheses of Lemma 2.11.1. First note that there is no loss in assuming that for all  $S \in \mathcal{S}$ ,  $f|_S$  is a homeomorphism to  $S'$ . We also have an inverse proper homotopy equivalence,  $g : V' \longrightarrow V$ . Again we can assume that  $g|_{S'}$  is a homeomorphism to  $S$ , indeed the inverse to  $f|_S$ .

Now let  $R \in \mathcal{R}$ . Let  $V_R$  be the corresponding cover of  $V$ , and identify  $R \subseteq V_R$ . Each component,  $E$ , of  $V_R \setminus \text{int } R$  corresponds to an end of  $V_R$ , whose relative boundary,  $\partial_0 E$ , is a component of  $\partial_0 R$ , which is a lift of an element of  $\mathcal{S}'$ . Note that the inclusion of  $\partial_0 E$  into  $E$  is a homotopy equivalence relative to  $\partial V$ , and so  $E$  deformation retracts onto  $\partial_0 E$ .

Let  $V'_R$  be the cover of  $V'$  given by the image of  $\pi_1(R)$  in  $\pi_1(V)$  induced by  $f$ . Thus,  $f$  lifts to a proper homotopy equivalence,  $\hat{f} : V_R \longrightarrow V'_R$ . Let  $R' \subseteq V'_R$  be the set of points to which  $\hat{f}|_R$  maps with degree 1. This is a submanifold of  $V'_R$ , with relative boundary  $\hat{f}(\partial_0 R)$ . Note that  $\hat{f}|_{\partial_0 R}$  is a homeomorphism from  $\partial_0 R$  to  $\partial_0 R'$ . Given an end,  $E$ , of  $V_R$  (as above), let  $E' \subseteq V'_R$  be the set of points to which  $\hat{f}|_E$  maps with degree 1. This is an end of  $V'_R$  with relative boundary,  $\partial_0 E'$ ,

a component of  $\partial_0 R'$ . If the ends  $E_1, E_2 \subseteq V_R$  are distinct, then  $E'_1 \cap E'_2 = \emptyset$ . (Since the boundary of  $E'_1 \cap E'_2$  can have at most one component.) Therefore, the ends  $E'$  are precisely the components of  $V'_R \setminus R'$ . Since each such  $E'$  has exactly one relative boundary component, it follows that  $R'$  must be connected.

We claim that the inclusion of  $R'$  into  $V'_R$  is a homotopy equivalence. Note that its inclusion is  $\pi_1$ -injective since the inclusion of each component of  $\partial_0 R'$  is. We claim that it is also  $\pi_1$ -surjective. To see this, let  $V'_{R'}$  be the cover of  $V'_R$  corresponding to  $R'$ , and lift  $R' \subseteq V'_R$  to this cover. Let  $V_{R'}$  be the corresponding cover of  $V_R$ . Now  $\hat{g} : V' \rightarrow V$  be the lift of  $g$ . This is again a proper homotopy equivalence. Let  $R'' \subseteq V_{R'}$  be the set of points to which  $g$  maps with degree 1 modulo 2. This is a submanifold, with relative boundary  $\partial_0 R''$ . Again,  $\hat{g}|_{\partial_0 R'}$  maps  $\partial_0 R'$  homeomorphically to  $\partial_0 R''$ . Now the quotient map,  $V_{R'} \rightarrow V_R$ , is bijective on  $\partial_0 R'$ . (It is the inverse of  $\hat{g} \circ \hat{f}|_{\partial_0 R'}$ .) Since  $R \hookrightarrow V_R$  is a homotopy equivalence,  $R$  lifts to a connected submanifold,  $R''' \subseteq V_{R'}$ , with  $\partial_0 R''' = \partial_0 R''$ . (In fact,  $R''' = R''$ .) Now the covering map  $R''' \rightarrow R$  is the identity on the boundary, and so the cover must have degree 1. In other words, the map  $V_{R'} \rightarrow V_R$  is a homeomorphism, proving the claim.

Let  $\pi_R : V_R \rightarrow R$  and  $\pi_{R'} : V'_R \rightarrow R'$  be the respective covering maps. Recall that  $\pi_{R'}|_{\text{int } R'}$  is injective. We claim that  $\pi_{R'}(\text{int } R) \cap \bigcup \mathcal{S}' = \emptyset$ . For suppose  $S \in \mathcal{S}$ , with  $S' \cap \pi_{R'}(\text{int } R') \neq \emptyset$ . Then  $S' \subseteq \pi_{R'}(\text{int } R')$ . Now  $S \cap \pi_R(R) = \emptyset$ , so  $S \subseteq \pi_R(E)$ , for some end  $E$  of  $V_R$ . Since  $f$  is a homotopy equivalence relative to  $\partial V$ ,  $S$  must be homotopic into  $R$ , and so lifts to a surface,  $\hat{S} \subseteq V_R$ . Since  $S \cap R = \emptyset$ , we have  $\hat{S} \subseteq E$ , for some end,  $E$ , of  $V_R$ . Now  $E$  deformation retracts onto  $\partial_0 E$  relative to  $\partial V$ , and so in particular,  $\hat{S}$  is homotopic into  $S_0 = \partial_0 E$ . By Lemma 2.11.3, we see that  $\hat{S}$  is parallel to  $S_0$  in  $V_R$ , and so  $S$  is parallel to  $\pi_R S_0 \in \mathcal{S}$  in  $V$ , contrary to our non-parallel hypotheses. This proves the claim.

It now follows that  $\pi_{R'}(\text{int } R)$  is a component of  $V' \setminus \bigcup \mathcal{S}'$ . In other words,  $R' \in \mathcal{R}'$ .

In summary, we have a map  $[R \mapsto R'] : \mathcal{R} \rightarrow \mathcal{R}'$  such that  $\hat{f}|_R : V_R \rightarrow V'_{R'}$  is a homotopy equivalence. Given that  $\hat{f}|_{\partial_0 R}$  is a homeomorphism to  $\partial_0 R'$ , we can properly homotope  $f$  so that  $f(\text{int } R) = \text{int } R'$ . We do this for all  $R$ . It then follows that  $f^{-1}(\bigcup \mathcal{S}') = \mathcal{S}$ , and it in turn follows that  $[R \mapsto R']$  is bijective. (We can also see this by applying the same construction to  $g : V' \rightarrow V$ .)

This proves Lemma 2.11.1.

We will also need to allow for parallel surfaces.

For this, we will assume that  $V$  is not a surface bundle over a circle. Suppose that  $S_1, S_2$  are parallel surfaces, bounding product region  $P$ . An orientation on  $S_1$  determines an orientation on  $S_2$  and we say they are **consistently oriented**. Now  $P$  also inherits an orientation from  $M$ , which in turn induces an orientation on  $\partial_0 P = S_1 \sqcup S_2$ . We write  $S_1 < S_2$  if this orientation agrees on  $S_1$  (and hence differs on  $S_2$ ).

Suppose now that  $f : V \rightarrow V'$  is a proper homotopy equivalence relative to  $\partial V$ . We assume this to be orientation preserving (hence of degree 1). Suppose that  $S'_1, S'_2$  are proper surfaces in  $V'$ , and that  $f|_{S_i}$  is homotopic to  $S'_i$ . If  $S_1 \parallel S_2$ , then  $S'_1 \parallel S'_2$ . Let  $P' \subseteq V'$  be the product region bounded by  $S'_1 \sqcup S'_2$ . Note that  $f$  induces orientations on  $S'_1$  and  $S'_2$ , and they are consistently oriented. We say that  $f$  **respects the order** on the parallel surfaces, if given  $S_1 < S_2$ , we have  $S'_1 < S'_2$ . Note that this doesn't depend on the choice of orientations on  $S_i$ , since reversing both their orientations, reverses their order, as well as the induced orders and orientations on  $S'_i$ .

We can property homotope  $f$  so that  $f|_{S_i}$  is a homeomorphism to  $S'_i$ . In this case,  $f|_P$  has degree  $\pm 1$  to  $P'$ . Saying that  $f$  respects order is equivalent to saying that it has degree 1 to  $P$ . Given that  $f : V \rightarrow V'$  has degree 1, this is equivalent to saying that  $f|(V \setminus \text{int } P)$  maps with degree 0 to  $P'$  (and with degree 0 to  $V' \setminus \text{int } P'$ ). In particular, we deduce:

**Lemma 2.11.4.** *Suppose  $f|_{S_i}$  is a homeomorphism to  $S'_i$  for  $i = 1, 2$ . If  $f(V \setminus P) \neq V'$ , then  $f$  preserves the order on  $S_1, S_2$ .*

In fact, it is sufficient that there is some point to which  $f|(P \setminus V)$  maps with degree 0.

Another criterion, in the case where  $V$  is non-compact, is the following. Suppose there is a proper ray,  $\alpha$ , based at a point of  $S_1$ , such that  $S'_2 \cap f(\alpha) \neq \emptyset$ . Then  $f$  respects the order of  $S_1$  and  $S_2$ .

Yet another way to view this is to lift to the corresponding covers,  $V_S$  and  $V'_S$  of  $V$  and  $V'$ . Now  $f$  lifts to a proper homotopy equivalence,  $\hat{f} : V_S \rightarrow V'_S$ . An orientation in  $S_i$  determines an order on the two ends of  $V_S$ . Similarly, we get an orientation on the ends of  $V'_S$ . To say that  $f$  preserves order means that if  $S_1$  and  $S_2$  bound the positive and negative ends of  $V_S$ , then  $S'_1$  and  $S'_2$ , respectively bound the positive and negative ends of  $V'_S$ .

We can now state the more general result.

Suppose that  $\mathcal{S}$  and  $\mathcal{S}'$  are locally finite families of disjoint proper subsurfaces in  $V$  and  $V'$  respectively, and  $f : V \rightarrow V'$  is orientation preserving. We assume the same hypotheses as Lemma 2.11.1 except that we drop the condition that no two elements of  $\mathcal{S}$  are parallel. We assume instead that  $f$  respects the order on any pair of parallel elements of  $\mathcal{S}$ .

**Lemma 2.11.5.** *Lemma 2.11.1 (and the subsequent discussion of complementary regions) holds under these revised hypotheses. So therefore does Corollary 2.11.2.*

The proof is basically the same. We have to allow for the possibility that  $R$  is a product region between two parallel surfaces,  $S_1$  and  $S_2$ . In this case, we get a product region  $R'$  between  $S'_1$  and  $S'_2$ , and a homotopy equivalence,  $\hat{f} : V_R \rightarrow V'_R$ , as before. We previously used the non-parallel hypothesis to show that if  $R \in \mathcal{R}$  is any complementary region, then  $f(\text{int } R) \cap (\bigcup \mathcal{S}') = \emptyset$ . This is still true. For if not, as before we get a surface  $S \in \mathcal{S}$  which lifts to a surface in  $\hat{S} \subseteq E \subseteq V_R$ ,

parallel to  $S_0 = \partial_0 E$ , and such that  $S' \subseteq \text{int } R$ . Now  $S$  and  $S_0$  bound a product region  $P \subseteq V$  (not necessarily a complementary region). Also  $S'$  and  $S'_0$  are parallel, and bound a product region  $P' \subseteq R' \subseteq V'$ . Now  $P$  and  $R$  are on opposite sides of  $S_0$ , but  $P'$  and  $R'$  are on the same side of  $S'_0$ . Since  $f$  maps  $R$  with degree 1 to  $R'$  (since  $f$  is orientation preserving) we see that  $f$  must have degree  $-1$  to  $P'$ , so it does not respect the order of  $S$  and  $S_0$ , contrary to our hypothesis.

This proves Lemma 2.11.5.

Note that the above statements can be interpreted in terms of graphs of groups. A collection of disjoint proper subsurfaces,  $\mathcal{S}$ , determines a graph-of-groups decomposition of  $\pi_1(V)$ , where the vertex groups are of the form  $\pi_1(R)$  for  $R \in \mathcal{R}(\mathcal{S})$ , and the edge groups are of the form  $\pi_1(S)$  for  $S \in \mathcal{S}$ . Under the given assumptions, these lemmas tell us, in particular, that the homotopy equivalence  $f$  induces a graph-of-groups isomorphism on  $\pi_1$ , where the map on edges is precisely the bijection  $[S \mapsto S']$ . At the level of universal covers, we get an equivariant isomorphism of Bass-Serre trees — dual to the families  $\tilde{\mathcal{S}}$  and  $\tilde{\mathcal{S}}'$ , as in Corollary 2.11.2.

Finally for application in Chapter 4, we refine the conclusion under the following hypothesis.

**Definition.** We say that a family,  $\mathcal{S}$ , of disjoint proper subsurfaces of  $V$  is (topologically) **cobounded** if each complementary region is compact.

We again assume that  $V$  and  $V'$  are orientable aspherical 3-manifolds with (possibly empty) boundaries. We assume they are not compact surface bundles over the circle. (In our application they will be non-compact.) Let  $\mathcal{S}$  and  $\mathcal{S}'$  be locally finite families of proper subsurfaces in  $V$  and  $V'$  respectively. We suppose that we have a bijection  $[S \mapsto S'] : \mathcal{S} \rightarrow \mathcal{S}'$ . We also assume that  $\mathcal{S}$  is cobounded.

**Lemma 2.11.6.** . *Let  $f : V, \partial V \rightarrow V', \partial V'$  be a proper homotopy equivalence. Suppose that for all  $S \in \mathcal{S}$ , with  $f|_S$  homotopic to the inclusion of  $S'$  into  $V'$  (precomposed with a homeomorphism of  $S$  to  $S'$ ). We suppose that  $f$  respects the order on any pair of parallel surfaces in  $\mathcal{S}$ . Then  $f$  is properly homotopic to a homeomorphism  $f_0 : V \rightarrow V'$  such that  $f_0(S) = S'$  for all  $S \in \mathcal{S}$ .*

*Proof.* By Lemma 2.11.5, we already have  $f_0|_R : R \rightarrow R'$  a homotopy equivalence for all  $R \in \mathcal{R}$ . Since  $R$  and  $R'$  are haken, it follows that  $f_0|_R$  is homotopic to a homeomorphism from  $R$  to  $R'$ , which we can assume to be constant on  $\partial_0 R$  (see Theorem 1.6.5).  $\square$

## 2.12. CONTROLLING THE MAP ON THICK PARTS.

In this section, we return to the set-up of earlier sections. We shall show that the map  $f : \Theta(P) \rightarrow \Theta(M)$  as defined in Section 2.7 is universally sesquilipschitz (Proposition 2.12.9). To this end, we will take the results of Section 2.10, and get ourselves into a position to apply Lemma 2.9.4.

Recall that we have riemannian path-metrics,  $d$  and  $d'$  on  $\Theta(P)$  and  $\Theta(M)$  respectively, and that  $\rho = \rho_P$  and  $\rho' = \rho_M$  are the electric pseudometrics obtained by forcing each Margulis tube to have diameter 0. The map  $f$  is uniformly lipschitz from  $(\Theta(P), d)$  to  $(\Theta(M), d')$ , and hence also from  $(\Theta(P), \rho)$  to  $(\Theta(M), \rho')$ . We write  $G = \pi_1(\Theta(P)) \cong \pi_1(\Theta(M))$ . We shall say that a subset of  $\Theta(P)$  (or  $\Theta(M)$ ) is  $k$ -**small** if its diameter in the metric  $\rho$  (or  $\rho'$ ) is at most  $k$ . In what follows, we speak of a set as being “uniformly small” to mean that it is  $k$ -small for some  $k$  depending only on  $\xi(\Sigma)$ .

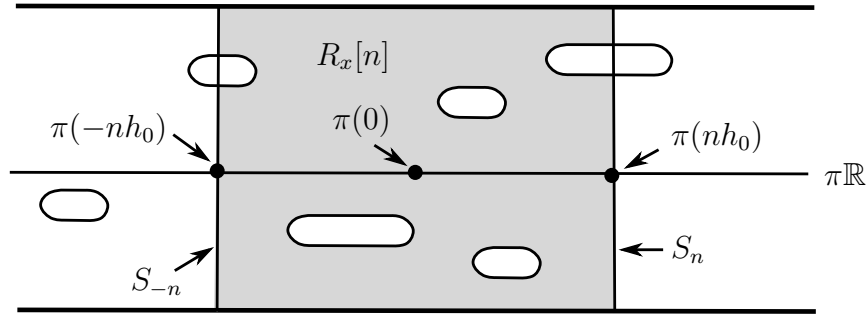
Given  $x \in \Theta(P)$ , we can find a uniformly small fibre  $S(x) \subseteq \Theta(P)$ . This can be achieved by taking a horizontal fibre, and then pushing it slightly off any Margulis tube. This may significantly increase its  $d$ -diameter, but only increases the  $\rho$ -diameter by an arbitrarily small amount. (Note that this is slightly different from the notion used in Section 2.10. Here we are assuming that  $S(x)$  lies in the thick part. This greatly simplifies the description of various topological operations. The cost is that we can no longer assume that  $S(x)$  varies continuously in  $x$ , but that will not matter to us in this section.) By Theorem 1.6.1, we can find a proper surface  $S'(x) \subseteq \Theta(M)$  in an arbitrarily small neighbourhood of  $f(S(x))$ , and homotopic to  $f(S(x))$  in  $\Theta(M)$ . Note that  $S'(x)$  is a fibre in the product space  $\Psi(M)$ . It is also uniformly small in  $\Theta(M)$ .

We now use the following construction of expanding bands in  $\Theta(P)$ . We fix a constant,  $h_0$ , to be defined shortly. Given  $x \in \Theta(P)$ , set  $R_x[0] = S(x)$ , and  $R'_x[0] = S'(x)$ . Let  $\pi : \mathbb{R} \rightarrow \Theta(P)$  be a bi-infinite geodesic (in the sense of a globally length minimising path parameterised by arc length) respecting the ends of  $\Theta(P)$ . Since  $\pi$  must cross  $R_x[0]$ , we can assume that  $\pi(0) \in R_x[0]$ . Given  $n \in \mathbb{Z}$ , let  $S_n = S(\pi(nh_0))$  and  $S'_n = S'(\pi(nh_0))$ . If  $h_0$  is large enough, the surfaces  $S_n$  will all be disjoint and occur in the correct order in  $\Psi(P)$ . Let  $R_x[n] = \Theta(P) \cap [S_{-n}, S_n]$ . In other words, it is the compact region of  $\Theta(P)$  bounded by  $S_{-n}$  and  $S_n$ . (Figure 2.25.) This gives an increasing sequence,  $R_x[0] \subseteq R_x[1] \subseteq R_x[2] \subseteq \dots$  of bands that eventually exhaust  $\Theta(P)$ . Now applying Lemma 2.6.2, again if  $h_0$  is large enough, the surfaces  $S'_n$  are all disjoint and occur in the correct order in  $\Theta(M)$ , and we similarly construct bands  $R'_x[n] = \Theta(M) \cap [S'_{-n}, S'_n]$ . We write  $CR_x[n]$  and  $CR'_x[n]$  for the closures of  $\Theta(P) \setminus R_x[n]$  and  $\Theta(M) \setminus R'_x[n]$  respectively. We can assume that:

**Lemma 2.12.1.** *For all  $x \in \Theta(M)$  and all  $n \in \mathbb{N}$  we have*

- (1)  $f(R_x[n]) \cap CR'_x[n+1] = \emptyset$ ,
- (2)  $f(CR_x[n+1]) \cap R'_x[n] = \emptyset$ .

*Proof.* This is a simple consequence of the discussion of end-separating sets in Section 2.10, and the fact that  $\sigma$  is a quasi-isometry (Lemma 2.10.2).  $\square$

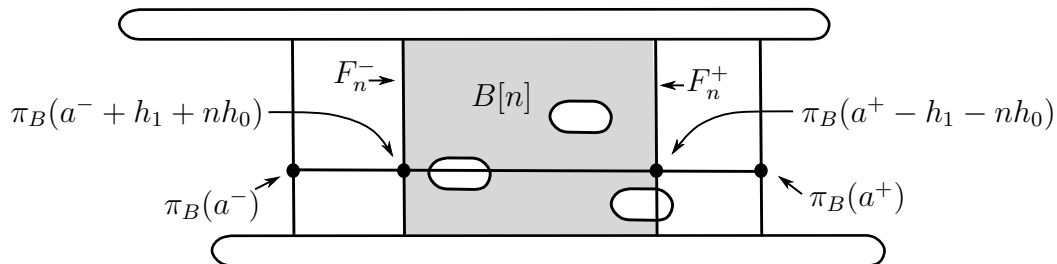
FIGURE 2.25. Definition of  $R_x[n]$ 

Note in particular, that  $f(R_x[n]) \cap f(CR_x[n+2]) = \emptyset$ . It will also be convenient to fix some  $\eta > 0$  smaller than the systoles of  $\Theta(P)$  and  $\Theta(M)$ , and we can refine Lemma 2.12.1 slightly to say that  $d'(f(R_x[n]), CR'_x[n+1]) \geq \eta$  and  $d'(f(CR_x[n+1]), R'_x[n]) \geq \eta$ . We also may as well assume that  $d(f(x), f(R_x[1])) \geq \eta$ .

We can also make a stronger statement concerning the nesting of the regions  $R'_x[n]$ . See Lemma 2.12.5 below.

We have a similar process of shrinking bands. Let  $B$  be a maximal band in  $\Psi(P)$ , with base surface  $\Phi$ . Let  $\rho_B$  be the electric pseudometric on  $B$  described in Section 2.10. (Recall that this is essentially obtained by taking the path-metric induced by  $d$ , and then metrically collapsing each Margulis tube whose interior meets  $B$  to have diameter 0. We may need to modify the metric near  $\partial_H B$  to take account of the fact that we cannot easily control the local geometry of the surfaces  $\partial_{\pm} B$ . It is formally defined by passing to the appropriate covering space.) In this case, we use “small” to refer to diameter with respect to the metric  $\rho_B$ . Now each  $x \in B$  lies in some uniformly small surface  $F(x)$  in  $B \cap \Theta(P)$  that is a fibre for  $B$ . Applying Lemma 1.6.1 again, we can find a surface  $F'(x)$  in an arbitrarily small neighbourhood of  $f(F(x))$  in  $\Theta(M)$ , and homotopic to  $f(F(x))$  in  $\Theta(M)$ .

Now let  $\pi_B : [a^-, a^+] \rightarrow B \cap \Theta(P)$  be a shortest geodesic from  $\partial_- B$  to  $\partial_+ B$  in  $B \cap \Theta(P)$ , with respect to the metric  $\rho_B$ . We write  $h(B) = a^+ - a^-$  for the length of this geodesic. We fix  $h_0$  and  $h_1$  as described below, and set  $h(n) = 2h_1 + (2n+1)h_0$ . Suppose  $h(B) \geq h(m)$  for some  $m \in \mathbb{N}$ . For each  $n = 0, 1, \dots, m$ , let  $F_{n,\pm} = F(\pi_B(a^{\pm} \mp (h_1 + nh_0)))$ , and let  $F'_{n,\pm} = F'(\pi_B(a^{\pm} \mp (h_1 + nh_0)))$ . If  $h_0$  is big enough then the surfaces  $F_{0,-}, F_{1,-}, \dots, F_{m,-}, F_{m,+}, \dots, F_{1,+}, F_{0,+}$  are all disjoint and occur in this order in  $B$ . Let  $B[n] = B_{\Phi}[n] = [F_{n,-}, F_{n,+}] \cap \Theta(M)$  be the compact region of  $\Theta(P)$  bounded by  $F_{n,-}$  and  $F_{n,+}$ . Thus  $B[n] \subseteq B$ . (Figure 2.26.) In fact, we have  $B[m] \subseteq \dots \subseteq B[1] \subseteq B[0] \subseteq B \cap \Theta(P)$ . By applying Lemma 2.10.6, if  $h_1$  is big enough we can assume that  $f(B[0])$  lies inside a band  $A$  in  $\Theta(M)$ . Moreover, by Lemma 2.10.7, we can assume that  $f(\Theta(P) \setminus B)$  does not enter  $A$ . We are thus effectively reduced to considering the map  $f|B[0]$  into  $A$ . Applying Corollary 2.11.2 and Lemma 2.11.5, we can assume that the surfaces  $F'_{0,-}, F'_{1,-}, \dots, F'_{m,-}, F'_{m,+}, \dots, F'_{1,+}, F'_{0,+}$  are disjoint and occur in this order in  $B$ .


 FIGURE 2.26. Definition of  $B[n]$ 

We set  $B'[m] = B'_\Phi[n] = [F_{n,-}, F_{n,+}] \cap \Theta(M)$ , and so  $B'[m] \subseteq \cdots \subseteq B'[1] \subseteq B'[0] \subseteq A \cap \Theta(M)$ . We write  $CB[n]$  and  $CB'[n]$  for the closures of  $\Theta(P) \setminus B[n]$  and  $\Theta(M) \setminus B'[n]$  respectively. Again, if  $h_0$  and  $h_1$  are large enough, we have (see Lemma 2.10.7):

**Lemma 2.12.2.** *For each maximal band, and for all  $n$ , we have:*

- (1)  $f(B_x[n]) \cap CB'_x[n+1] = \emptyset$ ,
- (2)  $f(CB_x[n+1]) \cap B'_x[n] = \emptyset$ .

The above bands are defined provided  $h(n+1) \leq h(B)$ . If  $h(n) > h(B)$ , we can set  $B[n] = \emptyset$ . This is consistent with Lemma 2.12.2.

We can also assume that the bands  $B[0]$  lies inside a 1-collared band,  $B_0 \subseteq B$ . This means that the collection of bands  $B[0]$  that we construct will have a nesting property (see Lemma 2.12.4(1)).

Finally, we can carry out the expanding band construction within a band. Suppose that  $B$  is a maximal band with base surface  $\Phi$ . Suppose that  $x \in B_\Phi[m]$ . We set  $R_{\Phi,x}[0]$  to be a uniformly small fibre containing  $x$ , which we can assume lies in  $B_\Phi[n]$ . As before, we construct increasing sequences of bands  $R_{\Phi,x}[0] \subseteq R_{\Phi,x}[1] \subseteq \cdots \subseteq R_{\Phi,x}[n]$  in  $B \cap \Theta(P)$ , and  $R'_{\Phi,x}[0] \subseteq R'_{\Phi,x}[1] \subseteq \cdots \subseteq R'_{\Phi,x}[n]$  in  $A \cap \Theta(M)$ . We can assume that  $R_{\Phi,x}[n] \subseteq B_\Phi[m-n]$ . We write  $CR_{\Phi,x}[n]$  and  $CR'_{\Phi,x}[n]$  respectively for the closures of  $\Theta(P) \setminus R_{\Phi,x}[n]$  and  $\Theta(M) \setminus R'_{\Phi,x}[n]$ . As before, applying Lemma 2.10.7, we get:

**Lemma 2.12.3.** *For each maximal band, and for all  $n$ , we have:*

- (1)  $f(R_{\Phi,x}[n]) \cap CR'_{\Phi,x}[n+1] = \emptyset$ ,
- (2)  $f(CR_{\Phi,x}[n+1]) \cap R'_{\Phi,x}[n] = \emptyset$ .

Let  $\mathcal{F}$  be the set of all subsurfaces of  $\Sigma$  (as usual, defined up to homotopy). Given  $\Phi \in \mathcal{F}$ , let  $B_\Phi \subseteq \Psi(P)$  be the (possibly empty) maximal band with base surface  $\Phi$ . Given  $n \in \mathbb{N}$ , let  $\mathcal{F}[n] = \{\Phi \in \mathcal{F} \mid B_\Phi \neq \emptyset, h(B_\Phi) \geq h(n)\}$ . Let  $\mathcal{B}[n] = \{B_\Phi[n] \mid \Phi \in \mathcal{F}[n]\}$ , and let  $\mathcal{B}'[n] = \{B'_\Phi[n] \mid \Phi \in \mathcal{F}[n]\}$ .

**Definition.** We refer to elements of  $\mathcal{B}[n]$  and  $\mathcal{B}'[n]$  as *level  $n$  bands* in  $\Theta(P)$  and  $\Theta(M)$  respectively.

Given  $\Phi \in \mathcal{F}$ , we write  $\mathcal{F}_\Phi \subseteq \mathcal{F}$  for the set of proper subsurfaces of  $\Phi$ . Let  $\mathcal{F}_\Phi[n] = \mathcal{F}_\Phi \cap \mathcal{F}[n]$ ,  $\mathcal{B}_\Phi[n] = \{B_{\Phi'}[n] \mid \Phi' \in \mathcal{F}_\Phi\}$  and  $\mathcal{B}'_\Phi[n] = \{B'_{\Phi'}[n] \mid \Phi' \in \mathcal{F}_\Phi\}$ .

If we choose  $h_0$  and  $h_1$  large the following are immediate consequences of the earlier lemmas:

**Lemma 2.12.4.**

(1) If  $A, B \in \mathcal{B}[0]$  are distinct, and  $A \cap B \neq \emptyset$ , the base surfaces  $\pi_\Sigma A$  and  $\pi_\Sigma B$  are strictly nested (i.e. one is proper a subset of the other).

(2) Suppose that  $x \in \Theta(P)$  and  $A \in \mathcal{B}[0]$ . If  $A \cap R_x[n] \neq \emptyset$  then  $A \cap CR_x[n+1] = \emptyset$ .

(3) Suppose  $\Phi \in \mathcal{F}[n+1]$ ,  $x \in B_\Phi[n+1]$  and  $A \in \mathcal{B}_\Phi[0]$ . If  $A \cap R_{\Phi,x}[n] \neq \emptyset$  then  $A \cap CR_{\Phi,x}[n+1] = \emptyset$ .

(4) Suppose  $\Phi \in \mathcal{F}[n+1]$  and  $A \in \mathcal{B}_\Phi[0]$ . If  $A \cap CB_\Phi[n] \neq \emptyset$  then  $A \cap B_\Phi[n+1] = \emptyset$ .

(Figures 2.27–2.30.)

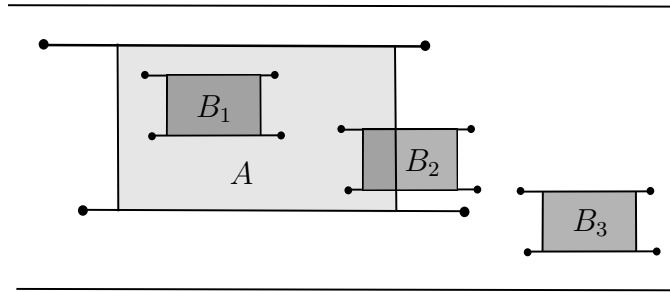


FIGURE 2.27. Lemma 2.12.4 (1):  $A, B_1, B_2, B_3 \in \mathcal{B}[0]$

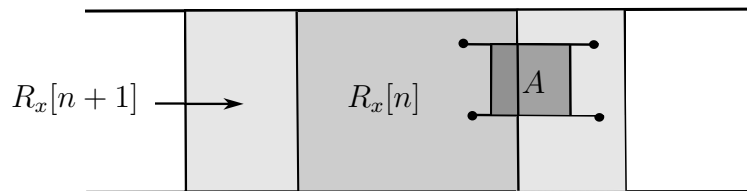


FIGURE 2.28. Lemma 2.12.4 (2)

**Lemma 2.12.5.** *The same statement holds in  $\Theta(M)$ , with  $R'_x[n]$  replacing  $R_x[n]$  and  $B'[n]$  replacing  $B[n]$  etc.*

We also have:

**Lemma 2.12.6.** *Suppose that  $p, q \in \mathbb{N}$ .*

(1) *If  $x \in \Theta(P)$ , the volume of  $R_x[p] \setminus \bigcup \mathcal{B}[q]$  is bounded above in terms of  $p$  and  $q$ .*



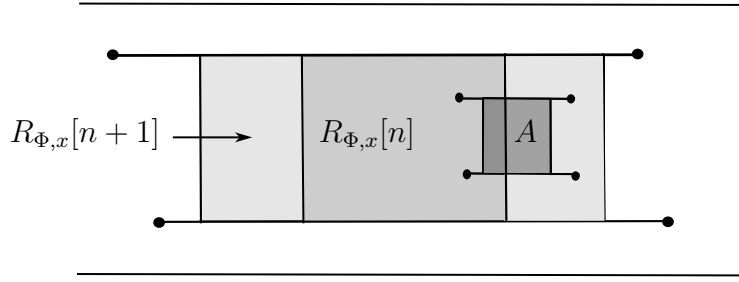


FIGURE 2.29. Lemma 2.12.4 (3)

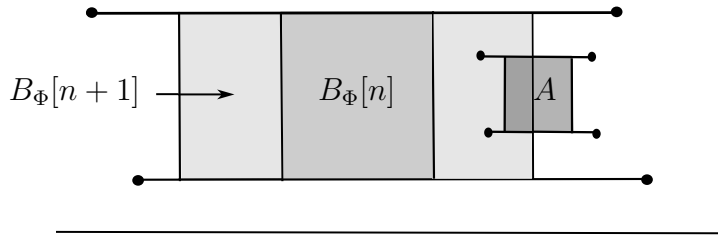


FIGURE 2.30. Lemma 2.12.4 (4)

(2) If  $\Phi \in \mathcal{F}[p]$  and  $x \in B_{\Phi}[p]$ , then the volume  $R_{\Phi,x}[p] \setminus \bigcup \mathcal{B}_{\Phi}[q]$  is bounded above in terms of  $p$  and  $q$ .

*Proof.* The riemannian notions of distance and volume (with respect to  $\rho$ ) are linearly bounded in terms of the combinatorial notions used in Section 2.4. This is therefore a direct corollary of Lemma 2.4.5.  $\square$

**Lemma 2.12.7.** Suppose that  $p, q \in \mathbb{N}$ .

(1) If  $x \in \Theta(P)$ , the volume of  $R'_x[p] \setminus \bigcup \mathcal{B}'[q]$  is bounded above in terms of  $p$  and  $q$ .

(2) If  $\Phi \in \mathcal{F}[p]$  and  $x \in B_{\Phi}[q]$ , then the volume  $R'_{\Phi,x}[p] \setminus \bigcup \mathcal{B}'_{\Phi}[q]$  is bounded above in terms of  $p$  and  $q$ .

*Proof.*

(1) By Lemmas 2.12.1(1) and 2.12.2(2) and the fact that  $f$  is surjective, we have

$$R'_x[p] \setminus \bigcup \mathcal{B}'[q] \subseteq f(R_x[p+1] \setminus \bigcup \mathcal{B}[q+1]).$$

Since  $f$  is uniformly lipschitz, the volume of the right hand side is bounded by Lemma 2.12.6(1), and the result follows.

(2) By Lemmas 2.12.1(1) and 2.12.2(2) we have:

$$R'_{\Phi,x}[p] \setminus \bigcup \mathcal{B}'_{\Phi}[q] \subseteq f(R_{\Phi,x}[p+1] \setminus \bigcup \mathcal{B}_{\Phi}[q+1])$$

and the result follows by Lemma 2.12.6(2).  $\square$

In fact, using Lemma 2.9.1, we see that we can also bound the volume of an  $\eta$ -neighbourhood of these sets in terms of  $\eta$ .

We can now set about verifying the hypotheses of Lemma 2.9.4. Fix some constant  $\eta$  less than the injectivity radius of  $\Theta(M)$ .

**Proposition 2.12.8.** *Suppose  $x, y \in \Theta(P)$  and  $d'(f(x), f(y)) \leq \eta$ . Then there is a path,  $\alpha$ , in  $\Theta(P)$ , of bounded diameter with respect to the metric  $d$ , such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $\Theta(M)$  of bounded diameter with respect to the metric  $d'$ .*

The conclusion of Proposition 2.12.8 determines a homotopy class of path from  $x$  to  $y$  in  $\Theta(P)$ , which we shall refer to as the **right homotopy class**.

The basic strategy is to start with any path  $\alpha$  from  $x$  to  $y$  in the right homotopy class. (Such a path exists, since  $f$  is a homotopy equivalence from  $\Theta(P)$  to  $\Theta(M)$ .) We first push this into a region of bounded depth about  $x$ , and then push it off all bands of a given bounded depth. Lemma 2.12.6 then gives a bound on the diameter of such a path in  $(\Theta(P), d)$ . By our choice of  $\alpha$ ,  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $\Theta(M)$ . We now push this disc into a region of bounded depth, and then off bands of bounded depth. Lemma 2.12.7 then bounds the diameter of this disc in  $(\Theta(M), d')$ . In practice we will only need to construct bands up to depth 10 for the following discussion. (The proof Lemma 2.12.7 eventually takes us up to depth 11.)

Let us first deal with the case where  $x, y \notin \bigcup \mathcal{B}[6]$ . We connect  $x$  to  $y$  by a path,  $\alpha$ , in the right homotopy class. We abbreviate  $R[n] = R_x[n]$ . Now  $d'(f(x), f(CR[1])) \geq \eta$  and so  $y \in R[1]$ . We first claim that we can push  $\alpha$  into  $R[2]$ .

To see this we pass to the universal covers,  $\tilde{f} : \tilde{\Theta}(P) \rightarrow \tilde{\Theta}(M)$ . Let  $\mathcal{S}[i] = \{\partial_- R[i], \partial_+ R[i]\}$  and  $\mathcal{S}'[i] = \{\partial_- R'[i], \partial_+ R'[i]\}$ . Let  $\mathcal{S} = \mathcal{S}[1] \cup \mathcal{S}[2]$  and  $\mathcal{S}' = \mathcal{S}'[1] \cup \mathcal{S}'[2]$ . Now  $R[1] \subseteq R[2] \subseteq R[3]$ , and so applying Lemma 2.12.1, given any two distinct surfaces,  $S, S' \in \mathcal{S}$ , we can construct a ray  $\beta$  from  $S$  to infinity such that  $f(\beta) \cap f(S') = \emptyset$ . (We only really need to do this if  $S$  and  $S'$  are parallel in  $\Theta(P)$ .) We have verified the hypotheses of Lemma 2.11.6 for the map  $f : \Theta(P) \rightarrow \Theta(M)$ . In particular, the lifts of the surfaces in  $\mathcal{S}$  have the same separation properties in  $\tilde{\Theta}(P)$  as the corresponding lifts of surfaces in  $\mathcal{S}'$  to  $\tilde{\Theta}(M)$  (as described by Corollary 2.11.2).

Let  $\tilde{\alpha}$  be a lift of the path  $\alpha$  to  $\tilde{\Theta}(P)$  connecting  $\tilde{x}$  to  $\tilde{y}$ . A lift,  $\delta$ , of the short geodesic  $[f(x), f(y)]$  connects  $\tilde{f}(\tilde{x})$  to  $\tilde{f}(\tilde{y})$  in  $\tilde{\Theta}(M)$ . Now if  $\alpha \subseteq R[1]$ , there is nothing to prove. If not, let  $z, w \in \alpha$  be the first and last intersection of  $\alpha$  with  $\partial_H R[1]$  (the relative boundary of  $R[1]$  in  $\Theta(P)$ ). We have  $z \in S_0 \in \mathcal{S}[1]$  and  $w \in S_1 \in \mathcal{S}[1]$ . Let  $\beta, \gamma$  be the subpaths of  $\alpha$  from  $x$  to  $z$  and  $y$  to  $w$  respectively, and let  $\tilde{z}, \tilde{w}, \tilde{\beta}, \tilde{\gamma}, \tilde{S}_0$  and  $\tilde{S}_1$  be the lifts to  $\tilde{\Theta}(P)$ . Now  $\beta, \gamma \subseteq R[1]$  and so, by Lemma 2.12.1,  $f(\beta), f(\gamma) \subseteq R'[2]$ . Thus  $\tilde{f}(\tilde{\beta}) \cup \delta \cup \tilde{f}(\tilde{\gamma})$  is a path in  $\tilde{\Theta}(M)$  connecting  $\tilde{f}(\tilde{z})$  to  $\tilde{f}(\tilde{w})$  and not meeting  $\bigcup \tilde{\mathcal{S}}'[2]$ . Thus  $\tilde{f}(\tilde{S}_0)$  and  $\tilde{f}(\tilde{S}_1)$  are not

separated by any element of  $\tilde{\mathcal{S}}'[2]$ . Since the corresponding surfaces  $\tilde{S}'_0$  and  $\tilde{S}'_1$  lie in arbitrarily small neighbourhoods of  $\tilde{f}(\tilde{S}_1)$  and  $\tilde{f}(\tilde{S}'_1)$ , these are not separated by any element of  $\tilde{\mathcal{S}}'[2]$ . As described above, the surfaces in  $\tilde{\Theta}(P)$  have the same separation properties, and so  $\tilde{S}_0$  and  $\tilde{S}_1$  are not separated by any element of  $\mathcal{S}[2]$ . We can thus connect  $\tilde{z}$  to  $\tilde{w}$  by a path in  $\tilde{\Theta}(P)$  not meeting  $\bigcup \mathcal{S}[2]$ . Together with the paths  $\tilde{\beta}$  and  $\tilde{\gamma}$ , this gives a path from  $\tilde{x}$  to  $\tilde{y}$ . Projecting back down to  $\Theta(P)$ , this gives a path from  $x$  to  $y$  in  $R[2]$  in the right homotopy class, as claimed.

In fact, we can refine the above observation slightly. Note that every time  $\tilde{\alpha}$  crosses some component of  $\mathcal{S}[2]$  it must eventually cross back again, and so we can replace the intervening path by a path in an arbitrarily small neighbourhood of this component. Projecting to  $\Theta(P)$ , we see that we can find a new path  $\alpha$  in the right homotopy class in  $R[2]$  and in an arbitrarily small neighbourhood of our original  $\alpha$  union  $\partial_H R[2]$ . We refer to this operation as “pushing  $\alpha$  into  $R[2]$ ”.

Our next job is to push  $\alpha$  off every level 7 band. The general procedure is as follows. The general procedure is as follows. Suppose that  $B[7] \in \mathcal{B}[7]$ . By our initial assumption,  $x, y \notin B[6]$ . We can now apply the above argument, with  $CB[6]$  playing the role of  $R[2]$  and  $CB[7]$  playing the role of  $R[3]$  to push  $\alpha$  off  $B[7]$ . In other words, we replace  $\alpha$  by another path in the right homotopy class in  $CB[7]$ , and in a small neighbourhood of our previous  $\alpha$  union  $\partial_H B[7]$ . Our new path might now leave  $R[2]$ , however, since the pushing operations took place inside  $B[6]$  and so certainly inside  $B[0]$ , Lemma 2.12.4(2) ensures that the resulting path lies inside  $R[3]$ .

We want to perform this construction for all level 7 bands, however there is a risk that the various “pushing” operations may interfere with each other. We therefore proceed by (reverse) induction on the complexity of the bands. By Lemma 2.12.4(1), any two level 0 bands of the same complexity are disjoint, and therefore the pushing operations on such bands can be performed simultaneously (or more precisely, in any order). We thus start with the level 7 bands of complexity  $\xi(\Sigma) - 1$ , and then move onto those of complexity  $\xi(\Sigma) - 2$  and continue all the way down to bands of complexity 1 (observing that there are no 3HS bands). The pushing operations of a given complexity may affect those already performed at a higher complexity, but Lemma 2.12.4 parts (1) and (4) ensure that we will never enter a level 8 band. Again, Lemma 2.12.4(2) ensures we remain inside  $R[3]$ . We thus end up with a path  $\alpha \subseteq R[3] \setminus \bigcup \mathcal{B}[8]$  in the right homotopy class.

Now by Lemmas 2.12.1 and 2.12.2,  $f(\alpha) \subseteq R'[4] \setminus \bigcup \mathcal{B}'[9]$ . Since  $\alpha$  lies in the right homotopy class,  $f(\alpha) \cup [f(x), f(y)] \subseteq R'[4] \setminus \bigcup \mathcal{B}'[9]$  bounds (the continuous image of) a disc  $D$  in  $\Theta(M)$ . Now the boundaries,  $\partial_{\pm} R'[4]$  are incompressible in  $\Theta(M)$ , and so we can push  $D$  into  $R'[4]$ , so that the resulting disc lies in a small neighbourhood of our original disc union  $\partial_H R'[4]$ .

Next, we push  $D$  off all level 10 bands in  $\Theta(M)$ , by reverse induction on complexity as before. For this we only need to observe that the boundaries of bands are incompressible. By Lemma 2.12.5, we end up with a disc  $D$  lying in  $R'[5] \setminus \bigcup \mathcal{B}'[10]$ .

In summary, we have found  $\alpha \subseteq R[3] \setminus \bigcup \mathcal{B}[8]$  such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $R'[5] \setminus \bigcup \mathcal{B}'[10]$ . Using Lemma 2.12.6, we see that the diameter of  $\alpha$  in  $(\Theta(P), d)$  is bounded. Using Lemma 2.12.7, and the subsequent remark about the  $\eta$ -neighbourhood, we see that the diameter of the disc is bounded in  $(\Theta(M), d')$ . This proves Proposition 2.12.8 in this case.

All the above was done under the assumption that  $x, y \notin \bigcup \mathcal{B}[6]$ .

We now move on the case where  $x$  or  $y$  lies in some level 6 band. Among all bands in  $\mathcal{B}[6]$  that meet  $\{x, y\}$  we choose one, say  $B[6]$ , of minimal complexity. We can assume that  $x \in B[6]$ . Let  $\Phi = \pi_\Sigma B[6]$  be the base surface. By the minimal complexity assumption, we see that  $x, y \notin \bigcup \mathcal{B}_\Phi[6]$ . Let  $R_\Phi[n] = R_{\Phi, x}[n]$ . Since  $x \in B[6]$ , we get that  $R[5]$  exists and lies inside  $B[0]$ .

We can now carry out the above construction, with  $R_\Phi[n]$  replacing  $R[n]$ , and with  $\mathcal{B}_\Phi[n]$  replacing  $\mathcal{B}[n]$ . In this way, we get a path  $\alpha \subseteq R_\Phi[3] \setminus \bigcup \mathcal{B}_\Phi[8]$  such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $R'_\Phi[5] \setminus \bigcup \mathcal{B}'_\Phi[10]$ . By Lemmas 2.12.6 and 2.12.7 again, we see that these have bounded diameter in  $(\Theta(P), d)$  and  $(\Theta(M), d')$  respectively.

This proves Proposition 2.12.8.

Finally, putting Proposition 2.12.8 together with Lemma 2.9.4, we get:

**Proposition 2.12.9.** *The map  $f : (\Theta(P), d) \longrightarrow (\Theta(M), d')$  (constructed as in Section 2.8) is uniformly universally sesquilipschitz.*

In other words, it is lipschitz and it lifts to a quasi-isometry,  $\tilde{f} : \tilde{\Theta}(P) \longrightarrow \tilde{\Theta}(M)$ . The constants involved only depend on  $\xi(\Sigma)$ .

### 2.13. THE DOUBLY DEGENERATE CASE.

In this section, we gather our constructions together to show that two doubly degenerate hyperbolic 3-manifolds with the same pairs of end invariants are isometric (Theorem 2.13.11).

Let  $M$  be a doubly degenerate 3-manifold. In other words,  $\Psi(M)$  is homeomorphic to  $\Sigma \times \mathbb{R}$ , and both ends are degenerate. We write  $e_-$  and  $e_+$  for the positive and negative ends of  $\Psi(M)$  respectively. This gives us two end invariants,  $a(e_-), a(e_+) \in \partial \mathcal{G}$ .

**Lemma 2.13.1.**  $a(e_-) \neq a(e_+)$ .

Given the description of  $\partial \mathcal{G}(\Sigma)$  in [Kla], this is equivalent to the fact proven in [Bon] that the two end invariants, considered as laminations, are distinct. One can also prove Lemma 2.13.1 from the results given here.

*Proof.* Suppose for contradiction that  $a(e_-) = a(e_+) = a$ , say. Let  $(\gamma_i^\pm)_{i \in \mathbb{N}}$  be a sequence of curves,  $\gamma_i^\pm \rightarrow a$ , all with bounded-length representatives in  $M$ , and tending out the end  $e_\pm$ . Let  $(\gamma_{i,j})_j$  be a geodesic from  $\gamma_i^-$  to  $\gamma_i^+$  in  $\mathcal{G}(\Sigma)$ . Then, by

a-priori bounds (Theorem 1.6.12),  $\gamma_{i,j}^*$  all have bounded length. Since the distance between  $\gamma_{i,j}^*$  and  $\gamma_{i,j+i}^*$  is bounded in the electric pseudometric on  $\Psi(M)$ , we see that for all sufficiently large  $i$ , there is some  $j(i)$  such that  $\gamma_{i,j(i)}^*$  meets some fixed compact subset of  $M$ . Write  $\delta_i = \gamma_{i,j(i)}$ . Now since each  $\delta_i^*$  has bounded length, there are only finitely many possibilities for  $\delta_i$ . But (using the hyperbolicity of  $\mathcal{G}(\Sigma)$ ),  $\delta_i \rightarrow a$  in  $\mathcal{G}(\Sigma) \cup \partial\mathcal{G}(\Sigma)$ , giving a contradiction. This proves Lemma 2.13.1.  $\square$

We are now in a position to construct our model space for  $M$ .

To begin, Theorem 2.4.1 associates to the pair  $a(e_-), a(e_+)$  a complete annulus system,  $W = \bigcup \mathcal{W} \subseteq \Psi = \Sigma \times \mathbb{R}$ . (There might, of course, be many annulus systems satisfying the conditions of Theorem 2.4.1. We arbitrarily choose one of them.) The construction of Section 2.5 now gives us a riemannian manifold  $(\Psi(P), d)$  — first open up each annulus into a torus, and then glue in a Margulis tube. Thus,  $\Psi(P)$  is also homeomorphic to  $\Sigma \times \mathbb{R}$ , and there is a natural (topological) proper homotopy equivalence of  $\Psi(P)$  with  $\Psi(M)$ . Each component of  $\partial\Psi(P)$  is a bi-infinite cylinder isometric to  $S^1 \times \mathbb{R}$ , in the induced path metric. We now construct  $P$  by gluing in a standard  $\mathbb{Z}$ -cusp to each such boundary component (a quotient of a horoball in  $\mathbb{H}^3$  by a  $\mathbb{Z}$ -action). Thus  $P$  is a complete riemannian manifold with empty boundary.

To relate the geometry of  $P$  to the geometry of  $M$ , we will again need the A-priori Bounds Theorem (Theorem 1.6.2 here) as well as the following strengthening to allow for subsurfaces. Recall that  $l_M(\gamma)$  is the length of the closed geodesic representative of  $\gamma$  in  $M$ .

**Theorem 2.13.2.** *Suppose that  $\Phi \subseteq \Sigma$  is a subsurface. Suppose that  $\alpha, \beta, \gamma \in \mathbf{C}(\Phi)$  and that  $\gamma$  lies in a tight geodesic from  $\alpha$  to  $\beta$  in  $\mathcal{G}(\Phi)$ . Then  $l_M(\gamma)$  is bounded above in terms of  $\xi(\Sigma)$ ,  $l_M(\alpha)$ ,  $l_M(\beta)$  and  $\max\{l_M(\delta)\}$  as  $\delta$  ranges over the boundary components  $\partial^\Sigma\Phi = \partial\Phi \setminus \partial\Sigma$ .*

This is proven in this form in [Bow4]. A similar result was shown in [Mi4].

Note that we can allow for  $\Phi = \Sigma$  — this is Theorem 1.6.11. We do not require that  $M$  is doubly degenerate for this result, only that  $\Psi(M)$  is homeomorphic to  $\Sigma \times \mathbb{R}$ , possibly with a number of cusps corresponding to accidental parabolics removed. If  $\alpha$  is homotopic to such a parabolic, we set  $l_M(\alpha) = 0$ . We need not worry about this in this section, though it cannot be avoided in general — see Section 3.1.

By induction, we see that if  $\alpha, \beta, \gamma \in \mathbf{C}(\Sigma)$  and  $\gamma \in Y^\infty(\{\alpha, \beta\})$  as defined in Section 2.2, then  $l_M(\gamma)$  is bounded above in terms of  $\xi(\Sigma)$ ,  $l_M(\alpha)$  and  $l_M(\beta)$ .

Let  $a = a(e_-)$  and  $b = a(e_+)$ . In Section 2.4, we constructed the annulus system  $\mathcal{W}$  out of a sequence of sets of the form  $Y(\alpha_i, \beta_i)$  as  $\alpha_i \rightarrow a$  and  $\beta_i \rightarrow b$ . By the defining property of an end invariants, we can choose  $\alpha_i$  and  $\beta_i$  so that  $l_M(\alpha_i)$  and  $l_M(\beta_i)$  remain bounded, and so by Theorem 2.13.2, all the curves we construct will have bounded length in  $M$ . However, the choice of these sequences might

depend on  $M$ . To see that all the curves have bounded length for any choice of sequences, we need Theorem 1.6.12, which we can equivalently state as follows:

**Theorem 2.13.3.** *Given any  $r \geq 0$  there is some  $r' \geq 0$  such that if  $\alpha, \beta, \gamma \in \mathbf{C}(\Sigma)$  and  $\gamma$  lies on a tight geodesic from  $\alpha$  to  $\beta$  and  $d(\alpha, \gamma) \geq r'$  and  $d(\beta, \gamma) \geq r'$ , then  $l_M(\gamma)$  is bounded above in terms of  $\xi(\Sigma)$ ,  $\min\{l_M(\delta) \mid \delta \in N(\alpha, r)\}$  and  $\min\{l_M(\epsilon) \mid \epsilon \in N(\gamma, r)\}$ .*

Suppose that  $\alpha'_i \rightarrow a$  and  $\beta'_i \rightarrow b$  are any two sequences tending to these end invariants, and that  $\gamma$  lies in some tight geodesic from  $\alpha'_i$  to  $\beta'_i$  for infinitely  $i$ . Now for large enough  $i$ , the geodesics from  $\alpha_i$  to  $\beta_i$  and from  $\alpha'_i$  to  $\beta'_i$  lie a bounded distance apart in arbitrarily large metric balls about  $\gamma$ . In particular, the hypotheses of Theorem 2.13.3 apply for suitable  $\alpha, \beta$ , in the geodesic from  $\alpha'_i$  to  $\beta'_i$ , and it follows that  $\gamma^*$  has bounded length. It now follows that all curves which lie in infinitely many of the sets  $Y(\alpha'_i, \beta'_i)$  have bounded realisations in  $M$ .

In summary, we can now take any sequences  $\alpha_i \rightarrow a$  and  $\beta_i \rightarrow b$  in  $\mathcal{G}(\Sigma) \cup \partial\mathcal{G}(\Sigma)$ , and construct the annulus system,  $\mathcal{W}$ , as above. (This makes no reference to  $M$  apart from its end invariants.) By the previous paragraph, all the closed geodesics in  $M$  arising will have bounded length. In other words:

**Proposition 2.13.4.** *There is some constant  $L \geq 0$  such that if  $\Omega \in \mathcal{W}$ , then  $l_M(\Omega^*) \leq L$ , where  $\Omega^*$  is the closed geodesic in  $M$  in the homotopy class of  $\Omega$ .*

This is precisely the hypothesis (APB) of Section 2.8 that allowed us to construct the map  $f : \Psi(P) \rightarrow \Psi(M)$ . In particular, Theorem 2.8.2 gives us a partition,  $\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1$  such that  $f : \Theta(P) \rightarrow \Theta(M)$  is a proper lipschitz homotopy equivalence of the thick parts. (We remark that the partition of  $\mathcal{W}$  and hence the definition of  $\Theta(P)$  might depend on  $M$ , but that does not affect the logic of the argument. The construction of  $P$  and  $\Psi(P)$  only made reference to the end invariants,  $a, b$ .)

**Lemma 2.13.5.** *The map  $f$  sends the positive (negative) end of  $\Psi(P)$  to the positive (negative) end of  $\Psi(M)$ .*

*Proof.* Let  $\Omega_i$  be a sequence of annuli tending out the positive end of  $\Psi(P)$ . Now  $f$  sends  $\partial T(\Omega_i)$  either to the associated geodesic  $\Omega_i^*$  in  $\Psi(M)$ , or else to the boundary of the Margulis tube about  $\Omega_i^*$ . In any case, since  $f$  is proper, the sequence  $\Omega_i^*$  must go out an end,  $e$ , of  $\Psi(M)$ . By construction of  $\mathcal{W}$ , the homotopy classes of  $\Omega_i$  tend to  $a(e_+)$  in  $\mathcal{G} \cup \partial\mathcal{G}$  and so by the definition of end invariant (Proposition 1.5.3), we see that  $a(e) = a(e_+)$ , and so, by Lemma 2.13.1,  $e = e_+$ , as required.  $\square$

This proves the end consistency assumption (EC) of Section 2.8, and so  $f(\Psi(P)) = \Psi(M)$ .

We remark that we have all that is needed to show that the collection of Margulis tubes in  $\Psi(M)$  is unlinked, that is Theorem 2.1.1. (We only need the constructions of Sections 2.8 and 2.10 for this.)

*Proof of Theorem 2.1.1 :* By Proposition 2.8.3, the set of tubes  $T_0(\Omega)$  are unlinked in  $\Psi(M)$ . But this includes all Margulis tubes with core curves less than some constant  $\eta > 0$  depending only on  $\xi(\Sigma)$ .  $\square$

We remark that, unlike [Ot3], this does not give us an effective computable,  $\eta$  explicitly in terms of  $\xi(\Sigma)$  (since it depends on the A-priori Bounds Theorem which is not effective).

Also, the fact that  $f$  has degree 1 was all that was needed to get us to Proposition 2.12.9, and so we see that the map  $f : \Theta(P) \rightarrow \Theta(M)$  is uniformly universally sesquilipschitz.

For the moment,  $f$  is only defined topologically on each of the Margulis tubes in  $\mathcal{T}(P)$ . If  $T \in \mathcal{T}(P)$ , then we have a lipschitz map  $f : \partial T \rightarrow \partial T'$ .

**Lemma 2.13.6.** *If  $T \in \mathcal{T}(P)$ , then  $f$  extends to a uniformly universally sesquilipschitz map,  $f : T \rightarrow T'$ .*

In other words, the extension,  $f$  is uniformly lipschitz and its lift to the universal covers,  $\tilde{f} : \tilde{T} \rightarrow \tilde{T}'$  is a quasi-isometry.

*Proof.* Let  $\tilde{\Theta}(P)$  and  $\tilde{\Theta}(M)$  be the universal covers of  $\Theta(P)$  and  $\Theta(M)$ , and let  $\hat{\Theta}(P) = \tilde{\Theta}(P)/H$  and  $\hat{\Theta}(M) = \tilde{\Theta}(M)/H$  be the covers corresponding to the subgroup  $H$  of  $G = \pi_1(\Theta(P)) \cong \pi_1(\Theta(M))$  generated by the longitude of  $T$ . We can identify  $\partial\tilde{T}$  and  $\partial\tilde{T}'$  with boundary components of  $\hat{\Theta}(P)$  and  $\hat{\Theta}(M)$  respectively. In the induced path metrics, they are euclidean cylinders whose longitudes have length uniformly bounded above and below.

By Proposition 2.12.9, the map  $\tilde{f} : \tilde{\Theta}(P) \rightarrow \tilde{\Theta}(M)$  is a lipschitz quasi-isometry, and so therefore is its projection,  $\hat{f} : \hat{\Theta}(P) \rightarrow \hat{\Theta}(M)$ . By Lemma 2.5.3,  $\partial\tilde{T}$  is quasi-isometrically embedded in  $\hat{\Theta}(P)$ , and so we can conclude that  $\hat{f}|_{\partial\tilde{T}}$  is a quasi-isometry from  $\partial\tilde{T}$  to  $\partial\tilde{T}'$  in the induced euclidean path metrics.

We are therefore in the situation described by Lemma 2.6.4 and the subsequent remark. In particular, there is a universally sesquilipschitz homotopy from  $f|_{\partial T} : \partial T \rightarrow \partial T'$  to a bilipschitz homeomorphism  $g : \partial T \rightarrow \partial T'$ . By Lemma 2.6.8, such a map  $g$  extends to a bilipschitz homeomorphism  $g : T \rightarrow T'$ .  $\square$

Now we can carry out the sesquilipschitz homotopy between  $f|_{\partial T}$  and  $g|_{\partial T}$  in a uniformly small neighbourhood of  $\partial T$  in  $T$ , and then use  $g$  to extend over  $T$ . This way, we extend  $f$  to a universally sesquilipschitz map  $f : T \rightarrow T'$ .

Performing this for each tube  $T \in \mathcal{T}(P)$  we get a lipschitz map  $f : \Psi(P) \rightarrow \Psi(M)$ . One can show this to be universally sesquilipschitz (cf. Proposition 2.13.8), but we are really interested in a further extension of  $f$  to the whole model space  $P$ . For this we still need to deal with the cusps.

Let  $R$  be a cusp of  $P$ , i.e. the closure of a component of  $P \setminus \Psi(P)$ . We have a corresponding cusp in  $R'$  in  $M$ , the closure of a component of  $M \setminus \Psi(M)$ . We have a proper lipschitz map  $f|_{\partial R} : \partial R \rightarrow \partial R'$ , between bi-infinite euclidean cylinders.

**Lemma 2.13.7.** *The map then  $f|_{\partial R}$  extends to a uniformly universally sesquilipschitz map,  $f : R \rightarrow R'$ .*

*Proof.* The argument is similar to that for Lemma 2.13.6. Using Lemma 2.5.3 as before, we see that  $f|_{\partial \hat{R}}$  is a uniform quasi-isometry to  $\partial \hat{R}'$ . (In this case, we lift to  $\hat{R}$  rather than to the universal cover of  $\hat{R}$ .) Note that  $\partial R$  and  $\partial R'$  are both uniformly quasi-isometric to the real line, under horizontal projection. We can apply Lemma 2.6.3 directly to see that there is a bounded homotopy to a bilipschitz homeomorphism of the real line. Thus (as in Lemma 2.6.4) we get a universally sesquilipschitz homotopy from  $f|_{\partial R}$  to a bilipschitz homeomorphism  $g : \partial R \rightarrow \partial R'$ . (We use that fact that a lipschitz map from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  which sends horizontal lines to horizontal lines and is bilipschitz in the vertical direction must, in fact, be bilipschitz.) The extension of  $g$  over  $R$  is now trivial — just send rays isometrically to rays.  $\square$

Performing this for each cusp we get a proper lipschitz homotopy equivalence,  $f : P \rightarrow M$ .

**Proposition 2.13.8.** *The map  $f : P \rightarrow M$  is uniformly universally sesquilipschitz.*

*Proof.* In other words, we claim that the lift  $\tilde{f} : \tilde{P} \rightarrow \tilde{M}$  is a uniform quasi-isometry. Since  $\tilde{f}$  is surjective, it is enough to put an upper bound on  $d(x, y)$  whenever  $d'(\tilde{f}(x), \tilde{f}(y)) < \eta$  for some fixed  $\eta > 0$ . But  $\tilde{P}$  and  $\tilde{M}$  are equivariantly decomposed into pieces, namely the lifts of thick parts, Margulis tubes, and cusps. We have shown that  $\tilde{f}$  respects this decomposition and that  $\tilde{f}$  restricted to each of the pieces is a uniform quasi-isometry. Moreover, we can assume that any two distinct pieces are distance at least  $\eta$  apart in  $\tilde{M}$ . The result now follows.  $\square$

We can summarise what we have shown as follows:

**Theorem 2.13.9.** *Given two distinct  $a, b \in \partial \mathcal{G}$ , we can construct a complete riemannian manifold,  $P$ , homeomorphic to  $\text{int}(\Sigma) \times \mathbb{R}$ , such that if  $M$  is a doubly degenerate hyperbolic 3-manifold with base surface  $\Sigma$  and end invariants  $a, b$ , then there is a uniformly universally sesquilipschitz map from  $P$  to  $M$ .*

Here “uniform” means that the constants depend only on  $\xi(\Sigma)$ . (I don’t know if this dependence is computable.)

To complete the proof we note:

**Lemma 2.13.10.** *If  $M = \mathbb{H}^3/\Gamma$  is doubly degenerate, then the limit set of  $\Gamma$  is all of  $\partial \mathbb{H}^3$ .*



This is a well known consequence of tameness. From our discussion it can easily be seen from the fact that any point of  $M$  lies on a homotopically non-trivial curve of bounded length. In fact, we observed in Section 1.4 that the geometrically finite ends of  $M$  correspond to the components of  $D(\Gamma)/\Gamma$ . In this case, there are none.

As a consequence we have:

**Theorem 2.13.11.** *Suppose  $M, M'$  are doubly degenerate hyperbolic 3-manifolds with the same base surface and end invariants. Then  $M$  and  $M'$  are isometric.*

*Proof.* The argument is now standard. We can use the same model space  $P$  for both  $M$  and  $M'$ . The universally sesquiplipschitz maps,  $P \rightarrow M$  and  $P \rightarrow M'$  give us an equivariant quasi-isometry between  $\tilde{M}$  and  $\tilde{M}'$ , both isometric to  $\mathbb{H}^3$ . This extends to an equivariant quasiconformal map  $\partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$ . Given Lemma 2.13.10 here, the result of Sullivan [Su] (given as Theorem 1.6.8 here) now tells us that this is in fact conformal. Thus the two actions on  $\mathbb{H}^3$  are conjugate by an isometry of  $\mathbb{H}^3$ , and so  $M$  is isometric to  $M'$ .  $\square$

In other words, we have proven the Ending Lamination Theorem (Theorem 1.5.4) for doubly degenerate product manifolds.

## 2.14. THE PROOF OF THE MAIN THEOREM IN THE INDECOMPOSABLE CASE.

In this section we describe how earlier arguments can be adapted to construct a model space for an indecomposable (orientable) complete hyperbolic 3-manifold. This will enable us to complete the proof of the Ending Lamination Theorem in that case. With some further modifications, the decomposable case can also be dealt with similarly.

### 2.14.1. Topological observations.

We have already referred to topological finiteness in Sections 1.2 and 1.4. We recall the definition.

**Definition.** A 3-manifold  $\Psi$  with boundary,  $\partial\Psi$ , is *topologically finite* if we can embed  $\Psi$  in a compact 3-manifold,  $\bar{\Psi}$ , with boundary,  $\partial\bar{\Psi}$ , so that  $\partial\Psi$  is a subsurface of  $\partial\bar{\Psi}$ , and  $\bar{\Psi} = \Psi \cup \partial\bar{\Psi}$ .

In other words, we can compactify  $\Psi$  by adjoining the “ideal” boundary  $\partial_I\Psi = \partial\bar{\Psi} \setminus \partial\Psi$ . In fact, the topology of the pair  $(\bar{\Psi}, \Psi)$  is determined by  $\Psi$ , though here we can regard  $\bar{\Psi}$  as part of the structure associated to  $\Psi$ .

Suppose that  $\Psi$  is topologically finite. In this case the existence of a Scott Core (Theorem 1.4.1 here) is elementary, and (in the indecomposable case, as here) it is equivalent to the following definition:

**Definition.** By a *core* of  $\Psi$  we will mean a compact submanifold,  $\Psi_0 \subseteq \Psi$ , such that  $\Psi \setminus \Psi_0$  is homeomorphic to  $\partial_I\Psi \times \mathbb{R}$ .

(In fact, in this case,  $\Psi \setminus \text{int } \Psi_0$  is homeomorphic to  $\partial_I \Psi \times [0, \infty)$  with  $\partial_I \Psi$  identified with the relative boundary of  $\Psi_0$  in  $\Psi$ .)

We write  $\partial_H \Psi_0$  for the relative, or *horizontal* boundary of  $\Psi_0$  in  $\Psi$ , and  $\partial_V \Psi_0 = \Psi_0 \cap \partial \Psi$  for the *vertical* boundary. Thus  $\partial \Psi_0 = \partial_H \Psi_0 \cup \partial_V \Psi_0$ . The ends of  $\Psi$  are in bijective correspondence with the components of  $\partial_H \Psi_0$ , and are just products.

**Definition.** An end  $e \cong \Sigma \times [0, \infty)$  is incompressible in  $\Psi$  if its inclusion into  $\Psi$  is  $\pi_1$ -injective.

(This is clearly independent of the choice of product neighbourhood of  $e$ .)

**Definition.** We say that  $\Psi$  is *indecomposable* if all its ends are incompressible.

Via Dehn's lemma, this is equivalent to saying that there is no disc in  $\bar{\Psi}$  whose boundary lies in  $\partial_I \Psi$ . It is also equivalent to the definition given in Section 1.4.

### 2.14.2. *The hyperbolic manifold.*

Now suppose that  $M$  is a complete orientable hyperbolic 3-manifold, and that  $\pi_1(M)$  is finitely generated. We shall also assume that  $M$  is not elementary, i.e. that  $\pi_1(M)$  is not abelian. (For elementary 3-manifolds, the Ending Lamination Theorem is elementary.) Let  $\Psi(M)$  be the non-cuspidal part of  $M$ . Here we will be assuming that  $\Psi(M)$  is indecomposable, and so  $\Psi(M)$  is topologically finite by [Bon].

We also remark that indecomposability is equivalent to saying that  $\pi_1(M)$  does not split as a free product relative to the maximal parabolic subgroups. (This is the formulation given in [Bon].)

Let  $\Psi_0$  be a core for  $\Psi(M)$ . We note that each component of  $\partial \Psi(M)$  is either a bi-infinite euclidean cylinder, which meet  $\Psi_0$  in a compact annular component of  $\partial_V \Psi_0$ ; or else is a torus and a component of  $\partial_V \Psi_0$ . These components bound  $\mathbb{Z}$ -cusps and  $(\mathbb{Z} \oplus \mathbb{Z})$ -cusps respectively in  $M$ . We also note that no component of  $\partial_H \Psi_0$  is a disc or annulus.

We write  $\mathcal{E}(M)$  for the set of ends of  $\Psi(M)$ , and recall the partition of  $\mathcal{E}(M)$  as  $\mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$ , as discussed in Section 1.5. Let  $C = C(M)$  be the convex core of  $M$ , and let  $C(r)$  be its  $r$ -neighbourhood. If  $r > 0$ ,  $\partial C(r)$  is a  $C^1$ -submanifold of  $M$ . Let  $F$  be a component of  $\Psi(M) \setminus \text{int } C(r)$ . Its boundary,  $\partial_H F$ , is a component of  $C(r)$ . Let  $\Psi(F) = F \cap \Psi(M)$ . This is a neighbourhood of a geometrically finite end of  $\Psi(M)$ . In fact, each end of  $\Psi(M)$  has a neighbourhood of this form. We write  $\partial_H \Psi(F) = \partial_H F \cap \Psi(M)$ , and  $\partial_V \Psi(F) = \Psi(F) \cap \partial \Psi(M)$ . Note that  $\Psi(F)$  is homeomorphic to  $\Sigma \times [0, \infty)$ , with  $\partial_H \Psi(F)$  identified with  $\Sigma \times \{0\}$ , and  $\partial_V \Psi(F)$  identified with  $\partial \Sigma \times [0, \infty)$ .

By Ahlfors's finiteness theorem, each geometrically finite end has associated to it a Riemann surface of finite type, which can be thought of as a geometrically finite end invariant. For the moment, however we will not be using this structure.

We now describe the main aim of this section. Suppose that  $\Psi$  is a topologically finite 3-manifold, with a decomposition of its ends,  $\mathcal{E} = \mathcal{E}_F \sqcup \mathcal{E}_D$ , so that no base surface of an end is a disc, annulus, sphere or torus, and no base surface in  $\mathcal{E}_F$  is a 3HS. Suppose that to each  $e \in \mathcal{E}_D$  there is associated an element,  $a(e)$ , in the boundary of the corresponding curve graph. We will associate to  $\Psi$ ,  $(a(e))_{e \in \mathcal{E}_D}$ , a “model” manifold  $P$ . This will be a complete riemannian manifold together with a submanifold,  $\Psi(P)$ , homeomorphic to  $\Psi$ , such that each component of  $P \setminus \Psi(P)$  is either a “standard”  $\mathbb{Z}$ -cusp or “standard”  $(\mathbb{Z} \oplus \mathbb{Z})$ -cusp.

We will show:

**Theorem 2.14.1.** *Let  $M$  be a tame indecomposable hyperbolic 3-manifold with non-cuspidal part  $\Psi(M)$ . Let  $P$  be the model manifold referred to above, constructed from  $\Psi(M)$ , the partition of its ends,  $\mathcal{E}(M) = \mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$  into geometrically finite and degenerate, and the collection  $(a(e))_{e \in \mathcal{E}_D(M)}$  of degenerate end invariants. Then there is a universally sesquilateral map from  $P$  into  $M$ .*

Note that here we are no longer claiming that the constants of our sesquilateral map are uniform. They might depend on the geometry as well as the topology of  $M$ . We suspect that some uniform statement could be made, but one would need to take into account the geometrically finite end invariants when constructing the model space. In any case, this would considerably complicate the construction.

The doubly degenerate case of Section 2.13, is where  $\Psi(M) \cong \Sigma \times \mathbb{R}$  and  $\mathcal{E}_F(M) = \emptyset$ . In this case, we can take the constants to depend only on  $\xi(\Sigma)$ .

The basic idea is to construct a model for each end of  $\Psi(M)$ , and then put any riemannian metric on the core  $\Psi_0$ . The only requirement of the latter is that it should match up with the metric we have already on the boundaries of the model ends.

### 2.14.3. Geometrically finite ends.

Let  $\Sigma$  be a compact surface. We want to associate to  $\Sigma$  a geometrically finite model  $P_\Sigma$ . Here we just give a very crude model that only depends on the topological type of  $\Sigma$ . A more sophisticated model, which takes into account an end invariant (Riemann surface) is described in [Mi4].

Let us fix any finite-area hyperbolic structure on  $\text{int } \Sigma$ . This is given by the quotient,  $\mathbb{H}^2/H$ , of a properly discontinuous action of  $H = \pi_1(\Sigma)$  on  $\mathbb{H}^2$ . We embed  $\mathbb{H}^2$  as a totally geodesic subspace of  $\mathbb{H}^3$  and extend the action to  $\mathbb{H}^3$  (so that it preserves setwise each component of  $\mathbb{H}^3 \setminus \mathbb{H}^2$ ). Let  $P_\Sigma$  be the quotient of one of the half-spaces bounded by  $\mathbb{H}^2$ . We write  $\Psi(P_\Sigma)$  for the non-cuspidal part of  $P_\Sigma$ . Thus each component of  $\partial\Psi(P_\Sigma)$  is a euclidean half-cylinder, which (at least for notational convenience) we can assume to be isometric to  $S^1 \times [0, \infty)$ .

Note that  $P_\Sigma$  has a product structure as  $\partial P_\Sigma \times [0, \infty)$ , where the first co-ordinate of  $x \in P_\Sigma$  is the nearest point to  $x$  in  $\partial P_\Sigma$ , and the second co-ordinate,  $t = t(x)$ , is the distance of  $x$  from  $\partial P_\Sigma$ . Thus  $P_\Sigma$  is a warped riemannian product where

the distances in the horizontal (constant  $t$ ) direction are expanded by a factor of  $\cosh t$ .

A geometrically finite end of  $M$  has qualitatively similar geometry. This is well understood. We only give an outline here.

Let  $e \in \mathcal{E}_F(M)$ , and let  $F$ ,  $\Psi(F)$ , etc. be as defined above, so that  $\Psi(F)$  is a neighbourhood of the end,  $e$ . Note that  $\partial_H F$  is component of  $\partial C(r)$ . Now  $F$  has a product structure  $F \cong \partial F \times [0, \infty)$  defined exactly as with  $P_\Sigma$ . In this case, the horizontal expansion at distance  $t$  from  $\partial F$  need not be constant, but will be bounded between two constants, namely,  $k_-(t) = \cosh(t+r)/\cosh(r)$  and  $k_+(t) = \sinh(t+r)/\sinh(r)$ . We note that the ratios of both  $k_-(t)$  and  $k_+(t)$  with  $\cosh t$  are bounded above and below (in terms of  $r$ ).

Now  $\partial F$  meets each  $\mathbb{Z}$ -cusp in a constant curvature cusp and so we can find a bi-lipschitz homeomorphism  $g : \partial P_\Sigma \rightarrow \partial F$ . We can now extend, using the product structures, to a homeomorphism,  $g : P_\Sigma \rightarrow F$ , which, by the above observations will also be bilipschitz. Unfortunately, this need not send  $\Psi(P_\Sigma)$  to  $\Psi(F)$ , though it is not hard to modify it so that it does. One way to describe this procedure is as follows.

We fix some positive  $k < 1$  as described below, and choose  $g$  so that for each  $s \leq 1$  it sends any horocycle of length  $s$  in  $\partial P_\Sigma$  to a horocycle of length  $ks$  in  $\partial F$ . Now given any  $t \geq 0$ , the level  $t$  surfaces in  $P_\Sigma$  and  $F$  meet the  $\mathbb{Z}$ -cusps in cusps of constant curvature determined by  $t$  and  $r$ . Under the above construction,  $g$  will send a horocycle of length  $s \leq 1$  in such a surface in  $P_\Sigma$  to a horocycle in the corresponding surface in  $F$ , and so the length of the image horocycle is bounded above and below by fixed multiples of  $ks$ . By choosing  $k$  sufficiently small, we can assume that this length is always less than the Margulis constant. Thus,  $g$  sends each  $\mathbb{Z}$ -cusp in  $P_\Sigma$  to the corresponding  $\mathbb{Z}$ -cusp in  $F$ . We can now modify  $g$  by post-composing with the projection of such a cusp in  $F$  to its boundary, using nearest point projection in the level surfaces in  $F$ . This projection will have bounded expansion on  $g(\Psi(F))$ . The resulting map  $f : \Psi(P_\Sigma) \rightarrow \Psi(F)$  is bilipschitz.

We have shown:

**Lemma 2.14.2.** *If  $e \in \Psi(F)$  is a geometrically finite end of  $\Psi(M)$  with base surface  $\Sigma$ , then there is a bilipschitz map  $f : \Psi(P_\Sigma) \rightarrow \Psi(F)$ .*

#### 2.14.4. Degenerate ends.

We next want to construct models for degenerate ends. We can use the following variation on Theorem 2.4.3. The proof is essentially the same, indeed a more direct application of Lemma 2.4.2. Given a compact surface  $\Sigma$ , write  $\Psi_+ = \Sigma \times [0, \infty)$  and  $\partial_H \Psi_+ = \Sigma \times \{0\}$ .

**Lemma 2.14.3.** *Given a complete multicurve,  $\alpha$ , and some  $a \in \partial\mathcal{G}(\Sigma)$  we can find a complete annulus system  $W = \bigcup \mathcal{W} \subseteq \Psi_+$  with  $\pi_\Sigma(W \cap \partial_H \Psi_+) = \alpha$ , and satisfying the conditions (P1)–(P4) of Theorem 2.4.3.*

We need to interpret condition (P1) which said that  $\mathbf{C}(\mathcal{W}) \subseteq \bar{Y}^\infty(Y)$ . Here we can take  $Y$  to be the limit of the sets  $Y^\infty(\mathbf{C}(\alpha) \cup \{\beta_i\})$  where  $\beta_i \in \mathbf{C}(\Sigma)$  is some sequence converging to  $a$ . Here we are using the local finiteness properties of hierarchies (see Lemma 2.2.2), as we did in Section 2.13 (cf. Theorem 2.13.2).

Now let  $\mathcal{W}_I = \{\Omega \in \mathcal{W} \mid \Omega \cap \partial_H \Psi_+ = \emptyset\}$  and  $\mathcal{W}_\partial = \mathcal{W} \setminus \mathcal{W}_I$ . Let  $\Lambda(\mathcal{W})$  be the space obtained by opening out each annulus of  $\mathcal{W}$  as before. We have a natural map,  $p : \Lambda(\mathcal{W}) \rightarrow \Psi_+$ . Each  $\Omega \in \mathcal{W}_I$  gives us a solid torus,  $\Delta(\Omega)$ , and each  $\Omega \in \mathcal{W}_\partial$  gives us an annulus,  $A(\Omega)$  with boundary  $p^{-1}(\Omega \cap \partial_H \Psi_+)$ .

Now let  $\Psi(P_a) = \Lambda(\mathcal{W}, \mathcal{W}_I)$  be the space obtained by gluing in a solid torus,  $T(\Omega)$ , to each  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}_I$ . (We won't need to define a space  $P_a$ , but will write  $\Psi(P_a)$  for the sake of maintaining consistent notation.) We write  $\partial_H \Psi(P_a) = p^{-1}(\partial_H \Psi_+) \cup \bigcup_{\Omega \in \mathcal{W}_\partial} A(\Omega)$ . In other words, it consists of all the (3HS) components of  $\partial_H \Psi_+ \setminus \alpha$  connected by annuli  $A(\Omega)$ , so as to recover  $\Sigma$  up to homeomorphism. In fact,  $(\Psi(P_a), \partial_H \Psi(P_a)) \cong (\Sigma \times [0, \infty), \Sigma \times \{0\})$ . (Note that it also depends on a choice of multicurve  $\alpha \subseteq \Sigma$ . This choice is not important, but it will be convenient to impose some restrictions later.)

We can now put a riemannian metric,  $d$ , on  $\Psi(P_a)$ , exactly as we did with  $\Psi(P)$ , by giving each  $T(\Omega)$  the structure of a Margulis tube. It also has a pseudometric,  $\rho$ , obtained by deeming each  $T(\Omega)$  to have diameter 0. Near the boundary,  $\partial_H \Psi(P_a)$ , these metrics may be a bit of a mess, but in a neighbourhood of the end of  $\Psi(P_e)$  they will have all of the properties, (W1)–(W9) laid out in Section 2.5.3.

### 2.14.5. Construction of the model space.

We are now in a position to describe the model space,  $P$ . The only data we need is the topology of  $\Psi(M)$ , the partition of its ends as  $\mathcal{E}(M) = \mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$ , and the assignment of degenerate end invariants,  $(a(e))_{e \in \mathcal{E}_D(M)}$ .

Let  $\Psi_0(P)$  be a homeomorphic copy of the core,  $\Psi_0(M)$ , of  $\Psi(M)$ . We have a decomposition of its boundary into the horizontal and vertical parts,  $\partial\Psi_0(P) = \partial_H \Psi_0(P) \cup \partial_V \Psi_0(P)$ . For each  $e \in \mathcal{E}_F(M)$ , we take a copy  $\Psi(P_e) := \Psi(P_{\Sigma(e)})$  of the geometrically finite model, for the base surface  $\Sigma(e)$ , and glue  $\partial_H \Psi(P_e)$  to the corresponding component of  $\partial_H \Psi_0(P)$ . If  $e \in \mathcal{E}_D(M)$ , we take a copy of the degenerate model,  $\Psi(P_e) := \Psi(P_{a(e)})$  and again glue  $\partial_H \Psi(P_e)$  to the corresponding component of  $\partial_H \Psi_0(P)$ . This case involves making a choice of multicurve,  $\alpha \subseteq \Sigma(e)$ , to construct  $\Psi(P_{a(e)})$ . In principle we could take any multicurve, but to avoid some technical complications, we will take it so that no component of  $\alpha$  is homotopic in  $\Psi_0(P)$  into the vertical boundary,  $\partial_V \Psi_0(P)$  (i.e. so that no curve of  $\alpha$  ends up being an accidental parabolic). In this way, we have constructed a topological copy  $\Psi(P)$ , of  $\Psi(M)$ . We have already some riemannian metric on

$\partial_H \Psi_0(P)$ . The model ends were such that the boundary curves of  $\partial_H \Psi_0(P)$  all have unit length. Each component of  $\partial_V \Psi_0(P)$  is either an annulus bounded by two such curves, which we can take to be isometric to  $S(1) \times [0, 1]$ ; or else a torus, which we can take to be a unit square euclidean torus,  $S(1) \times S(1)$  (with any marking). This gives a riemannian structure to  $\partial \Psi_0(P)$ , which we extend to a riemannian metric on  $\Psi_0(P)$ . We can choose the metric in a neighbourhood of the boundary curves of  $\partial_H \Psi_0(P)$  so that the boundary,  $\partial \Psi(P)$ , is smoothly embedded in  $\Psi(P)$ .

Finally, to construct  $P$ , we note that each component of  $\partial \Psi(P)$  is either a square torus, in which case, we glue in a standard  $(\mathbb{Z} \oplus \mathbb{Z})$ -cusp, or else a bi-infinite cylinder isometric to  $S(1) \times \mathbb{R}$  (made up from an annular component of  $\partial_V \Psi(P)$  together with the vertical boundary components of two model ends), in which case we glue in a standard  $\mathbb{Z}$ -cusp.

#### 2.14.6. *The map to $M$ .*

This gives us our model space,  $P$ . We now define a map  $f : P \rightarrow M$ , in a series of steps as follows.

First, for each  $e \in \mathcal{E}_F(M)$ , Lemma 2.14.2 gives us a universally sesquilipschitz map  $f : \Psi(P_e) \rightarrow \Psi(F)$ , where  $\Psi(F)$  is the neighbourhood of  $e$  as defined in Subsection 2.14.5.

Now suppose that  $e \in \mathcal{E}_D(M)$ . We want to construct a map  $f : \Psi(P_e) \rightarrow \Psi(M)$ . This is best done by passing to the cover,  $\hat{\Psi}(M)$  of  $\Psi(M)$  corresponding to the end,  $e$ . Note that  $\hat{\Psi}(M) \subseteq \Psi(\hat{M})$ , where  $\hat{M}$  is the cover of  $M$  corresponding to  $e$ . (These need not be equal, since a cusp of  $M$  may open out in  $\hat{M}$ .) Now  $\hat{M}$  is a product manifold with base surface  $\Sigma(e)$ , so that  $\Psi(\hat{M})$  is homeomorphic to  $\Sigma \times \mathbb{R}$ , possibly with accidental parabolic cusps removed. In any case, the A-priori Bounds Theorem (Theorem 2.13.4) applies in this case. This means that if  $\Omega \in \mathcal{W}$  then  $l_M(\Omega)$  is bounded above in terms of  $\xi(\Sigma)$ ,  $\max\{l_M(\delta) \mid \delta \in \mathbf{C}(\alpha)\}$ , and the length bound in the definition of a simply degenerate end (Proposition 2.9.1). Here  $l_M$  denotes the length of the homotopic closed geodesic in  $M$ , or equivalently, in  $\hat{M}$ . If this happens to be parabolic, we set it equal to 0.

We are now in a position to apply the construction of Sections 2.7 and 2.8. This gives us a partition of  $\mathcal{W}_I$  as  $\mathcal{W}_0 \sqcup \mathcal{W}_1$ , and a map  $f : \Psi(P_e) \rightarrow \hat{M}$  which is lipschitz on the ‘‘thick part’’,  $\Theta(P_e) = \Psi(P_e) \setminus \bigcup_{\Omega \in \mathcal{W}_0} \text{int } T(\Omega)$ , but only, for the moment, defined topologically on the thin part — the union of the Margulis tubes  $T(\Omega)$  for  $\Omega \in \mathcal{W}_0$ .

There are a couple of minor complications in this procedure, which are most simply resolved by observing that we only need to have  $f$  defined geometrically on a neighbourhood of the end of  $\Psi(P_e)$  — any lipschitz extension to the remainder of  $\Psi(P_e)$  will do. The first complication is that some of the annuli in  $\mathcal{W}_I$  may correspond to accidental parabolics. The construction will still work in this case,

but in any case, there are only finitely many such  $\Omega \in \mathcal{W}_I$ . Secondly we note that, by construction,  $f$  maps each component of  $\partial_V \Psi(P_e)$  the corresponding component of  $\Psi(\hat{M})$ , but it is still conceivable that  $f(\Psi(P_e))$  might enter other components of  $\hat{M} \setminus \hat{\Psi}(M)$ . As before,  $f$  is proper, and sends  $P_e$  out an end of  $\hat{\Psi}(M)$ , and this end cannot contain any such regions. This problem can therefore only arise in a compact subset of  $\Psi(P_e)$  and so can be fixed by the earlier observation. We now end up with a map to  $\hat{\Psi}(M)$ , which descends to a map  $f : \Psi(P_e) \rightarrow \Psi(M)$ .

Since  $f : \Psi(P_e) \rightarrow \Psi(M)$  is proper, it must send  $\Psi(P_e)$  out some end  $e'$  of  $\Psi(M)$ . If  $e \neq e'$ , then the corresponding base surfaces must be homotopic in  $\Psi(M)$ . Thus, applying Waldhausen's Cobordism Theorem (Theorem 1.6.2 here), we see that, in fact,  $\Psi(M)$  is just a product  $\Sigma \times \mathbb{R}$ , and so we are in the doubly degenerate situation dealt with in Section 2.13. We saw there that  $a(e) \neq a(e')$  giving a contradiction. In other words, we have shown  $\Psi(P_e)$  must get sent out the corresponding end of  $\Psi(M)$ .

We now have  $f$  defined on each of the ends of  $\Psi(P)$  and hence on all of  $\partial_H \Psi_0(P)$ . We now extend to any lipschitz map of  $\Psi_0(P)$  into  $\Psi(M)$ , in the right homotopy class, such that each component of  $\partial_V \Psi_0(P)$  gets sent to the corresponding component of  $\Psi(M)$ .

This gives us a proper, end-respecting homotopy equivalence  $f : \Psi(P) \rightarrow \Psi(M)$ , for the moment only defined topologically on the Margulis tubes of the degenerate ends. In particular,  $f$  is surjective.

Suppose  $e \in \mathcal{E}_D(M)$ . Since  $f$  is proper, we can find a neighbourhood  $\Psi(M_e) \cong \Sigma(e) \times [0, \infty)$  of this end in  $\Psi(M)$  such that  $f^{-1}\Psi(M_e) \subseteq \Psi(P_e)$ . We can also find a neighbourhood,  $\Psi_1(P_e) \subseteq \Psi(P_e)$  so that  $f(\Psi_1(P_e)) \subseteq \Psi(M_e)$ . We can also assume that all the properties (W1)–(W9) of Subsection 2.5.3 hold in  $\Psi_1(P_e)$ . To understand this end, we are thus effectively reduced to considering the map  $f|_{\Psi_1(P_e)}$  into  $\Psi(M_e)$ . Since we only need to control the geometry of the map in some neighbourhood of the end, we can deem any finite set of Margulis tubes in  $\Psi(M_e)$ , and their preimages in  $\Psi_1(P_e)$ , to lie in the the respective ‘‘thick parts’’. In particular, we can assume that  $\partial_H \Psi(M_e)$  and  $\partial_H \Psi_1(P_e)$  lie in the thick part, and that  $f(\partial_H \Psi_1(P_e))$  is homotopic in  $\Psi(M_e)$  to  $\partial_H \Psi(M_e)$ . Now all the arguments of Section 2.8 and 2.10 go through as before. For the pushing argument of Section 2.12, we need to assume that our points lie sufficiently far out the end, in order to push our path into a band, but we only need to verify the sesquilipschitz property on some neighbourhood of the end.

We can thus extend  $f$  to a uniformly lipschitz map on each of the Margulis tubes in  $\Psi_1(P_e)$ , and we deduce that  $f|_{\Psi_1(P_e)}$  is universally sesquilipschitz to  $\Psi(M_e)$ . We can take  $f$  to be any lipschitz map in the right homotopy class in the remaining Margulis tubes in  $\Psi(P_e)$ .

We are now ready to show:

**Lemma 2.14.4.** *The map  $f : \Psi(P) \rightarrow \Psi(M)$  is universally sesquilipschitz.*

*Proof.* By construction,  $f$  is a proper lipschitz map. For each end  $e \in \mathcal{E}(M)$  we can choose any product neighbourhood,  $\Psi_1(M_e)$ , so that any two distinct sets of the form  $\Psi_1(M_e)$  are distance  $\eta > 0$  apart for some constant  $\eta > 0$ . If  $e \in \mathcal{E}_D(M)$  we can also take  $\Psi_1(M_e) \subseteq \Psi(M_e)$  as defined above. Let  $\Psi_1(P) \subseteq \Psi(P)$  be a core containing the preimage of an  $\eta$ -neighbourhood of  $\Psi(M) \setminus \bigcup_{e \in \mathcal{E}(M)} \Psi_1(M_e)$ . Since  $\Psi_1(M)$  is compact, the map  $f|_{\Psi_1(P)}$  is sesquilipschitz onto its range.

We want to show that the lift of  $f$  to the universal covers of  $\Psi(P)$  and  $\Psi(M)$  is a quasi-isometry. It is sufficient to bound the distance between two points in the domain that get sent to points at most  $\eta$  apart in the range. But this is now easy given that there are such bounds on each component of the lifts of  $\Psi_1(P)$  and each  $\Psi_1(P_e)$ .  $\square$

We finally need to define  $f : P \rightarrow M$ . In other words, we need to extend  $f$  over each cusp  $R$  of  $P$ . Let  $R'$  be the corresponding cusp in  $M$ .

If  $R$  is a  $(\mathbb{Z} \oplus \mathbb{Z})$ -cusp, we simply extend the bilipschitz homeomorphism,  $f|_{\partial R} : \partial R \rightarrow \partial R'$  to a bilipschitz homeomorphism  $f : R \rightarrow R'$  by sending each geodesic ray to a geodesic ray.

Suppose  $R$  is a  $\mathbb{Z}$ -cusp. Then  $\partial R$  is a bi-infinite cylinder, and each of its ends is a vertical boundary components of a model end. For a geometrically finite model end, such a boundary component will be geodesically embedded. To deal with the general situation, we need to pass to the covers of  $\Psi(P)$  and  $\Psi(M)$  corresponding to  $\partial R$ . In a degenerate end, the same argument as Lemma 2.13.7 shows that its intersection with the lift of  $\partial R$  is quasi-isometrically embedded in the lift of the end. Since the two ends lift to disjoint sets, it now follows that  $\partial R$  is quasi-isometrically embedded in the cover of  $\Psi(P)$ . Therefore the map  $f|_{\partial R} : \partial R \rightarrow \partial R'$  is a quasi-isometry with respect to the induced euclidean path metrics. We can thus extend  $f|_{\partial R}$  to a universally sesquilipschitz map  $f : R \rightarrow R'$  exactly as in Section 2.13.

We have now defined  $f : P \rightarrow M$ .

*Proof of Theorem 2.14.1 :* We know that  $f : P \rightarrow M$  is a proper lipschitz homotopy equivalence, that it respects the decompositions of  $P$  and  $M$  into non-cuspidal parts and cusps, and that it is universally sesquilipschitz between the non-cuspidal parts and between corresponding cusps. It now follows easily that  $f$  is itself universally sesquilipschitz.  $\square$

As a consequence, we immediately get:

**Proposition 2.14.5.** *Suppose that  $M, M'$  are complete indecomposable hyperbolic 3-manifolds and that there is a homeomorphism from  $M$  to  $M'$  that sends cusps of  $M$  into cusps of  $M'$  and conversely. Suppose that the induced map between the non-cuspidal parts sends each geometrically finite end to a geometrically finite end and each degenerate end to a degenerate end. Suppose that (under the induced homeomorphisms of base surfaces) the end invariants of corresponding pairs of*



*degenerate ends are equal. Then there is an equivariant quasi-isometry between the universal covers of  $M$  and  $M'$ .*

*Proof.* We can use the same model manifold  $P$  for both  $M$  and  $M'$ . Theorem 2.14.1 tells us that there are universally sesquipschitz maps  $P \rightarrow M$  and  $P \rightarrow M'$ . The lifts then give us equivariant quasi-isometries between the universal covers.  $\square$

Note that if there are no geometrically finite ends, then the limit set is all of  $S^2$  (cf. Lemma 2.13.10) Sullivan's theorem (given as Theorem 1.6.8 here) tells us immediately that  $M$  and  $M'$  are isometric, exactly as in Theorem 2.13.11.

#### 2.14.7. Incorporating the geometrically finite end invariants.

In general we need to take account of the geometrically finite end invariants:

**Theorem 2.14.6.** *Let  $M, M'$  be as in Proposition 2.14.5, and assume, in addition that the corresponding geometrically finite end invariants are also equal. Then the homeomorphism between  $M$  and  $M'$  is homotopic to an isometry.*

One way to prove Theorem 2.14.6 would be to construct a model space using geometrically finite model ends that take account of the end invariants as in [Mi4]. In this case, one would show that the quasiconformal extension of the quasi-isometry given by Proposition 2.14.5 would be conformal. This is the approach taken in [BroCCM].

Given Proposition 2.14.5 as stated, one can also proceed as follows.

Write  $M = \mathbb{H}^3/\Gamma$  and  $M' = \mathbb{H}^3/\Gamma'$ , where  $\Gamma \cong \pi_1(M) \cong \pi_1(M') \cong \Gamma'$ , and write  $D(\Gamma)$  and  $D(\Gamma')$  for the discontinuity domains. By Ahlfors's finiteness theorem,  $D(\Gamma)/\Gamma$  and  $D(\Gamma')/\Gamma'$  are (possibly disconnected) Riemann surfaces of finite type. Our indecomposability assumption tells us that each component of either discontinuity domain is a disc.

Now Proposition 2.14.5 gives us a quasi-isometry from  $\mathbb{H}^3$  to itself which is equivariant with respect to these actions. This extends to an equivariant quasiconformal map,  $f : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$ . This maps  $D(\Gamma)$  to  $D(\Gamma')$  and descends to a quasiconformal map  $\bar{f} : D(\Gamma)/\Gamma \rightarrow D(\Gamma')/\Gamma'$ . Since the geometrically finite end invariants are equal,  $\bar{f}$  is homotopic to a conformal map  $\bar{g} : D(\Gamma)/\Gamma \rightarrow D(\Gamma')/\Gamma'$ . We can now lift  $\bar{g}$  to an equivariant conformal map  $g : D(\Gamma) \rightarrow D(\Gamma')$ . We set  $g$  to be equal to  $f$  on the limit sets. We thus get an equivariant bijection  $g : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$ , which is conformal on the discontinuity domains and a homeomorphism of limit sets.

If we knew that  $g$  were quasiconformal, we would see that it was conformal, since, as noted in Section 1.3 a map which is quasiconformal everywhere and conformal almost everywhere is conformal. This would then complete the proof. However, it is not immediately clear even that  $g$  is continuous. We are saved by the following:

**Lemma 2.14.7.** *Suppose that  $U \subseteq \mathbb{C}$  is a proper simply connected domain. Suppose that  $f : U \rightarrow U$  moves each point a bounded distance with respect to the Poincaré metric. Then the extension of  $f$  by the identity of  $\mathbb{C} \setminus U$  is continuous. Moreover, if  $f|_U$  is quasiconformal, then its extension is quasiconformal, where the quasiconformal constant depends only on the given displacement bound.*

Here of course, the “Poincaré metric” refers to the unique complete curvature  $-1$  metric in the conformal class.

*Proof.* Write  $d_e$  for the euclidean metric. Suppose that  $z \in U$  and  $d(z, \partial U) = r$ . Using the Koebe Quarter Theorem to compare with the Poincaré metric on the disc of radius  $r$  centred at  $z$ , we get the well known estimate  $|ds| \geq \frac{2}{r}|dz|$ , where  $|ds|$  is the infinitesimal Poincaré metric on  $U$ . Suppose  $a, b \in U$ . Now take a Poincaré geodesic from  $a$  to  $b$ , and parameterise it by euclidean arc-length. Integrating along this path, we deduce that if  $a, b \in U$  are distance at most  $k$  apart in the Poincaré metric, then  $d_e(a, b) \leq (e^{2k} - 1) \max\{d_e(a, \partial U), d_e(b, \partial U)\}$ . Continuity of  $g$  at  $\partial U$  now follows easily. The fact that it is quasiconformal follows, for example, using the above estimate to control the metric quasiconformal distortion of  $g$  on  $\partial U$ .  $\square$

We can elaborate on this as follows:

**Lemma 2.14.8.** *Let  $\mathcal{U}$  be a family of disjoint simply connected open domains of the Riemann sphere,  $\mathbb{C}^\infty$ . Let  $f : \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$  be a map (not a-priori assumed continuous) such that  $f|(\mathbb{C}^\infty \setminus \bigcup \mathcal{U})$  is the identity, and such that for all  $U \in \mathcal{U}$ ,  $f(U) = U$  and  $f|_U$  moves each point a uniformly bounded distance with respect to the Poincaré metric on  $U$ . Then  $f$  is (continuous and) quasiconformal.*

*Proof.* This follows similarly as in the proof of Lemma 2.14.7.

Alternatively, we could enumerate the elements of  $\mathcal{U} = \{U_1, U_2, \dots\}$ , and apply Lemma 2.14.7 to each  $U_i$ . For each  $n$ , the composition of the first  $n$  maps is uniformly quasiconformal. Let  $n \rightarrow \infty$  and apply the fact that the space of uniformly quasiconformal maps fixing at least three points is compact.  $\square$

*Proof of Theorem 2.14.6 :* Let  $h = g^{-1} \circ f : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$ , where  $f, g$  are the maps defined above. By construction,  $h$  is  $\Gamma$ -equivariant, and quasiconformal on  $D(\Gamma)$  and the identity on the limit set. Let  $U$  be a component of  $D(\Gamma)$ , conformally a disc. Now it is well known that a quasiconformal map of the disc is a quasi-isometry of the Poincaré metric. (This is based on the fact that the modulus of any embedded annulus that separates two points from infinity is bounded above in terms of the hyperbolic distance between them.) It thus extends to a homeomorphism of the ideal boundary. Since the map  $h$  is equivariant with respect to a finite co-area action, it follows that it must be the identity on the ideal boundary, and hence moves every point a bounded distance in the Poincaré metric. Lemma 2.14.8 now tells us that  $h$  is continuous and quasiconformal on  $\partial\mathbb{H}^3$ . Since  $f$  is quasiconformal, it follows that  $g$  is quasiconformal. But it is

conformal on  $D(\Gamma)$  and hence, applying Sullivan's result [Su] (see Theorem 1.6.8) it is conformal everywhere. Thus there is a hyperbolic isometry conjugating the  $\Gamma$ -action to the  $\Gamma'$ -action. This descends to an isometry from  $M$  to  $M'$ .  $\square$

This now completes the proof of the Ending Lamination Theorem in the decomposable case.

### 3. THE GENERAL CASE OF THE ENDING LAMINATION THEOREM

We now move on to the general case of the Ending Lamination Theorem.

There are several new issues to be addressed. Some are relatively straightforward, though others require more work. First, the issue of marking of end invariants in this case is more subtle. We have already discussed this in Section 1.5. Also there will some adjustment to be made when applying Sullivan's Theorem to complete the argument. This will be discussed in Section 3.8.

The main complication, however, arises from the fact that we cannot simply reduce to surface groups by lifting to an appropriate cover. As a consequence, many results we used before cannot be quoted directly. Instead we have to find a geometric means of "isolating" the ends of our manifold. We can then observe that the relevant techniques can be generalised. (In fact, our discussion here applies equally well to the indecomposable case.)

#### 3.1. COMPRESSIBLE ENDS.

Let  $M$  be any complete hyperbolic 3-manifold with  $\pi_1(M)$  finitely generated, so that  $\Psi(M)$  is topologically finite. Let  $e \in \mathcal{E}(M)$ . The geometrically finite case will be very similar to that dealt with earlier, so we assume for the moment that  $e \in \mathcal{E}_D(M)$ . Write  $\Sigma = \Sigma(e)$ . As before, we have  $\mathcal{G}(e) \equiv \mathcal{G}(\Sigma(e)) = \mathcal{G}(\Sigma)$ .

We know that every 3HS end is geometrically finite, so  $\xi(\Sigma) \geq 1$ .

We will need to give some special attention to the case where  $\xi(\Sigma) = 1$ . However, it is worth noting that this can arise only in a very specific case.

**Proposition 3.1.1.** *Suppose  $M$  has a compressible end,  $e$ , with  $\xi(\Sigma(e)) = 1$ . Then  $\Sigma(e)$  is a 4HS. In fact, if  $\Psi$  is the noncuspidal part of  $M$ , and  $\bar{\Psi}$  is its compactification, then  $\bar{\Psi}$  is a genus-2 handlebody, with  $\partial_V \bar{\Psi} = \partial \bar{\Psi} \setminus \partial_I \bar{\Psi}$  consisting of two annuli, representing free generators of  $\pi_1(\bar{\Psi})$ .*

In particular,  $\Phi$  is unique up to homeomorphism, and  $\pi_1(M) \cong \pi_1(\bar{\Psi})$  has a free generating set consisting of a pair of parabolic elements.

*Proof.* Let  $E \cong \Sigma(e) \times [0, \infty)$  be a neighbourhood of the end  $e$ . By hypothesis,  $\pi_1(\partial E) \rightarrow \pi_1(\Phi)$  is not injective, so by Dehn's Lemma, there is an embedded disc,  $D \subseteq \Phi$ , with  $\partial D = D \cap E = D \cap \partial E$  homotopically non-trivial in  $\partial E$ .

If  $\Sigma(e)$  were a 1HT, we could choose free generators,  $a, b$ , of  $\pi_1(\Sigma(e))$  with  $a$  representing the free homotopy class of  $\alpha$ . Thus,  $a$ , and hence also  $[a, b]$ , is trivial in  $\pi_1(\Psi)$ . But  $[a, b]$  represents a peripheral curve, and hence parabolic, giving a contradiction.

Therefore,  $\Sigma(e)$  is a 4HS. Let  $N$  be a regular neighbourhood of  $E \cup D$  in  $\Psi$ . Then the relative boundary of  $N$  in  $\Psi$  consists of two annuli,  $A_1$  and  $A_2$ . Since  $\Phi$  is anannular, there are annuli,  $B_1, B_2 \subseteq \partial\Psi$ , such that  $A_i \cup B_i$  bounds a solid torus,  $T_i$ . Now  $\Psi = N \cup T_1 \cup T_2$ . Thus,  $\Psi$  is homeomorphic to  $N$ , and we see that it has the form described.  $\square$

Most of our discussion of degenerate ends with focus on the case where  $\xi(\Sigma(e)) \geq 2$ .

By our definition of a degenerate end, every neighbourhood of  $e$  meets some closed geodesic in  $M$ . In fact, it is a consequence of tameness that every neighbourhood  $E \cong \Sigma \times [0, \infty)$  of  $e$  contains a closed geodesic of  $M$ , which is homotopic to a simple closed curve of  $\Sigma$ . (See [Th1, Bon, Cana] for example, and the discussion below.) We refer to such curves as “simple”.

Based on this (or one of a number of equivalent statements) we will show:

**Proposition 3.1.2.** *There is some constant,  $L_0 = L_0(\xi(\Sigma))$  depending only on  $\xi(\Sigma)$  such that there is a geodesic ray  $(\gamma_i)_{i \in \mathbb{N}}$  in  $\mathcal{G}(e)$  such that for all  $i$ ,  $\gamma_i$  is represented by a closed curve in  $E$  of length at most  $L_0$ .*

In fact, as we shall see, we can take each of these representatives to be a closed geodesic in  $M$ .

Given this, we fix some such constant  $L_0 = L_0(\xi)$ , and make the following (somewhat artificial) definition:

*Notation.* Let  $\mathbf{a}(e) = \mathbf{a}(M, e) \subseteq \partial\mathcal{G}(e)$  be the set of points  $a \in \partial\mathcal{G}(e)$  such that there is some geodesic sequence  $(\gamma_i)_i$  in  $\mathcal{G}(e)$  satisfying the conclusion of Proposition 3.1.2 for some fixed  $L_0 = L_0(\xi(\Sigma))$  and tending to  $a$ .

Thus Proposition 3.1.2 tells us that  $\mathbf{a}(e)$  is non-empty. We will see later that  $\mathbf{a}(e)$  is a singleton (Proposition 3.7.2), and give some other, more natural, descriptions of this element (see for example, Proposition 3.7.3).

A key step in achieving this will be the following:

**Proposition 3.1.3.** *There is some constant,  $L = L(\xi(\Sigma))$  depending only on  $\xi(\Sigma)$  such that if  $(\gamma_i)_{i \in \mathbb{N}}$  is any tight geodesic ray in  $\mathcal{G}(e)$  tending to some element of  $\mathbf{a}(M, e)$ , then, for all sufficiently large  $i$ ,  $\gamma_i$  is represented by a curve of length at most  $L$  in  $E$  (which can be taken to be a closed geodesic in  $M$ ).*

Propositions 3.1.2 and 3.1.3 will be proven in Section 3.6. In the incompressible case, Proposition 3.1.3 is a variant of the A-priori Bounds Theorem of Minsky [Mi2], reproven in the incompressible case in [Bow3]. This result, as in the indecomposable case, is key to establishing that the map from the model space to

the hyperbolic 3-manifold is lipschitz. From that point on, only a few relatively minor modifications of the proof given in the indecomposable case are required. These are discussed in Section 3.7.

In order to reduce to the incompressible case, we will need a purely topological observation, which we can give at this point (cf. Lemma 2.7.1 of [Bow6]).

**Lemma 3.1.4.** *Suppose that  $N$  is a 3-manifold with boundary,  $\partial N$ , and that  $F \subseteq \partial N$  is a compact subsurface such that no boundary component of  $F$  is homotopically trivial in  $N$ . Suppose that  $K \subseteq N$  is a closed subset, carrying all of  $\pi_1(N)$ . Then  $F$  is  $\pi_1$ -injective in  $N \setminus K$ .*

Note that, in fact, it would be enough to assume that  $K$  carries the image of  $\pi_1(F)$  in  $M$  (defined up to conjugacy), since we can lift to the cover corresponding to  $F$ , and replace  $K$  by its preimage.

*Proof.* We begin with a preliminary observation. Suppose that  $S \subseteq \partial N$  is a compact subsurface with the image of  $\pi_1(S)$  trivial in  $\pi_1(N)$ . homotopic to a point in  $N$ . Then  $S$  is planar (i.e. genus 0). This can be shown by using Dehn's Lemma [Hem1] to cut  $S$  into 3-holed spheres glued along boundary curves which bound disjoint embedded discs in  $N$ . These must be connected in a treelike fashion, since any closed cycle of adjacencies would give rise to curve in  $S$  that is non-trivial in  $N$ . In other words,  $S$  is planar as claimed.

Now suppose, for contradiction, that  $\pi_1(F)$  does not inject into  $\pi_1(N \setminus K)$ .

By Dehn's Lemma, there is an embedded disc  $D \subseteq N \setminus K$  with  $\partial D = D \cap F = D \cap \partial N$  a non-trivial curve in  $F$ . Let  $U$  be a small open product neighbourhood of  $D$  in  $N \setminus K$  so that  $U \cap \partial N = U \cap F$  is a small annular neighbourhood of  $\partial D$  in  $F$ . Let  $P = N \setminus U$ . Thus,  $K \subseteq P$ .

If  $P$  is connected then it determines a splitting of  $\pi_1(N)$  as a free product  $\pi_1(N) \cong \pi_1(P) * \mathbb{Z}$ . But since  $K \subseteq P$  carries all of  $\pi_1(N)$ , this gives the contradiction that  $\pi_1(N)$  is conjugate into a proper factor.

Thus  $P$  has two components,  $P_0$  and  $P_1$ . We can suppose that  $K \subseteq P_1$ . Now  $\pi_1(N) \cong \pi_1(P_0) * \pi_1(P_1)$  and as above,  $\pi_1(N)$  is conjugate into  $\pi_1(P_1)$ . Thus  $\pi_1(P_0)$  is trivial, and so in particular the image of  $\pi_1(P_0 \cap F)$  is trivial in  $\pi_1(N)$ . It follows by the observation of the first paragraph, that  $P_0 \cap F$  is planar. Since  $\partial D$  is non-trivial in  $F$ ,  $P_0 \cap F$  cannot be a disc. It must therefore have another boundary component, say  $\beta \subseteq P_0 \cap \partial F$ . But, by hypothesis,  $\beta$  is non-trivial in  $N$ , giving a contradiction.  $\square$

Lemma 3.1.4 will be used in the proof of Lemma 3.4.1.

### 3.2. PLEATING SURFACES.

The notion of a ‘‘pleating surface’’ we use here is a slight variation of the notion of a ‘‘pleated surface’’, as introduced by Thurston [Th1] (see [CanaEG] for a more detailed discussion). A pleating (or pleated) surface is (in particular) a lipschitz

map of a surface into a hyperbolic 3-manifold in an appropriate homotopy class. Up until now, in this paper, we have not needed them explicitly. This is because the map we have constructed from the model space to the 3-manifold has furnished us with a supply of maps of this type, and we have made use of these instead. However, pleating surfaces were used in the proof of the A-priori Bounds Theorem, as given in [Bow3], which was needed to construct a lipschitz map from the model in the first place (as in Section 2.8 here). In order to adapt the a-priori bounds result to the compressible case, we will need to revisit this; hence the discussion of pleated surfaces we give here.

Let  $(M, d)$  be a complete hyperbolic 3-manifold. We fix a Margulis constant and let  $\Psi(M)$  and  $\Theta(M)$  be respectively the non-cuspidal and thick parts of  $M$  as before.

Given  $x, y \in \Psi(M)$ , let  $\rho(x, y)$  be the electric pseudometric on  $\Psi(M)$ , that is, the minimum length of  $\beta \cap \Theta(M)$  as  $\beta$  varies over all paths from  $x$  to  $y$  in  $\Psi(M)$ .

Let  $\Sigma$  be a compact surface and let  $\text{int}(\Sigma) = \Sigma \setminus \partial\Sigma$ . We say that a homotopy class of maps from  $\Sigma$  into  $M$  (or equivalently  $\text{int}(\Sigma)$  into  $M$ ) is **type-preserving** if it sends each boundary component of  $\Sigma$  (or equivalently each end of  $\text{int}(\Sigma)$ ) homotopically to a generator of a  $\mathbb{Z}$ -cusp of  $M$ . We refer to this as the **associated** cusp. We remark that if two proper type-preserving maps of  $\text{int}(\Sigma)$  are homotopic then they are, in fact, properly homotopic.

**Definition.** A **pleating surface** is a uniformly lipschitz type-preserving map  $\phi : (\text{int}(\Sigma), \sigma) \rightarrow (M, d)$ , where  $\sigma$  is a finite-area hyperbolic metric on  $\text{int}(\Sigma)$ .

By “uniformly lipschitz” we mean  $\mu$ -lipschitz for some  $\mu \geq 1$ . In many situations, as in the traditional notion of pleated surface, one can take  $\mu = 1$ , though here we want to allow for a larger constant depending only on  $\xi(\Sigma)$ . The metric  $\sigma = \sigma_\phi$  is regarded as part of the data of the pleating surface. Note that  $\phi$  is necessarily proper.

In practice, the pleating surfaces we deal with will all have the property that any ray going out a cusp of  $\text{int}(\Sigma)$  will, from a certain point on, get sent to a ray going out the associated cusp of  $M$ .

Given some  $\alpha \in \mathbf{C}(\Sigma)$ , we write  $\alpha_\Sigma^*$  for its geodesic realisation in  $(\text{int}(\Sigma), \sigma)$ . Similarly, we write  $\beta_M^*$  for the geodesic realisation of a homotopy class,  $\beta$ , of closed curves in  $M$  — that is assuming  $\beta$  is non-trivial and non-parabolic. If we fix a homotopy class of type-preserving maps  $\phi : \text{int}(\Sigma) \rightarrow M$ , we abbreviate  $\alpha_M^* = (\phi(\alpha))_M^*$  (so that the homotopy class of  $\phi$  is implicitly understood). The notation is taken to imply that this exists: in other words, that  $\alpha$  is not homotopic into a cusp.

**Definition.** A pleating surface,  $\phi : \text{int}(\Sigma) \rightarrow M$  is said to **realise**  $\alpha \in \mathbf{C}(\Sigma)$  if  $\phi|_{\alpha_\Sigma^*}$  maps  $\alpha_\Sigma^*$  locally isometrically and with degree  $\pm 1$  to  $\alpha_M^*$ .

By a **multicurve** in  $\Sigma$  we mean a non-empty (finite) set of elements of  $\mathbf{C}(\Sigma)$  which can be realised disjointly in  $\Sigma$ . We say that a pleating surface **realises** a multicurve if it realises every component thereof.

**Lemma 3.2.1.** *Suppose we are given a type-preserving homotopy class,  $\phi$ , from  $\text{int}(\Sigma)$  into  $M$ , and a multicurve,  $\gamma$ , in  $\Sigma$ , such that the  $\phi$ -image of each component of  $\gamma$  is non-trivial and non-parabolic in  $M$ . Then there is a pleating surface in this class realising  $\gamma$ .*

This is based on a construction of Thurston and Bonahon [Th1, Bon], which we outline as follows. We give a more detailed technical discussion in Section 3.3.2.

We first realise  $\alpha$  as some smooth curve, then choose any  $x \in \alpha$ , and extend this to an ideal triangulation of  $S$ , whose edges are loops based at  $x$  (including  $\alpha$ ) as well as properly embedded rays going out the cusps. We now choose any  $y \in \alpha_M^* = \phi(\alpha_S^*)$  and realise all these edges as geodesic loops or rays based at  $y$  (so that  $x$  gets sent to  $y$ , and  $\alpha$  to  $\alpha^*$ ). We then extend to  $S$  by sending 2-simplices homeomorphically to totally geodesic triangles in  $M$ . We pull back the metric to  $S$  to give a pseudometric  $\sigma$  on  $S$ . The realisation  $\phi : S \rightarrow M$  is then 1-lipschitz. This is the construction used by Bonahon in [Bon]. This gives a singular hyperbolic pseudometric on the domain, which would be sufficient for our purposes. To obtain a hyperbolic metric on the domain, we need to adjust this somehow. One way is to use the “spinning” construction of Thurston. Note that there is a real line’s worth of possibilities for  $y$  (that is, taking account of the based homotopy class of our realisation  $(S, x) \rightarrow (M, y)$ ). By sending  $y$  off to infinity we converge on a pleated surface in the traditional sense. We note that, in fact, the same argument can be applied to any multicurve as in the hypotheses of Lemma 3.2.1.

We next make a few fairly routine observations regarding pleating surfaces. Suppose that  $\phi : \text{int}(\Sigma) \rightarrow M$  is a  $\pi_1$ -injective pleating surface. Then  $\text{int}(\Sigma)$  with the preimages of the associated cusps removed has a **principal component**,  $F$ , which carries all of  $\pi_1(\Sigma)$ . In fact, by choosing the Margulis constant of  $M$  sufficiently small in relation to  $\xi(\Sigma)$ , we can assume that  $F$  contains each horocycle of length 1. Now it is well known that any simple closed geodesic in  $\text{int}(\Sigma)$  cannot cross any such horocycle, and so must lie inside  $F$ . As an immediate consequence, applying Lemma 3.2.1, we have:

**Lemma 3.2.2.** *Suppose we have a  $\pi_1$ -injective homotopy class,  $\phi : \text{int}(\Sigma) \rightarrow M$ , and  $\alpha \in \mathbf{C}(\Sigma)$  such that  $\phi(\alpha)$  is non-parabolic in  $M$ . Then the closed geodesic in  $M$  in the class of  $\phi(\alpha)$  cannot enter any of the cusps of  $M$  associated to  $\phi$ .*

We will be applying this principle in a slightly different form — see Lemma 3.4.4.

Returning to our  $\pi_1$ -injective pleating surface  $\phi$ , another observation is that each component of  $\phi^{-1}\Theta(M)$  has bounded diameter. It follows easily that the  $\rho$ -diameter of  $\phi(F)$  is bounded. In fact:

**Lemma 3.2.3.** *There is some  $h_0 = h_0(\xi(\Sigma))$  such that if  $\phi : \text{int}(\Sigma) \rightarrow M$  is a  $\pi_1$ -injective pleating surface whose image,  $\phi(\text{int}(\Sigma))$ , only meets the associated cusps. Then the  $\rho$ -diameter of  $\phi(S) \cap \Psi(M)$  is at most  $h_0$ .*

*Proof.* In this case, the principal component,  $F$ , is a component of  $\phi^{-1}\Psi(M)$ , and we have seen that  $\phi(F)$  has bounded  $\rho$ -diameter. Any other components will be homotopic into an associated cusp. These are easily dealt with, but in practice we won't need to worry about them.  $\square$

Pleating surfaces in this general form are somewhat awkward to deal with. We can perform a ‘‘tidying up’’ operation. At the cost of increasing the lipschitz constant by a bounded amount, we can assume that the principal component,  $F$ , is the same as  $\phi^{-1}\Psi(M)$  and that its boundary consists entirely of horocycles of length 1. From this point on, we are only interested in the metric restricted to  $F$ , which we can identify with the original surface  $\Sigma$ .

For some later discussion it will be convenient to modify the definition of ‘‘pleated surface’’ a little, by cutting away cusps and viewing it as a map to  $\Psi(M)$ . To this end we define a **truncated pleating surface** in  $E$  to be a uniformly lipschitz map  $\phi : (\Sigma, \sigma) \rightarrow (E, d)$ , where  $\sigma$  is some hyperbolic structure on  $\Sigma$  with each boundary component horocyclic of fixed length. (Geodesic of fixed length would do just as well.) We also assume that each boundary component gets mapped to an intrinsic euclidean geodesic in  $\partial_V E$ . We will discuss this further in Section 3.5.

We will also require a somewhat deeper result concerning pleating surfaces, namely a version of the Uniform Injectivity Theorem, the original being due to Thurston.

Let  $\mathbf{E} \rightarrow M$  be the projectivised tangent bundle of  $M$ . (We can think of this as the unit tangent bundle factored by the direction-reversing involution.) If  $\phi : S \rightarrow M$  is a pleating surface realising a multicurve  $\gamma$ , we can lift  $\phi|_{\gamma_S}$  to a simple curve  $\gamma_{\mathbf{E}} \subseteq \mathbf{E}$ . We write  $\psi = \psi_\phi : \gamma_S \rightarrow \gamma_{\mathbf{E}}$  for the lift of  $\phi|_{\gamma_S}$ . Thus,  $\psi$  is a homeomorphism and a local isometry with respect to the metrics  $\sigma$  and  $d_{\mathbf{E}}$ .

**Lemma 3.2.4.** *Given  $\xi, \mu, \eta, \epsilon > 0$ , there is some  $\delta > 0$  with the following property. Suppose that  $S = \text{int} \Sigma$  is a surface with  $\xi(\Sigma) = \xi$ . Suppose that  $\phi : S \rightarrow M$  is a  $\mu$ -lipschitz pleating surface realising  $\gamma$ , and let  $\psi : \gamma_S \rightarrow \gamma_{\mathbf{E}} \subseteq \mathbf{E}$  is the lift described above. Suppose that there is some  $\eta > 0$  such that the injectivity radius of  $M$  at each point of  $\gamma_M = \phi(\gamma_S)$  is at least  $\eta$ . Suppose in addition that there is a map  $\theta : N(\gamma_M, \eta) \rightarrow S$  such that  $\theta \circ \phi : N(\gamma, \eta/\mu) \rightarrow S$  is homotopic to the inclusion of  $N(\gamma, \eta/\mu)$  into  $S$ . If  $x, y \in \gamma_S$  with  $d_{\mathbf{E}}(\psi(x), \psi(y)) \leq \delta$ , then  $\sigma(x, y) \leq \epsilon$ .*

Note that the map  $\theta$  need only be defined up to homotopy.

Lemma 3.2.4 is an immediate consequence of the statement for laminations given as Proposition 5.1.1, and we postpone the discussion until then.



We now go back to discuss some constructions of pleating surfaces that will be needed later.

### 3.3. NEGATIVELY CURVED SPACES.

We will need to consider a notion of pleating surface in a broader context than that discussed in Section 3.2, in particular when the range is “negatively curved”, with upper curvature bound  $-1$ . Morally, having concentrated negative curvature can only work in our favour, though it introduces a number of technical complications. Most of what need can be phrased in terms of locally  $\text{CAT}(-1)$  metrics, though in practice, all our metrics will be hyperbolic polyhedral complexes. We remark that  $\text{CAT}(-1)$  geometry has been used in a related context in [Sol1], and also applied in [Bow6]. We begin by recalling some standard definitions.

#### 3.3.1. *General discussion.*

We recall the notion of “geodesic” and “geodesic space” from Section 1.3. Note that up to reparameterisation a geodesic is a path whose rectifiable length equals the distance between its endpoints. We can also define a **local geodesic** as a path such that each point in the domain has a neighbourhood on which the restriction of the path is geodesic. We shall abbreviate “closed local geodesic” to “closed geodesic” since there can be no confusion in that case.

For any  $k \in \mathbb{R}$ , we have the notion of a “ $\text{CAT}(k)$ ” (or “locally  $\text{CAT}(k)$ ”) space which satisfies the  $\text{CAT}(k)$  comparison axiom globally (or locally). We refer to [BridH] for a detailed account of such spaces. The Cartan-Hadamard Theorem in this context says that if  $k \leq 0$ , then a locally  $\text{CAT}(k)$  space is globally  $\text{CAT}(k)$  if and only if it is simply connected (in which case it is contractible). For a proper (complete locally compact)  $\text{CAT}(-1)$  space we have the usual classification of isometries into elliptic, parabolic and loxodromic. We are only interested here in discrete torsion-free groups, so there are no elliptics. This gives rise to the following “thick-thin” decomposition.

Let  $(R, d)$  be a proper locally  $\text{CAT}(-1)$  space and  $\eta > 0$ . Let  $\tau(R)$  be the set of  $x$  such that  $x$  lies in a homotopically non-trivial curve  $\gamma$  of length less than  $\eta$ . We write  $\tau_0(R) \subseteq \tau(R)$  for the set of  $x$  such that some such  $\gamma$  can be homotoped in  $R$  to be arbitrarily short. If  $\alpha$  is a closed geodesic, we write  $\tau(R, \alpha) \subseteq \tau(R)$  for the set of  $x$  such that some such  $\gamma$  can be homotoped to some multiple of  $\alpha$ . We write  $\tau_+(R)$  for the union of all  $\tau(R, \alpha)$  as  $\alpha$  varies over all closed geodesics in  $R$ . One can show that all of these sets are open, and that  $\tau(R) = \tau_0(R) \cup \tau_+(R)$ . (Without a lower curvature bound, these sets need not be disjoint.) We write  $\Theta(R) = \Theta_\eta(R) = R \setminus \tau(R)$  and  $\Psi(R) = \Psi_\eta(R) = R \setminus \tau_+(R)$ . (If  $R$  is a complete hyperbolic 3-manifold, these sets agree with those already defined.) We think of  $\Theta(R)$  and  $\Psi(R)$  respectively as the “thick” and “non-cuspidal” parts of  $R$ .

We next describe a means of constructing locally CAT( $-1$ ) spaces in a hyperbolic 3-manifold. (We restrict to 3 dimensions for simplicity, though the discussion can easily be generalised to higher dimensions.)

Let  $M$  be a complete hyperbolic 3-manifold. Given  $x \in M$ , let  $\Delta_x(M)$  be the unit tangent space at  $x$ , as in Section 3.2. The following notions were used in [Bow6].

**Definition.** A *polyhedron*,  $\Xi$ , in  $M$  is a locally finite embedded simplicial complex, all of whose simplices are (embedded) totally geodesic simplices.

Note that  $\Xi$  is necessarily closed in  $M$ .

Any point  $x \in \Xi$  determines a closed polyhedral subset,  $\Delta_x(M, \Xi) \subseteq \Delta_x(M)$ , of tangent vectors lying in  $\Xi$ . (We think of this as the “link” of  $x$  in  $\Xi$ .)

**Definition.** We say that  $\Xi$  is *balanced* at  $x \in \Xi$ , if  $\Delta_x(M, \Xi)$  is not contained in any open hemisphere of  $\Delta_x(M)$ .

We say that  $\Xi$  is *fat* at  $x$  if  $\Delta_x(M, \Xi)$  is connected. We say that  $\Xi$  is *balanced* (respectively *fat*) if it is balanced (respectively fat) at every point.

Let  $\Xi \subseteq M$  be a polyhedral set. Let  $U \subseteq M$  be a connected component of  $M \setminus \Xi$ . We put the induced path-metric on  $U$ , and let  $\Pi$  be its metric completion. We write  $\text{int } \Pi = U$  and write  $\partial \Pi = \Pi \setminus \text{int } \Pi$ . The inclusion of  $\text{int } P$  into  $M$  extends uniquely to a continuous map,  $\pi : \Pi \rightarrow M$ . Note that  $\Pi$  is also a “polyhedral space” in the sense that it admits a structure of a locally finite simplicial complex, where all simplices are compact hyperbolic simplices (that is, isometric to a compact simplex in hyperbolic space). This observation allows us to apply the usual “link” criterion for  $\Pi$  to be CAT( $-1$ ). In particular, we note:

**Lemma 3.3.1.** *If  $\Xi$  is fat and balanced, then  $\Pi(\Xi)$  is locally CAT( $-1$ ).*

*Proof.* Suppose  $x \in \Pi$ . The link of  $x$  in  $\Pi$  can be identified with the metric completion of a connected component of  $\Delta_{\pi x}(M) \setminus \Delta_{\pi x}(M, \Xi)$ . (Note that the elements of  $\pi^{-1}\pi x$  correspond to the set of such components.) Such a set is locally CAT(1) (since the intrinsic link of any point is isometric either to a real interval, or to a circle of length  $2\pi$ ). To see that is globally CAT(1), we need to show, in addition, that it contains no intrinsic closed geodesics of length strictly less than  $2\pi$ .

To see this, suppose that  $\alpha$  were such a closed geodesic in the link. Then its image  $\pi\alpha$  in  $\Delta_{\pi x}(M)$  is a closed path of length less than  $2\pi$ , and so  $\pi\alpha$  lies inside some open hemisphere,  $H \subseteq \Delta_{\pi x}(M)$ . Since  $H$  cannot contain  $\Delta_{\pi x}(M, \Xi)$ , we must have  $H \cap \Delta_{\pi x}(M, \Xi) = \emptyset$ . It follows that  $\alpha$  lies inside an open hemisphere in the link of  $x$  in  $\Pi$ . Therefore,  $\alpha$  could not have been an intrinsic geodesic after all.

Given that  $\Pi$  is (isometric to) a hyperbolic polyhedral space, it now follows that it is locally CAT( $-1$ ) as required.  $\square$

Now suppose that  $(\Pi, d_\Pi)$  is a proper locally  $\text{CAT}(-1)$  space. (Recall that “proper” means complete and locally compact. For a geodesic space, this is equivalent to saying that all closed bounded subsets are compact.) Much of the discussion of pleating surfaces in Section 3.2 applies with  $M$  replaced by  $\Pi$ . We can define a pleating surface as a type preserving uniformly lipschitz map  $\phi : (S, \sigma) \rightarrow (\Pi, d_\Pi)$  where  $\sigma$  is a finite area locally  $\text{CAT}(-1)$  metric on the surface  $S$ . In practice, we only need to consider a polyhedral space  $\Pi$  of the type constructed above, and piecewise riemannian metrics (or pseudometrics) on  $\text{int}(\Sigma)$ . Moreover, for applications, we can assume that the cusps of  $\text{int}(\Sigma)$  are isometric to standard hyperbolic cusps in some neighbourhood of the end. In this setting we will only consider  $\pi_1$ -injective pleating surfaces.

### 3.3.2. *Technical discussion of pleating surfaces in $\text{CAT}(-1)$ spaces.*

One can construct pleating surfaces in a locally  $\text{CAT}(-1)$  space in essentially the same way as for a hyperbolic manifold. However, there are a number of technical issues: there may be cone singularities, or a collapsing of part of the surface. Morally these phenomena work in our favour, in that they tend to concentrate negative curvature; and the only essential point is that curvature in the domain should be at most  $-1$ . Nevertheless, we need to make some sense of this. A key observation is that, for our applications here, one only really needs the map to be defined on a certain subset of the surface, for example, the 1-skeleton of a triangulation. To simplify the discussion here, we will only consider  $\pi_1$ -injective maps (which is all we need, when the range is not a manifold). The arguments easily generalise.

Let  $Y$  be a locally  $\text{CAT}(-1)$  space (in practice here, a hyperbolic polyhedral space). Its universal cover,  $\tilde{Y} = \tilde{Y}$  is therefore  $\text{CAT}(-1)$ . Let  $\Phi$  be a surface with boundary,  $\partial\Phi$ , and let  $\tilde{\Phi}$  be its universal cover. Let  $\phi : \Phi \rightarrow Y$  be a  $\pi_1$ -injective map (assumed for the moment to be defined up to homotopy). This lifts to a  $\pi_1(\Phi)$ -equivariant map,  $\tilde{\phi} : \tilde{\Phi} \rightarrow \tilde{Y}$ . It is simpler to perform the constructions in the universal cover, and assume implicitly that we do so equivariantly. In this way, they descend to maps from (subsets of)  $\Phi$  into  $Y$ .

Let us assume first that  $\Phi$  is compact with  $\partial\Phi \neq \emptyset$ , and moreover that each component of  $\partial\Phi$  corresponds to a closed geodesic in  $Y$  (that is, not a cusp). We can realise  $\phi$  so that  $\phi|_{\partial\Phi}$  is locally geodesic, and pull back the metric to  $\partial\Phi$ . Lifting  $\tilde{\phi} : \tilde{\Phi} \rightarrow \tilde{Y}$ , we see that each component of  $\partial\tilde{\Phi}$  gets mapped isometrically to a bi-infinite geodesic.

Now take any triangulation of  $\Phi$  with all vertices in  $\partial\Phi$ . Write  $K \subseteq \Phi$  for its 1-skeleton, and  $\tilde{K} \subseteq \tilde{\Phi}$  for its lift to  $\tilde{\Phi}$ . Clearly,  $\partial\Phi \subseteq K$ . In particular, we already have  $\tilde{\phi}$  defined on the set of vertices. We can extend  $\tilde{\phi}$  equivariantly to  $\tilde{K}$  by mapping each edge to the geodesic connecting the  $\tilde{\phi}$ -images of its endpoints. We suppose that each edge is either mapped homeomorphically or collapsed to a

point. This descends to a map  $\phi : \Phi \rightarrow Y$ . We get an induced path-pseudometric in  $K$  or  $\tilde{K}$ . Clearly  $\tilde{\phi}$  is 1-lipschitz with respect to this metric. In fact, we can make a stronger statement. First we put a “hyperbolic pseudometric” on  $\tilde{\Phi}$ . To this end, we consider the following general construction.

Suppose that  $(\{x_1, x_2, x_3\}, d_0)$  is a 3-point pseudometric space. We can find  $a_i, a_2, a_3 \in \mathbb{H}^2$  such that  $d_{\mathbb{H}^2}(a_i, a_j) = d_0(x_i, x_j)$  for all  $i, j$ . This choice is unique up to isometry of  $\mathbb{H}^2$ . We write  $T$  for the convex hull of  $\{a_1, a_2, a_3\}$ .

If  $\{x_1, x_2, x_3\}$  is a metric space, then  $T$  is a (bona fide) hyperbolic triangle, and we set  $\theta_i \in (0, \pi)$  for its angle at  $x_i$ . We also need to consider degenerate cases. Up to permutation of indices, these are:

- (1) The  $x_i$  are all distinct, and  $d_0(x_1, x_2) = d_0(x_1, x_3) + d_0(x_3, x_2)$ . We set  $\theta_1 = \theta_2 = 0$  and  $\theta_3 = \pi$ .
- (2)  $x_1 = x_2 \neq x_3$ . We set  $\theta_1 = \theta_2 = \pi/2$  and  $\theta_3 = 0$ .
- (3)  $x_1 = x_2 = x_3$ . We can define  $\theta_i$  arbitrarily (though  $\theta_1 = \theta_2 = \theta_3 = \pi/3$  would seem a natural choice).

Note that, in cases (1) and (2),  $T$  is a geodesic segment, and in case (3) it is a point.

Returning to  $\tilde{\Phi}$ , note that any 2-cell of the triangulation determines a 3-points set (its vertices) hence a (possibly degenerate) triangle in the hyperbolic plane. We can thus assign angles to each of its corners. Moreover, we can map it (linearly) to  $\mathbb{H}^2$ , and take the induced pseudometric. Piecing these together gives us a path-pseudometric,  $d_{\tilde{\Phi}}$ , on  $\tilde{\Phi}$ , which descends to a path-pseudometric,  $d_{\Phi}$ , on  $\Phi$ . We note:

**Lemma 3.3.2.** *The map  $\tilde{\phi} : (\tilde{K}, d_{\tilde{\Phi}}) \rightarrow \tilde{Y}$  is 1-lipschitz.*

*Proof.* Suppose  $x, y \in \tilde{K}$ . If  $x, y$  lie in some triangle of the triangulation, then it follows directly from the CAT(-1) property that  $d_{\tilde{Y}}(\tilde{\phi}x, \tilde{\phi}y) \leq d_{\tilde{\Phi}}(x, y)$ . In general, since  $d_{\tilde{\Phi}}$  is defined as the path-pseudometric, we can find a sequence,  $x = x_0, x_1, \dots, x_n \in \tilde{K}$  with  $d_{\tilde{\Phi}}(x, y) = \sum_{i=1}^n d_{\tilde{\Phi}}(x_{i-1}, x_i)$ , and with  $x_{i-1}, x_i$  lying in some triangle for all  $i$ . The general statement then follows directly from the triangle inequality.  $\square$

Note that it follows that  $\phi : (K, d_{\Phi}) \rightarrow Y$  is also 1-lipschitz.

Given that  $\tilde{\phi}$  is geodesic on each boundary component it also follows that each component of  $\partial\tilde{\Phi}$  is geodesic in the intrinsic metric,  $d_{\tilde{\Phi}}$ .

We can view  $d_{\tilde{\Phi}}$  as a degenerate hyperbolic structure on  $\tilde{\Phi}$ . (In fact, we can think of it as a limit of hyperbolic structures, where we take some  $\epsilon > 0$  and reassigning each 0-length edge to have length  $\epsilon$ . In this case, piecing together triangles gives us a hyperbolic metric. We then let  $\epsilon \rightarrow 0$ .)

The fact that boundary components are geodesic can be equivalently expressed by saying that if  $v \in \partial\Phi$  is a vertex, then the sum of the corresponding angles in the incident triangles at  $v$  is at least  $\pi$ .

It also follows from the Gauss-Bonnet theorem (or by a simple direct argument summing over the triangles and using the above observation) that the area of  $\Phi$  is at most  $-2\pi\chi(\Phi)$ , where  $\chi$  is the Euler characteristic. In particular, it is bounded above in terms of the complexity of  $\Phi$ .

There are some extensions and variations of the above construction. For example, suppose that  $\Phi$  is a compact surface (now possibly with  $\partial\Phi = \emptyset$ ), and  $\alpha \subseteq \Phi$  is a multicurve. Suppose we start with a  $\pi_1$ -injective map  $\phi : \Phi \rightarrow Y$ , initially defined only up to homotopy. Suppose that each component of  $\alpha \cup \partial\Phi$  corresponds to a closed geodesic in  $Y$ . We can first realise  $\phi$  so that each such component gets mapped locally injectively to the corresponding closed geodesic. We then apply the above construction to (the closure of) each component of  $\Phi \setminus \alpha$ . This gives us a triangulation of  $\Phi$ , with  $\alpha$  lying in its 1-skeleton,  $K$ , and a hyperbolic pseudometric on  $\Phi$ , together with realisation of  $\phi$  such that  $\phi|_K$  is 1-lipschitz with respect to this metric.

We can also allow  $\Phi$  to have a finite number of punctures, each of which corresponds to a cusp of  $Y$  (that is a parabolic in the action of  $\pi_1(\Phi)$  on  $\tilde{Y}$ ). For this we include the punctures in a triangulation, and allow edges to get sent to geodesics rays tending out one of the cusps.

One can proceed to realise  $\phi$ , extending over triangles, so that it is uniformly lipschitz. We will not give details here, since it can be bypassed for the applications we have in mind.

In particular, the bound on area gives rise to a bound on the diameter of each thick part of  $\Phi$ . This, in turn gives us bounds on the diameters of their images in  $Y$  in the electric metric. This bounds diameters of geodesics multicurves in  $Y$ , as measured in the electric metric. It is easily seen that this only requires the map to be defined and 1-lipschitz on the 1-skeleton of a triangulation.

A further variation of this construction is required for the proof of Lemma 3.4.6 where we realise a homotopy between a curve and its geodesic representative by a pleating surface, with domain an annulus. We can assume that the original curve is piecewise geodesic, and so a similar construction applies.

### 3.4. ISOLATING SIMPLY DEGENERATE ENDS.

Suppose that  $e \in \mathcal{E}_D(M)$ . We want to focus our attention on the intrinsic geometry of  $e$ , so that the arguments of the incompressible case can be applied much as before. We will do this by looking at the complement of a balanced polyhedral set. (A different trick of passing to branched covers was used by Canary [Cana], and could probably be used here also. This has the advantage of producing a riemannian metric. However it introduces other technical complications.)

Let  $E$  be a neighbourhood of the end  $e$  homeomorphic to  $\Sigma \times [0, \infty)$ . Recall that  $\partial_H E = \Sigma \times \{0\}$  is the relative boundary of  $E$  in  $\Psi(M)$ . Note that  $e$  determines

a homotopy class of type-preserving maps from  $\text{int}(\Sigma)$  into  $M$ , which we refer to as the *ambient fibre class*.

To get started we note:

**Lemma 3.4.1.** *There is some compact fat balanced polyhedral set,  $\Xi \subseteq M$ , such that  $e$  is incompressible in  $M \setminus \Xi$ .*

More formally this means that we can find a neighbourhood,  $E$ , of the end  $e$ , such that  $E \cap \Xi = \emptyset$ , and such that the inclusion of  $E$  into  $M \setminus \Xi$  is  $\pi_1$ -injective.

There are a number of ways of constructing such a  $\Xi$  (for example, taking the images of pleated surfaces, constructed as in [Bon]), as discussed in Section 3.2. Here is another way:

*Proof.* We first construct a balanced graph,  $\Xi_0$ , in  $M$  carrying all of  $\pi_1(M)$ .

To this end, let  $g_0$  be any non-trivial non-parabolic element of  $\pi_1(M)$ , and extend this to any finite generating set,  $g_0, \dots, g_n$ , of  $\pi_1(M)$ . We assume all the  $g_i$  to be non-trivial.

Let  $R$  be a wedge of  $n + 1$  circles,  $R = \alpha_0, \dots, \alpha_n$ . Let  $\phi : R \rightarrow M$  be a map which sends  $\alpha_0$  locally injectively to the closed geodesic in  $M$  corresponding to  $g_0$ , and sending each  $\alpha_i$  to a geodesic loop based at some point,  $p \in \phi(\alpha_0)$ , and corresponding to  $g_i$ .

Let  $\Xi_0 = \phi(R)$ . This is a graph in  $M$ , with geodesic edges. Note that, since  $\phi$  might not be injective, we may have introduced new vertices (in addition to  $p$ ). Nevertheless,  $\Xi_0$  is balanced (in the sense defined above).

Of course,  $\Xi_0$  will not be fat. But we can easily fatten it up. To begin, if  $x \in \Xi_0$  is a vertex, choose  $r > 0$  sufficiently small so that  $N(x, r)$  is an embedded hyperbolic ball. Now let  $P(x)$  be the convex hull (in  $N(x, r)$ ) of  $\Xi \cap N(x, r)$ . Note that  $P(x)$  has dimension at least 2 everywhere.

Now, if  $\epsilon$  is any edge of  $\Xi_0$ , let  $\tilde{\epsilon}$  be any lift to  $\mathbb{H}^3$ , and let  $\tilde{x}, \tilde{y}$  be its endpoints. Let  $P(\tilde{\epsilon})$  be the convex hull of the union of the lifts  $P(\tilde{x}) \cup P(\tilde{y})$ , and let  $P(\epsilon)$  be the projection of  $P(\tilde{\epsilon})$  back to  $M$ . Let  $\Xi$  be the union of all the  $P(\epsilon)$  as  $\epsilon$  ranges over all edges of  $\Xi_0$ . Thus,  $\Xi \supseteq \Xi_0$  is fat and balanced.

Applying Lemma 3.1.4, we see that  $e$  is incompressible in  $M \setminus \Xi$ , as required.  $\square$

We can now choose a neighbourhood,  $E \cong \Sigma \times [0, \infty)$  of  $e$ , such that  $E \cap \Xi = \emptyset$ . (Figure 3.1.)

Now let  $U$  be the component of  $M \setminus \Xi$  containing  $E$ . As described in Section 3.3, we put the induced path-metric on  $U$ , and let  $\Pi$  be its metric completion. We write  $\text{int } \Pi = U$  and  $\partial \Pi = \Pi \setminus U$ . Thus  $\pi : \Pi \rightarrow M$  is injective on  $\text{int } \Pi$ . For expository simplicity, we will assume that  $\pi$  is injective also on  $\partial \Pi$ . (This can always be arranged by some elaboration of Lemma 3.4.1, though it does not affect the argument in any essential way.) In this way, we can view  $\Pi$  as a subset of  $M$  containing  $E$ . By Lemma 3.3.1,  $\Pi$  is locally CAT(-1).

Recall, from Section 3.3, the definition of the thin part,  $\tau(\Pi) = \tau_0(\Pi) \cup \tau_+(\Pi)$ , of  $\Pi$ . The set  $\tau_+(\Pi)$  is a union of  $\tau(\Pi, \alpha)$  as  $\alpha$  ranges over all closed  $d_\Pi$ -geodesics.

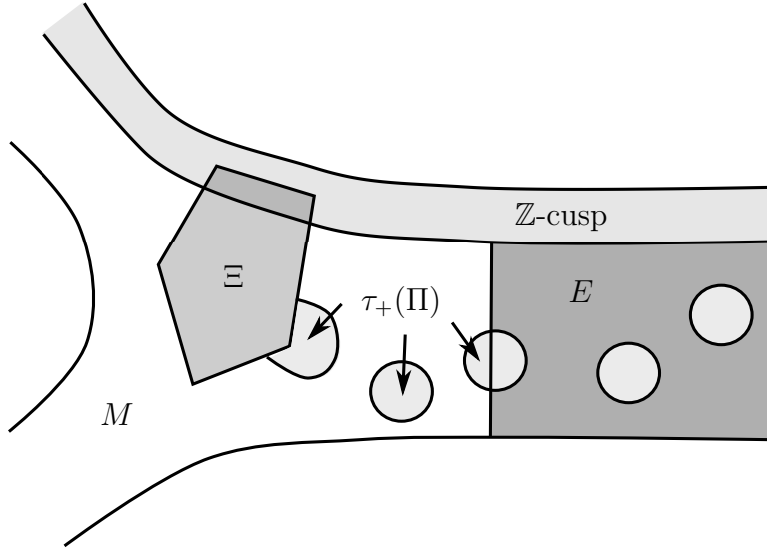


FIGURE 3.1. A schematic picture of  $E$

If  $\alpha \cap \partial\Pi = \emptyset$ , then  $\alpha$  is also a closed geodesic in  $M$  (in particular non-trivial in  $M$ ). If  $\tau(\Pi, \alpha) \cap \partial\Pi = \emptyset$ , then  $\tau(\Pi, \alpha)$  is a Margulis tube in  $M$  with respect to the hyperbolic metric. Suppose that  $\alpha, \beta$  are closed  $d_\Pi$ -geodesics with  $\tau(\Pi, \alpha) \cap \tau(\Pi, \beta) \cap U \neq \emptyset$ . Let  $x$  be a point of this intersection. By definition of the sets  $\tau(\Pi, \alpha)$  and  $\tau(\Pi, \beta)$ ,  $x$  lies in curves,  $\alpha'$  and  $\beta'$ , of length at most  $\eta$ , homotopic in  $\Pi$  respectively to non-zero powers of  $\alpha$  and  $\beta$ . The Margulis Lemma (applied in  $M$ ) now tells us that  $\alpha'$  and  $\beta'$  generate an abelian subgroup of  $\pi_1(M)$ . Such a group must be trivial, loxodromic or parabolic in  $M$ . Thus, if  $\alpha, \beta \subseteq U$ , then they are both hyperbolic geodesics and hence equal. Therefore, if  $\alpha \neq \beta$ , we conclude that at least one of  $\alpha$  or  $\beta$  must meet  $\partial\Pi$ .

Suppose  $x \in \tau_0(\Pi)$ . By definition,  $x$  lies in a non-trivial curve,  $\gamma \subseteq \Pi$ , of length less than  $\eta$ , which can be homotoped to be arbitrarily short in  $\Pi$ , hence also in  $M$ . It follows that  $x$  lies in a cusp of  $M$ . In other words, we have  $\tau_0(\Pi) \cap \Psi(M) = \emptyset$ . In particular,  $\tau_0(\Pi)$  contains no closed geodesic of  $M$  of length less than  $\eta$ . It follows that if  $\alpha$  is a closed  $d_\Pi$ -geodesic and  $\tau(\Pi, \alpha) \cap \tau_0(\Pi) \cap U \neq \emptyset$ , then  $\alpha \cap \partial\Pi \neq \emptyset$ .

Recall that  $E \cong \Sigma \times [0, \infty)$  is a neighbourhood of  $e$  contained in  $\text{int}(\Pi)$ . By discreteness, only finitely many  $\tau(\Pi, \alpha)$  can meet  $\partial_H E$ , and these include all those meeting both  $E$  and  $\partial\Pi$ . Thus, after shrinking  $E$ , we can assume that if  $\tau(\Pi, \alpha) \cap E \neq \emptyset$ , then  $\tau(\Pi, \alpha) \subseteq U$ , and so it is a Margulis tube in  $M$ . We also assume that  $E$  only meets the associated cusps of  $M$ . From this, it follows easily that  $\partial_V E = E \cap \Psi(M) = E \cap \Psi(\Pi)$ . Thus  $E \cap \Theta(M) = E \cap \Theta(\Pi)$ .

Recall that, in Section 2.1, we defined the ‘‘electric pseudometric’’,  $\rho = \rho_M$ , on  $M$  by collapsing the Margulis tubes. It will be more convenient, for the moment, to work with another pseudometric,  $\rho_E$ , where we also collapse the complement

of  $E$  in  $\Psi(M)$ . In other words,  $\rho_E(x, y)$ , is the minimum length of  $\beta \cap E \cap \Theta(M)$  as  $\beta$  ranges over all paths from  $x$  to  $y$  in  $\Psi(M)$ . We can also view this as a pseudometric on  $\Psi(\Pi)$ . Clearly  $\rho_E \leq \rho_M$ .

Given  $x \in E$ , we write  $D(x) = \rho_M(x, \partial_H E) = \rho_E(x, \partial_H E)$  for the **depth** of  $x$  in  $E$ . Note that  $D : E \rightarrow [0, \infty)$  is a proper continuous function. We note that if  $x, y \in E$  have depth greater than the  $\rho_M$ -diameter of  $\partial_H E$ , then  $\rho_M(x, y) = \rho_E(x, y)$ . Given a subset  $Q \subseteq E$ , write

$$D(Q) = \rho_E(Q, \partial_H E) = \inf\{D(x) \mid x \in Q\}$$

for the depth of  $Q$ . We write  $\text{diam}_E(Q)$  for the  $\rho_E$ -diameter of  $Q$ .

There is natural type-preserving  $\pi_1$ -injective homotopy class of maps  $S \rightarrow \Pi$ , determined by  $e$ , which we refer to as the **fibres class**.

Let  $h_0 = h_0(\Sigma)$ , be the constant given by Lemma 3.2.3.

**Lemma 3.4.2.** *If  $\phi : \text{int}(\Sigma) \rightarrow \Pi$  is a pleating surface in the fibres class, then  $\text{diam}_E(\phi(\text{int}(\Sigma)) \cap \Psi(\Pi)) < h_0$ .*

*Proof.* In applications, we only need this where  $(S, \sigma_\phi)$  is piecewise hyperbolic. In this case we can apply the Gauss-Bonnet Theorem, so the argument follows as with Lemma 3.2.3.  $\square$

In particular, we get:

**Lemma 3.4.3.** *If  $\phi : \text{int}(\Sigma) \rightarrow \Pi$  is a pleating surface in the fibres class, and  $\phi(\text{int}(\Sigma)) \cap E$  contains some point of depth at least  $h_0$  in  $E$ , then  $\phi(\text{int}(\Sigma)) \cap \Psi(\Pi) \subseteq E$ .*

Given  $\alpha \in \mathbf{C}(e) \equiv \mathbf{C}(\Sigma)$ , we write  $\alpha_\Pi^*$  for its realisation as a closed geodesic in  $\Pi$ . Thus if  $\alpha_\Pi^* \cap \partial\Pi = \emptyset$ , then  $\alpha_\Pi^* = \alpha_M^*$  is also a closed geodesic in  $M$ .

**Lemma 3.4.4.** *Suppose that  $\alpha \in \mathbf{C}(e)$  and  $\alpha_\Pi^* \cap E$  contains some point of depth at least  $h_0$ . Then  $\alpha_\Pi^* \subseteq E$ ,  $\alpha_\Pi^* = \alpha_M^*$ , and  $\text{diam}_E(\alpha_\Pi^*) < h_0$ .*

*Proof.* Let  $\phi : \text{int}(\Sigma) \rightarrow \Pi$  be a pleating surface in the fibres class realising  $\alpha_\Pi^*$ . By Lemma 3.4.3,  $\phi(\text{int}(\Sigma)) \cap \Psi(M) \subseteq E$ . Let  $F$  be the principal component of  $\phi^{-1}(\Psi(\Pi)) = \phi^{-1}(E) \subseteq S$ . A similar argument to that of Lemma 3.4.2 now applies. Assuming that we have chosen the Margulis constant,  $\eta$ , sufficiently small (depending on  $\xi(\Sigma)$ ), any simple geodesic in  $\text{int}(\Sigma)$  will lie in  $F$ . (One way to see this is to note that any point in the non-cuspidal part of  $\text{int}(\Sigma)$  lies in a non-trivial non-peripheral curve of bounded length in  $\text{int}(\Sigma)$ . This maps to a curve of bounded length in  $\Pi$ , which will lie outside all Margulis cusps for sufficiently small Margulis constant. We have only used the fact that the curvature on  $S$  is at most  $-1$ .) In particular,  $\alpha_S^* \subseteq F$ , and so  $\alpha_\Pi^* = \phi(\alpha_S^*) \subseteq E$ . Since  $E \cap \partial\Pi = \emptyset$ , it follows that  $\alpha_\Pi^* = \alpha_M^*$ . Finally,  $\text{diam}_E(\alpha_\Pi^*) \leq \text{diam}_E(\phi(\text{int}(\Sigma)) \cap \Psi(\Pi)) < h_0$ .  $\square$

**Lemma 3.4.5.** *Suppose that  $\xi(\Sigma) \geq 2$  and that  $\alpha, \beta \in \mathbf{C}(e)$  are adjacent in  $\mathcal{G}(e)$  and  $\alpha_\Pi^* \cap E$  contains some point of depth at least  $h_0$ . Then  $\beta_\Pi^* = \beta_M^* \subseteq E$ , and  $\text{diam}_E(\alpha_\Pi^* \cup \beta_\Pi^*) < h_0$ .*



*Proof.* The proof follows that of Lemma 3.4.4. This time we take  $\phi$  to realise the multicurve  $\{\alpha, \beta\}$ .  $\square$

We remark that, in retrospect, we see that the pleating surfaces arising in the proofs of Lemmas 3.4.4 and 3.4.5 do not meet  $\partial\Pi$ . If they were constructed by the folding procedure described in Section 3.2, then all the 2-simplices would be totally geodesic in  $M$ . We can then proceed to spin around the curves to give us pleated surfaces whose domains are hyperbolic surfaces.

**Lemma 3.4.6.** *For all  $l \geq 0$ , there is some  $h_1(l)$  such that if  $\alpha \in \mathbf{C}(e)$  is realised by a curve,  $\alpha_0$ , in  $E$  of length at most  $l$  and containing a point of depth at least  $h_1(l)$ , then  $\alpha_{\Pi}^* = \alpha_M^* \subseteq e$  and  $\rho_E(\alpha_0 \cup \alpha_{\Pi}^*) < h_1(l)$ .*

*Proof.* This is a fairly standard argument (cf. [Bon]). We can realise the homotopy between  $\alpha_0$  and  $\alpha_{\Pi}^*$  in  $\Pi$  by a map,  $\phi : A \rightarrow \Pi$ , where  $A$  is an annulus whose boundary components get mapped to  $\alpha_0$  and  $\alpha_{\Pi}^*$ . This can be constructed as for pleated surfaces. The pull-back (pseudo)metric on  $A$  is locally CAT(-1), and can be assumed piecewise riemannian. (In practice, we can approximate  $\alpha_0$  by a piecewise geodesic curve, so that the surface will have the form described in Subsection 3.3.2.) As with Lemma 3.4.2, we see that  $\text{diam}_E(\phi(A) \cap \Psi(\Pi))$  is bounded in terms of the area of  $A$ . (Note that there must be a path in  $\phi^{-1}(\Psi(\Pi))$  connecting the two boundary components.) Applying Gauss-Bonnet, this is in turn bounded in terms of the length of  $\partial A$  which is at most  $2l$ . In other words, there is some constant  $h' = h'(l)$  such that  $\text{diam}_E(\alpha_0 \cup \alpha_{\Pi}^*) \leq \text{diam}_E(\phi(A)) \leq h'$ . We can now set  $h_1 = h_0 + h'$ . It then follows that  $D(\alpha_{\Pi}^*) > h_0$ , and so by Lemma 3.4.4,  $\alpha_{\Pi}^* = \alpha_M^* \subseteq e$  and  $\text{diam}_E(\alpha_{\Pi}^*) < h_0$ , and the result follows.  $\square$

We have shown the essential properties we need. In order to restrict our attention to the end,  $e$ , we can perform a few “tidying up” operations on pleating surfaces. We will only be interested in pleating surfaces realising multicurves of depth at least  $h_0$ . As observed after Lemma 3.4.5, such a surface can be assumed to have domain a hyperbolic surface. Then, as discussed in Section 3.3, at the cost of increasing the lipschitz constant by a uniformly bounded amount, we can assume that the preimage of the non-cuspidal part is a core bounded by horocycles of some fixed length. We can assume that these horocycles get mapped to euclidean geodesics in  $\partial_V E$ . From this point on, the remainder of the manifold is of little interest to us, until we put the pieces back together again in Section 3.8. In particular, we can forget all about  $\Pi$  from this point onwards.

### 3.5. QUASIPROJECTIONS AND A-PRIORI BOUNDS.

The aim of this section is to give a generalisation of the A-priori Bounds Theorem to the compressible case. Specifically, we are aiming at Propositions 3.5.7 and 3.5.8, which together are variations of Theorems 2.13.2 and 2.13.3. For most

of the discussion, we assume that  $\xi(\Sigma) \geq 1$ . The case where  $\xi(\Sigma)$  can be dealt with by quite different arguments of [Bow3], as we discuss at the end.

We will describe a projection map which associates to any curve in  $\Sigma$ , another curve which has bounded length representative in the end. A similar projection map was a key ingredient in the argument presented in [Bow3]. Once its basic properties are established, the remainder of the argument is almost identical to that presented there.

Let  $e \in \mathcal{E}_D(M)$ , and let  $\Sigma = \Sigma(e)$ . Note that  $\xi(\Sigma) \geq 1$ . We have already observed that if  $\xi(\Sigma) = 1$ , then  $e$  is incompressible, so we can assume, in fact, that  $\xi(\Sigma) \geq 2$ . Let  $E \cong \Sigma \times [0, \infty)$  be a neighbourhood of  $e$  in  $\Psi(M)$ . Recall that  $\mathcal{G}(e)$  is naturally identified with  $\mathcal{G}(\Sigma(e))$ , as discussed in Section 1.5.

**Conventions.** Throughout this section, by a “pleating surface” we will mean a truncated pleating surface (as defined in Section 3.2) in the fibre class, whose image lies in  $E$ .

We note that Lemmas 3.4.2 and 3.4.3 apply equally well for a pleating surface interpreted in this way. We also note that Lemma 3.2.4 (Uniform Injectivity) applies in this situation. If an  $\eta$ -neighbourhood of the curve lies in  $E$ , then projection of  $E$  to  $\Sigma$  to this neighbourhood gives us the map  $\theta$  referred to there. For this we should insist that the curve has depth at least  $\eta$ , but after shrinking  $E$  slightly, we can forget this detail.

Let

$$J = \{\alpha \in \mathbf{C}(\Sigma) \mid \alpha_M^* \subseteq E\}.$$

If  $\alpha \in J$ , write  $l_M(\alpha)$  for the length of  $\alpha_M^*$ , and write  $D(\alpha) = D(\alpha_M^*) = \rho_E(\alpha_M^*, \partial_H E)$  for the depth of  $\alpha_M^*$  in  $e$ . If  $\alpha \in \mathbf{C}(\Sigma) \setminus J$ , we set  $D(\alpha) = 0$ . Let  $h_0$  be the constant of Lemma 3.4.5. It immediately follows from that lemma that:

**Lemma 3.5.1.** *If  $\alpha, \beta \in \mathbf{C}(\Sigma)$  are adjacent, then  $|D(\alpha) - D(\beta)| \leq h_0$ .*

We write

$$J(h, l) = \{\alpha \in J \mid D(\alpha) > h, l_M(\alpha) \leq l\}.$$

An immediate consequence of Lemma 3.5.1 applied inductively is:

**Lemma 3.5.2.** *If  $k \in \mathbb{N}$  and  $h \geq kh_0$ , then  $N(J(h, \infty), k) \subseteq J(h - kh_0, \infty)$ .*

Here  $N(Q, k)$  denotes the  $k$ -neighbourhood of  $Q \subseteq \mathbf{C}(\Sigma)$  with respect to the combinatorial metric,  $d_{\mathcal{G}}$ , on  $\mathcal{G}(\Sigma)$ .

Another consequence of Lemma 3.4.5 is that  $J(0, l)$  is a locally finite subset of  $\mathcal{G}(\Sigma)$  for any  $l \in (0, \infty)$  (that is, any bounded set meets  $J(0, l)$  in a finite set).

Now suppose that  $\alpha \in J(h_0, \infty)$ , and let  $(\Sigma, \sigma) \rightarrow E$  be a pleating surface in  $E$  realising  $\alpha$ , as given by Lemma 3.4.3. Let  $J_\sigma(l) \subseteq \mathbf{C}(\Sigma)$  be the set of curves whose geodesic realisations in  $(\Sigma, \sigma)$  have length at most  $l$ . It is a standard and relatively straightforward fact that the diameter of  $J_\sigma(l)$  in  $\mathcal{G}(\Sigma)$  is bounded in

terms of  $l$ . Moreover, there is some  $l_0 = l_0(\Sigma)$  such that  $J_\sigma(l_0) \neq \emptyset$ . We fix  $l_0$  and abbreviate  $J_\sigma = J_\sigma(l_0)$ . Suppose that  $\phi' : (\Sigma, \sigma') \rightarrow E$  is another such pleating surface realising  $\alpha$ .

**Lemma 3.5.3.** *Suppose that  $\phi : (\Sigma, \sigma) \rightarrow M$  and  $\phi' : (\Sigma, \sigma') \rightarrow M$  are pleating surfaces, both realising  $\alpha \in J(h_0, \infty)$ . Then  $\text{diam}_{\mathcal{G}(\Sigma)}(J_\sigma \cup J_{\sigma'})$  is bounded above in terms of  $\xi(\Sigma)$ .*

*Proof.* The proof follows as in Lemma 4.2 of [Bow3]. (The proof presented there was suggested to me by the referee of that paper.) Note that  $\phi(\Sigma)$  and  $\phi'(\Sigma)$  both lie in  $E$ , and so the argument goes through simply by replacing  $M$  by  $E$ . (The curves constructed there are homotopic in  $E$ , hence in  $\Sigma$ .)  $\square$

We therefore choose some pleating surface,  $\phi : (\Sigma, \sigma) \rightarrow M$ , realising  $\alpha$ , and  $\gamma \in J_\sigma$  and set  $\text{proj}(\alpha) = \gamma$ . This is then well defined up to bounded distance in  $\mathbf{C}(\Sigma)$ . If  $\alpha \in J(h_0, l_0)$  we can set  $\text{proj}(\alpha) = \alpha$ . Note that if  $\alpha$  and  $\beta$  are adjacent in  $\mathbf{C}(\Sigma)$ , then we can choose the same pleating surface for both, showing that  $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta))$  is bounded, as in [Bow3].

Suppose  $h \geq h_0 + h_1(l_0)$  where  $h_1$  is the function of Lemma 3.4.6. It follows that if  $\alpha \in J(h, \infty)$  then  $\text{proj}(\alpha) \in J(h - (h_0 + h_1(l_0)), r_0)$ .

Let us summarise what we have shown. For notational simplicity, we increase our original choice of constant  $h_0$  to  $h_0 + h_1(l_0)$ . In this way, it will serve for all the above purposes.

**Lemma 3.5.4.** *There is some  $h_0 = h_0(\xi(\Sigma))$ ,  $l_0 = l_0(\xi(\Sigma))$ ,  $k_0 = k_0(\xi(\Sigma)) \in \mathbb{N}$ , and a map  $\text{proj} : J(h_0, \infty) \rightarrow J(0, l_0)$  with the following properties:*

- (P1) *If  $\alpha \in J(h, \infty)$  for some  $h \geq h_0$ , then  $\text{proj}(\alpha) \in J(h - h_0, l_0)$ .*
- (P2) *If  $\alpha \in J(h_0, l_0)$ , then  $\text{proj}(\alpha) = \alpha$ .*
- (P3) *If  $\alpha, \beta \in J(h_0, \infty)$  are adjacent in  $\mathbf{C}(\Sigma)$ , then  $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta)) \leq k_0$ .*

(Figure 3.2.)

To deduce more about the map  $\text{proj}$ , we bring the hyperbolicity of  $\mathcal{G}(\Sigma)$  into play. We are specifically aiming at Lemma 3.5.6. This depends on the following general observation about quasiprojections in a hyperbolic graph.

**Lemma 3.5.5.** *For all  $k, s, t \geq 0$  there exist  $u, R \geq 0$  with the following property. Suppose that  $(\mathcal{G}, d)$  is a  $k$ -hyperbolic graph,  $A, B \subseteq V(\mathcal{G})$  and  $\omega : A \rightarrow B$  is a map satisfying  $\omega(x) = x$  for all  $x \in A \cap B$  and  $d(\omega(x), \omega(y)) \leq t$  whenever  $x, y \in A$  are adjacent. Suppose that  $x_0, x_1, \dots, x_n$  is a geodesic in  $\mathcal{G}$  with  $d(x_0, B) \leq s$ ,  $d(x_n, B) \leq s$  and  $N(x_i, u) \subseteq A$  for all  $i$ . Then for all  $i$ ,  $d(x_i, \omega(x_i)) \leq R$ .*

In the case where  $A = V(\mathcal{G})$ , this follows from the fact that the image of a quasiprojection (here  $\omega(V(\mathcal{G}))$ ) is quasiconvex. (This is discussed in [Bow3] for example.) Here we have to take account of the fact the quasiprojection is only defined on a certain subset.

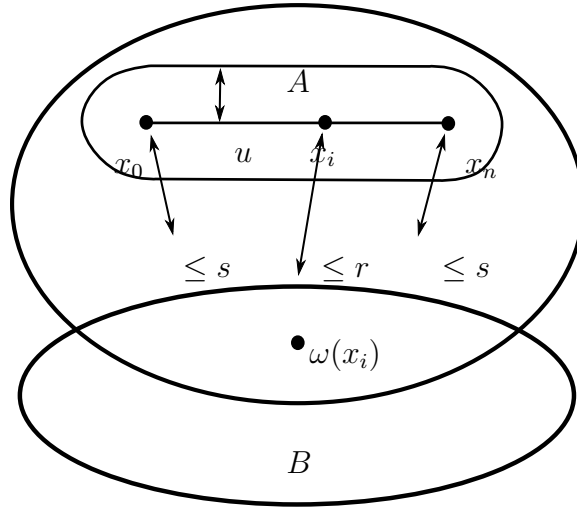


FIGURE 3.2. Lemma 3.5.4

*Proof.* We first make the observation that if  $x \in A$  and  $v$  is such that  $N(x, v) \subseteq A$ , and  $d(x, A \cap B) \leq v \leq u$  then  $d(x, \omega(x)) \leq tv$ . This follows by straightforward induction.

Now fix  $v \geq s, t$  to be determined shortly. Let  $\{p + 1, p + 2, \dots, q - 1\}$  be a maximal set of consecutive indices such that  $d(x_i, B) > v$  for all  $p + 1 \leq i \leq q - 1$ . In other words,  $d(x_p, B) \leq v, d(x_q, B) \leq v$  and  $d(x_i, B) > v$  whenever  $p < i < q$ . (We have used the assumption that  $v \geq s$ .) We can assume that  $u \geq v$  so that in particular,  $d(x_p, A \cap B) \leq v$  and  $d(x_q, A \cap B) \leq v$ . Let  $y_i = \omega(x_i)$ . Thus,  $d(x_p, y_p) \leq tv$  and  $d(x_q, y_q) \leq tv$  and  $d(y_i, y_{i+1}) \leq t$  for all  $i$ . Now the piecewise geodesic path  $[y_p, y_{p+1}] \cup \dots \cup [y_{q-1}, y_q]$  has length at most  $t(q - p)$  and lies outside a  $(v - t)$ -neighbourhood of the geodesic segment  $\underline{x} = x_p, x_{p+1}, \dots, x_q$ .

Let  $z_p$  be the last point on the geodesic  $[x_p, y_p]$  with  $d(\underline{x}, z_p) \leq v - t$ . Similarly define  $z_q$ . Let  $\zeta$  be the piecewise geodesic path  $[z_p, y_p] \cup [y_p, y_{p+1}] \cup \dots \cup [y_{q-1}, y_q] \cup [y_q, z_q]$  in  $\mathcal{G}$ . Now  $\text{length}(\zeta) \leq (q - p)t + 2vt = (q - p + 2v)t$ . Also  $d(z_p, z_q) \geq (q - p) - 2vt$ . By Lemma 1.6.10, it follows that  $\text{length}(\zeta) \geq e^{\mu(v-t)}d(z_p, z_q) - c$ , where  $\mu > 0$  and  $c \geq 0$  are fixed constants depending only on  $k$ . Therefore  $(q - p - 2vt)e^{\mu(v-t)} \leq (q - p + 2v)t + c$ . Now choose  $v$  so that  $e^{\mu(v-t)} \geq 2t$ , say. We then obtain  $2t(q - p - 2tv) \leq t(q - p + 2v) + c$ , and so  $q - p \leq (2 + 2t)v + c/t$ . This places an upper bound on  $q - p$ , which depends ultimately only on  $k, s$  and  $t$ . This in turn places an upper bound, say  $h$ , on  $d(x_i, B)$  for all  $i$  with  $p \leq i \leq q$ . We can assume that  $u \geq h$ .

Since this applies to any such segment,  $x_p, x_{p+1}, \dots, x_q$ , we conclude that  $d(x_i, B) \leq h$  for all  $i$ . Now by the first paragraph again, we see that for all  $i$ ,  $d(x_i, \omega(x_i)) \leq th$ . We therefore set  $R = th$ . □

We apply this as follows:

**Lemma 3.5.6.** *Given  $s \geq 0$ , there exist  $h_2, k_2 \geq 0$ , depending only on  $s$  and  $\xi(\Sigma)$  such that if  $\alpha_0, \alpha_1, \dots, \alpha_n$  is a geodesic in  $\mathcal{G}(\Sigma)$  with  $d_{\mathcal{G}}(\alpha_0, J(0, l_0)) \leq s$ ,  $d_{\mathcal{G}}(\alpha_n, J(0, l_0)) \leq s$  and with  $\alpha_i \in J(h_2, \infty)$  for all  $i$ , then for all  $i$ ,  $d_{\mathcal{G}}(\alpha_i, \text{proj}(\alpha_i)) \leq k_2$ .*

Here  $l_0$  is the constant involved in the definition of  $\text{proj} : J(h_0, \infty) \rightarrow J(0, l_0)$ . We can certainly choose  $h_2 \geq h_0$  so that  $\text{proj}(\alpha_i)$  is defined.

*Proof.* Let  $A = J(h_0, \infty)$  and  $B = J(0, l_0)$ , so that  $A \cap B = J(h_0, l_0)$ . We now apply Lemma 3.5.5, with  $\mathcal{G} = \mathcal{G}(\Sigma)$  and  $\omega = \text{proj} : A \rightarrow B$ . Let  $k = k(\xi(\Sigma))$  be the hyperbolicity constant of  $\mathcal{G}(\Sigma)$ , and let  $t = k_0$  be the constant of Lemma 3.5.4. Let  $u, R$  be the constants of Lemma 3.5.5 and  $h_2 = (u + 1)h_0$ . Then, by Lemma 3.5.2,  $N(J(h_2, \infty), u) \subseteq J(h_0, \infty)$ . In particular, by hypothesis,  $N(\alpha_i, u) \subseteq J(h_0, \infty) = A$  for all  $i$ . Thus, applying Lemma 3.5.5, we see that  $d_{\mathcal{G}}(\alpha_i, \text{proj}(\alpha_i)) \leq k_2$  for all  $i$ , where we set  $k_2 = R$ .  $\square$

The above generalises the discussion in Section 4 of [Bow3], and our proof of Lemma 3.5.5 broadly followed the argument there. In particular, after the proof of Lemma 4.2 of [Bow3] there is a statement similar to that of Lemma 3.5.6 above, though without any reference to the “depth” of closed geodesics. The projection map used there is defined in essentially the same way. This result was used in that paper in the proofs of Lemmas 5.2 and 5.3 and Theorems 1.3 and 1.4 thereof.

We now have all the ingredients to apply the arguments of [Bow3] to deduce the following version of the A-priori Bounds Theorem to the compressible case when  $\xi(\Sigma) \geq 2$ . We state it in general.

**Proposition 3.5.7.**

- (1)  $(\exists h)(\forall l)(\exists l')$  such that if  $(\gamma_i)_{i=0}^n$  is a tight geodesic in  $\mathcal{G}(\Sigma)$  with  $\gamma_0, \gamma_n \in J(h, l)$  and  $\gamma_i \in J(h, \infty)$  for all  $i$ , then  $\gamma_i \in J(0, l')$  for all  $i$ .
- (2)  $(\exists h)(\forall k, l)(\exists k', l')$  such that if  $(\gamma_i)_{i=0}^n$  is a tight geodesic in  $\mathcal{G}(\Sigma)$  with  $\gamma_0, \gamma_n \in N(J(h, l), k)$  and  $\gamma_i \in J(h, \infty)$  for all  $i$ , then  $\gamma_i \in J(h, l')$  for all  $i$  and with  $k' \leq i \leq n - k'$ .

(In fact,  $l'$  just depends on  $l$  and  $\xi(\Sigma)$  and in part (G2),  $k'$  just depends on  $k$  and  $\xi(\Sigma)$ .)

*Proof when  $\xi(\Sigma) \geq 2$ .* (Sketch) Propositions 3.5.7 (1) and 3.5.7 (2) respectively correspond to Theorems 1.3 and 1.4 of [Bow3]. These are proven in Section 8 thereof, and we can now follow through the arguments there. As mentioned above, this used a projection function defined in a similar way, and a statement corresponding to Lemma 3.5.6 was used in their proofs. All that needs to be observed in addition is that none of the constructions involved take us outside the set  $E$ . This is ensured by the initial hypothesis that the curves,  $(\gamma_i)_i$ , are all sufficiently deep inside  $E$ . (To avoid potential confusion, we point out that the symbol “ $E$ ” was used in [Bow3] to denote the projectivised tangent bundle.)

The basic idea in [Bow3] is that if the conclusion fails, then we could find very long curves in  $M$  which “fill up” subsurfaces of  $\Sigma$ , in a manner which would allow us to shortcut the tight geodesic of the hypotheses, and thereby derive a contradiction. This goes through as before. The homotopy classes of curves and subsurfaces should now be interpreted as intrinsic to  $E$  — the rest of the manifold,  $M$ , plays no role in this.

The argument of [Bow3] made use of a consequence of Thurston’s Uniform Injectivity Theorem (stated as Theorem 4.1 in [Bow3]). Here, we substitute Lemma 3.2.4.  $\square$

(Note that, in the incompressible case, Propositions 3.5.7(1) and 3.5.7(2) are respectively implied by Theorems 2.13.2 and 2.13.3. Indeed, if we neglect the requirement about depths of curves, they are just rephrasing of those statements.)

Proposition 3.5.7 is all we need to prove Propositions 3.1.2 and 3.1.3, as we will do in the next section. However, for their application we will also need a relative version of this, or at least of part (G1) of the statement. Let  $\Phi$  be a connected proper  $\pi_1$ -injective subsurface of  $\Sigma$ . We shall assume here that any boundary component of  $\Phi$  that is homotopic to a boundary component of  $\Sigma$  equals that boundary component. We write  $\partial_\Sigma\Phi$  for the relative boundary of  $\Phi$  in  $\Sigma$ , and write  $\mathbf{C}(\partial_\Sigma\Phi) \subseteq \mathbf{C}(\Sigma)$  for the set of components of  $\partial_\Sigma\Phi$ . (It is possible that two such components might get identified in  $\mathbf{C}(\Sigma)$ .) We can also identify  $\mathbf{C}(\Phi)$  as a subset of  $\mathbf{C}(\Sigma)$ . Note that, by Lemma 3.5.2, if for some  $h \geq h_0$ ,  $\mathbf{C}(\partial_\Sigma\Phi) \cap J(h, \infty) \neq \emptyset$ , then  $\mathbf{C}(\Phi) \subseteq J(h - h_0, \infty)$ .

The following is a generalisation of Theorem 2.8.5 to the compressible case.

**Proposition 3.5.8.** *( $\exists h$ )( $\forall l$ )( $\exists l'$ ) such that if  $\Phi \subseteq \Sigma$  is a proper subsurface with  $\mathbf{C}(\partial_\Sigma\Phi) \subseteq J(h, l)$  and  $(\gamma_i)_{i=0}^n$  is a tight geodesic in  $\mathcal{G}(\Phi)$  with  $\gamma_0, \gamma_n \in J(0, l)$ , then  $\gamma_i \in J(0, l')$  for all  $i$ .*

*Proof when  $\xi(\Sigma) \geq 2$ .* By Lemma 3.4.6, we see that if  $\gamma$  is any curve in  $\Phi$ , then  $\gamma$  is realised as a closed geodesic,  $\gamma_M^*$ , in  $M$ , with  $\gamma_M^* \subseteq E$ , and with  $\rho(\gamma_M^*, \alpha_M^*)$  bounded above for any  $\alpha \in \partial_\Sigma\Phi$ . In particular,  $\gamma$  is non-trivial in  $M$ . The arguments of [Bow3] now apply.

(In fact, alternatively and perhaps more simply, by Dehn’s Lemma, one can see that  $\Phi$  is incompressible in  $M$ , since if it were contained a trivial curve, it would have to contain a trivial simple closed curve. This means that one can apply the result of [Bow3] directly.)  $\square$

So far, in this section, we have assumed that  $\xi(\Sigma) \geq 2$ . We finish with an account of how this works when  $\xi(\Sigma) = 1$ . Note that, by Proposition 3.1.1, if  $\Sigma$  is a 1HT, then the end  $e$  is incompressible, so the account of Chapter 2 applies. We can therefore take  $\Sigma$  to be 4HS — though in fact the discussion can be applied to both cases.

We first note that Proposition 3.5.7 holds in this case. This follows from Section 9 of [Bow3], as with Theorem 1.6.11 here. Note that the relevant statements of

[Bow3] in this case apply to any type-preserving representation of  $\pi_1(\Sigma)$  into  $PSL(2, \mathbb{C})$  (that is, sending peripheral elements to parabolics). Here, length is formally defined in terms of traces. In particular, it does not require the map to be injective. (A similar discussion applies to Proposition 3.5.8 if we allow  $\xi(\Phi) = 1$ .)

We can also define a projection map,  $\text{proj} : J(h_0, \infty) \rightarrow J(0, l_0)$  exactly as with Lemma 3.4.4, and properties (P1) and (P2) hold for the same reason. (The proof given of (P3) now longer works, but we do not need it in this case.)

The remaining fact we need is:

**Lemma 3.5.9.** *If  $\gamma, \delta \in \mathcal{G}(\Sigma)$  are adjacent, then  $\rho_E(\gamma^*, \delta^*)$  is bounded above in terms of  $\max\{l_M(\gamma), l_M(\delta)\}$ .*

Note that it follows that  $|D(\gamma) - D(\delta)|$  is similarly bounded above. (In fact, the bound can be made uniform, independent of  $l_M(\gamma)$  and  $l_M(\delta)$ , as a more careful analysis of the proof would show. But the weaker statement will serve for our purposes.)

Lemma 3.5.9 follows from the constructions of Subsections 2.8.1 and 2.8.2. We just consider the case where  $\Sigma$  is a 4HS.

The situation corresponds combinatorially to a 4HS block as described in Subsection 2.8.2. This is a genus-3 handlebody, with the curves  $\gamma, \delta$  corresponding the core curves of the non-vertical annuli in its boundary. The four vertical annuli correspond to cusps. We described how to cut such a block into two truncated octahedra. We take one of these octahedra, and lift everything to the universal covers. Now two opposite vertices,  $x, y$ , of this octahedron correspond to loxodromic axes which project to  $\gamma^*$  and  $\delta^*$  respectively. The remaining four vertices correspond to parabolic fixed points. We now cut the octahedron into four 3-simplices, meeting along an 1-simplex with endpoints  $x, y$ . Consider one such 3-simplex. This has vertices  $x, y, z, w$ , where  $z, w$  correspond to parabolic points. Let  $Q(x), Q(y), Q(z), Q(w)$  be the convex sets described in Subsection 2.8.1. (Thus,  $Q(z), Q(w)$  are horoballs, and each of  $Q(x), Q(y)$  is either a loxodromic axis or a Margulis region, depending on whether the translation distance is greater than or less than the Margulis constant.) We now follow the argument in that subsection. Note that the 2-simplices  $xzw$  and  $yzw$  both satisfy condition (C4) of Lemma 2.7.6. Applying Lemma 2.7.4 respectively to these, we bound the lengths of the arcs,  $\beta(xz), \beta(xw), \beta(yz), \beta(yw)$ , in terms of the translation distances of  $\gamma^*$  and  $\delta^*$ , i.e.  $l_M(\gamma)$  and  $l_M(\delta)$ . Now as with Lemma 2.7.5, we bound the length of  $\beta(xy)$ . Projecting back to  $M$ , we bound  $\rho_E(\gamma^*, \delta^*)$  as required. (What we have just described is a subset of the argument leading to Lemma 2.7.6.)

### 3.6. SHORT GEODESICS IN DEGENERATE ENDS.

In this section, we give proofs of Propositions 3.1.2 and 3.1.3, and describe a variant of the “a-priori bounds” result for our version of hierarchies.

As observed in Section 3.1 we can restrict attention to the case where  $\xi(\Sigma) \geq 2$ .

Let  $e \in \mathcal{E}_D(M)$ . Choose a neighbourhood  $E \cong [0, \infty)$  of  $e$  in  $\Psi(M)$  as in Section 3.5, and let  $D : E \rightarrow [0, \infty)$  be the depth function as defined in Section 3.4. Let  $\text{proj} : J(h_0, \infty) \rightarrow J(0, l_0)$  be the quasiprojection defined in Section 3.5. We recall that for all  $\alpha \in J(h_0, \infty)$ ,  $|D(\alpha) - D(\text{proj}(\alpha))| \leq h_0$ . Also, if  $\alpha, \beta \in \mathbf{C}(\Sigma)$  are adjacent in  $\mathcal{G}(\Sigma)$  and  $\alpha \in J(h_0, \infty)$ , then  $\beta \in J$  and we have  $|D(\alpha) - D(\beta)| \leq h_0$  and  $|D(\text{proj}(\alpha)) - D(\text{proj}(\beta))| \leq h_0$  and, at least when  $\sigma(\Sigma) \geq 2$ , also  $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta)) \leq k_0$ . These statements all follow directly from Lemma 3.5.6.

In Section 1.4, we defined “degenerate” as the negation of geometrically finite. In other words, every neighbourhood of the end meets some closed geodesic in  $M$ . In view of the Tameness Theorem, this can be considerably strengthened. In particular it is “geometrically tame” (or “simply degenerate”). One of several equivalent ways of saying this was given by Proposition 1.5.2. Here is another which we will use as the starting point of our discussion.

**Lemma 3.6.1.** *If  $e \in \mathcal{E}_D(M)$ , then there is a sequence of elements,  $(\gamma_i)_{i \in \mathbb{N}}$ , in  $J$  with  $D(\gamma_i) \rightarrow \infty$ , such that the geodesics  $\gamma_i^*$  all lie in  $\Psi(M)$  and tend out the end  $e$ .*

See [Th1, Bon, Cana, Bow6] for a discussion of this.

Note that there is no loss in assuming that each  $\gamma_i$  lies in  $J(0, l_0)$  (on replacing  $\gamma_i$  by  $\text{proj}(\gamma_i)$ , as given by Lemma 3.5.4, and applying Lemma 3.5.2).

*Remark.* In particular, this implies Proposition 1.5.2. Conversely, we could derive Lemma 3.6.1 from the conclusion of Proposition 1.5.2. To see this, note that if  $(\gamma_i)_i$  is a sequence of curves with bounded-length length representatives tending out the end, then their geodesic representatives,  $\gamma_i^*$  also tend out the end (by applying Lemma 3.4.6).

Given that  $J(0, l_0)$  is locally finite in  $\mathcal{G}(\Sigma)$  (by Lemma 3.4.5) the existence of such a sequence in  $J(0, l_0)$  is equivalent to saying that  $J(0, l_0)$  is infinite, or that it is unbounded.

The next step is to interpolate to get a “coarse path” of curves tending out the end. We consider first the case when  $\xi(\Sigma) \geq 2$ .

**Lemma 3.6.2.** *There exist  $k_0 = k_0(\xi(\Sigma))$  and  $l_0 = l_0(\xi(\Sigma))$  such that there is a sequence  $(\alpha_i)_i \in J(0, l_0)$  with  $d_{\mathcal{G}}(\alpha_i, \alpha_{i+1}) \leq k_0$  for all  $i$  with  $D(\alpha_i) \rightarrow \infty$ .*

*Proof.* Let  $h_0, k_0, l_0$  be as given by Lemma 3.5.4. From the observation above, we know that  $J(0, l_0)$  is infinite. Indeed,  $J(j, l_0)$  is infinite, for all  $h \geq 0$ .

Now choose any  $\beta_0 \in J(h_0, l_0)$ , and let  $D_0 = D(\beta_0) \geq h_0$ . Let  $\mathcal{H}$  be the graph with vertex set  $V(\mathcal{H}) = J(D_0, l_0)$  and with  $\alpha, \beta \in V(\mathcal{H})$  deemed adjacent if  $d_{\mathcal{G}}(\alpha, \beta) \leq k_0$ . Note that  $\mathcal{H}$  is a locally finite graph.

Let  $B = \{\beta \in V(\mathcal{H}) \mid D(\beta) \leq D_0 + 3h_0\}$ . Thus,  $B \subseteq V(\mathcal{H})$  is finite. We claim that each point of  $V(\mathcal{H})$  can be connected to  $B$  by some path in  $\mathcal{H}$ . To see this, suppose  $\beta \in V(\mathcal{H})$ , and let  $\beta_0, \beta_1, \dots, \beta_n = \beta$  be any path in  $\mathbf{C}(\Sigma)$  from  $\beta_0$



to  $\beta$ . Thus  $|D(\beta_i) - D(\beta_{i+1})| \leq h_0$  for all  $i$ . If  $\beta \notin B$ , there is some  $p$  so that  $D(\beta_i) \geq D_0 + h_0$  for all  $i \geq p$  and with  $D(\beta_p) \leq D_0 + 2h_0$ . Let  $\delta_i = \text{proj}(\beta_i)$  for all  $i \geq p$ . Then  $\delta_p \in B$ ,  $\delta_n = \beta$ , and  $d_G(\delta_i, \delta_{i+1}) \leq k_0$  for all  $i \geq p$ . We see that  $\delta_p, \delta_{p+1}, \dots, \delta_n$  connects  $B$  to  $\beta$  in  $\mathcal{H}$  as claimed.

In summary, we see that  $\mathcal{H}$  is infinite, locally finite, and has finitely many components. It therefore has a component of infinite diameter, which then contains an infinite arc  $(\alpha_i)_{i \in \mathbb{N}}$ . Note that  $D(\alpha_i) \rightarrow \infty$  as required.  $\square$

We can now prove the statements of Section 3.1, starting with Proposition 3.1.2. In the above notation, this says that, for some  $L \geq 0$  there is a sequence of elements of  $J(0, L)$  that form a geodesic in  $\mathbf{C}(\Sigma)$ .

*Proof of Proposition 3.1.2 :* We begin with the sequence,  $(\alpha_i)_{i \in \mathbb{N}}$ , given by Lemma 3.6.2. We want to replace this by a geodesic  $(\gamma_i)_{i \in \mathbb{N}}$ , perhaps at the cost of increasing the length bound for the closed geodesic representatives in  $M$ .

For each pair,  $i, j \in \mathbb{N}$ , let  $\pi(i, j) \subseteq \mathbf{C}(\Sigma)$  be a tight geodesic in  $\mathcal{G}(\Sigma)$  from  $\alpha_i$  to  $\alpha_j$ . Let  $m(i, j) = \min\{D(\delta) \mid \delta \in \pi(i, j)\}$  (where  $m(i, j) = 0$  if some curve of  $\pi(i, j)$  lies outside  $J(0, \infty)$ ). (Figure 3.3.) Note that  $\pi(i, i) = \{\alpha_i\}$ , so

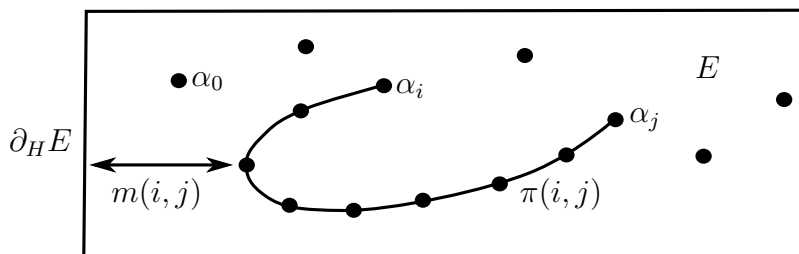


FIGURE 3.3. The dots represent curves in  $E$ . The path  $\pi(i, j)$  is a sequence of curves.

$m(i, i) \rightarrow \infty$  as  $i \rightarrow \infty$ . By Proposition 3.5.7(1), there is some  $h \geq 0$  such that if  $m(i, j) \geq h$ , then  $\pi(i, j) \subseteq J(0, L)$ . Here  $h$  and  $L$  depend only on  $\xi(\Sigma)$ .

By construction,  $d_G(\alpha_j, \alpha_{j+1}) \leq k_0$ , and so, by the hyperbolicity of  $\mathcal{G}(\Sigma)$ , the geodesics  $\pi(i, j)$  and  $\pi(i, j + 1)$  lie a bounded distance apart (depending only on  $\xi(\Sigma)$ ) for each  $i$  and  $j$ . From this it follows (using Lemma 3.5.1) that  $|m(i, j) - m(i, j + 1)|$  is bounded by some constant  $m_0 = m_0(\xi(\Sigma))$  for all  $i, j$ .

We distinguish two cases:

Case (1):  $(\exists i)(\forall j \geq i)(m(i, j) \geq h)$ .

We fix some such  $i$ . Then,  $\pi(i, j) \subseteq J(0, L)$  for all  $j \geq i$ . Let  $\gamma_k^j$  be the  $k$ th curve in  $\pi(i, j)$ , so that  $\gamma_0^j = \alpha_i$ . By Lemma 3.5.1,  $D(\gamma_k^j)$  is bounded above in terms of  $D(\alpha_i)$  and  $k$ , and so, as  $j$  varies, there are only finitely many possibilities for any given  $k$ . After passing to a diagonal subsequence, we can therefore suppose that for each  $k$ ,  $\gamma_k^j$  eventually stabilises on some curve  $\gamma_k \in J(0, L)$  as  $j \rightarrow \infty$ . Thus,

$(\gamma_k)_k$  gives us our required geodesic.

Case (2):  $(\forall i)(\exists j \geq i)(m(i, j) < h)$ .

Now  $m(i, i) \rightarrow \infty$ , and we can assume that  $m(i, i) \geq h + m_0$  for all  $i$ . For any  $i$ ,  $|m(i, j) - m(i, j + 1)| \leq m_0$  for all  $j$ , and so there is some  $j(i)$  such that  $h \leq m(i, j(i)) \leq h + m_0$ . Note that  $\pi(i, j(i)) \subseteq J(0, L)$ . Now let  $\delta_0^i \in \pi(i, j(i))$  be such that  $D(\delta_0^i) \leq h + m_0$ . Let  $(\delta_k^i)_{k=0}^{p_i}$  be the subpath of  $\pi(i, j(i))$  going backwards from  $\delta_0^i$  to  $\delta_{p_i}^i = \alpha_i$ . (Figure 3.4.) As in Case (1), we see that for

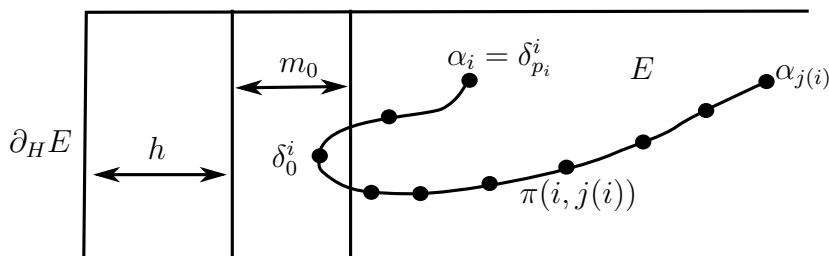


FIGURE 3.4. The dots represent curves in  $E$ .

any given  $k$ , there are only finitely many possibilities for the curve  $\delta_k^i$  as  $i \rightarrow \infty$ . Therefore, taking a diagonal sequence, we can suppose that  $\delta_k^i$  stabilises on some curve  $\delta_k \in J(0, L)$ . Thus  $(\delta_k)_k$  is the required geodesic.  $\square$

*Proof of Proposition 3.1.3 when  $\xi(\Sigma) \geq 2$ .* Let  $(\gamma_i)_i$  be a tight geodesic converging on some point,  $a \in \mathbf{a}(M, e) \in \partial \mathcal{G}(\Sigma)$ . By definition of  $\mathbf{a}(M, e)$ , there is a geodesic  $(\gamma'_i)_i$  lying in  $J(0, L_0)$  and converging to  $a$ . By hyperbolicity,  $(\gamma_i)_i$  and  $(\gamma'_i)_i$  eventually remain a uniformly bounded distance apart. In other words, up to shifting the indices, we can assume that  $d_{\mathcal{G}}(\gamma_i, \gamma'_i) \leq k_1 = k_1(\xi(\Sigma))$  for all  $i$ . Since  $\gamma'_i \in J(0, L)$  for all  $i$ , we can apply Proposition 3.5.7(2) to deduce that  $\gamma_i \in J(0, L')$  for all sufficiently large  $i$ , where  $L'$  depends only on  $L, k$  and  $\xi(\Sigma)$ , and hence ultimately only on  $\xi(\Sigma)$ .  $\square$

A combination of Propositions 3.5.7 and 3.5.8 now gives:

**Proposition 3.6.3.** *For all  $h$  there is some  $h'$  and for all  $l$  there is some  $r'$  such that if  $Q \subseteq J(h', l)$ , then  $Y(Q) \subseteq J(h, r')$ .*

If we iterate this procedure  $2\xi(\Sigma)$  times, then we arrive a set containing  $Y^\infty(Q)$ , as defined in Section 2.2. In particular, we see:

**Theorem 3.6.4.** *There is some  $h_1$ , such that if  $Q \subseteq J(h, l)$ , then  $Y^\infty(Q) \subseteq J(0, l')$ , where  $l'$  depends only on  $r$  and  $\xi(\Sigma)$ .*

In particular, this means that all curves in  $Y^\infty(Q)$  can be realised as closed geodesics of bounded length in  $M$ , which go out the end  $e$ . In particular, they are

all non-trivial and non-peripheral.

Finally, we consider the case where  $\xi(\Sigma) = 1$ .

*Proof of Theorem 3.1.1 when  $\xi(\Sigma) = 1$ .* As before, we start with a sequence of curves,  $(\alpha)_i$ , with  $l(\alpha_i)$  with  $\alpha_i^*$  tending out the end  $e$ . Thus  $D(\alpha_i) \rightarrow \infty$ . Let  $(\beta(i, j))_{j=1}^{p_i}$  be a geodesic from  $\alpha_0$  to  $\alpha_i$  in  $\mathcal{G}(\Sigma)$ . By Proposition 3.5.7,  $l(\beta(i, j))$  is bounded, so by Lemma 3.5.9,  $|D(\beta(i, j)) - D(\beta(i, j+1))|$  is bounded. It follows that  $p_i \rightarrow \infty$ . Moreover, for any given  $j$ ,  $D(\beta(i, j))$  is bounded, so the closed geodesics  $\beta(i, j)$  lie in compact subset of  $E$ . Therefore, there are only finitely many possibilities for  $\beta(i, j)$ . Taking a diagonal subsequence, we can suppose that  $\beta(i, j) = \gamma_j$  for some fixed curve  $\gamma_j$ . The sequence  $(\gamma_j)_j$  is now the required geodesic.  $\square$

Note that in this case, there are no proper subsurfaces, so Theorem 3.6.4 is immediate.

We are now ready to adapt the model to this case, as we describe in the next section.

### 3.7. THE MODEL SPACE OF A DEGENERATE END.

Let  $M$  be a complete hyperbolic 3-manifold, with  $\pi_1(M)$  finitely generated, and let  $e \in \mathcal{E}_D(M)$ . Let  $\Sigma = \Sigma(e)$ . We can assume that  $\xi = \xi(\Sigma) \geq 2$ .

We describe a model,  $\Psi(P_e) \cong \Sigma \times [0, \infty)$ , and a proper map,  $f_e : \Psi(P_e) \rightarrow \Psi(M)$ , which sends  $\Psi(P_e)$  out the end  $e$ . It will be a proper homotopy equivalence into a neighbourhood,  $E \cong \Sigma \times [0, \infty)$ , of  $e$  in  $\Psi(M)$ . It then follows that  $f_e$  maps surjectively to another neighbourhood  $E_0 \subseteq E \subseteq \Psi(M)$  of  $e$ .

Only the initial discussion, as far as Proposition 3.7.1, is directly relevant to the proof of the Ending Lamination Theorem (completed in Section 3.8). The remainder establishes that the end invariant is well defined, and that it has a number of natural descriptions.

Let  $E \cong \Sigma \times [0, \infty)$  be a neighbourhood of  $e$  in  $\Psi(M)$ . Let  $\rho_E$  be the electric pseudometric on  $E$ , as defined in Section 3.4. We recall Proposition 3.1.2, namely that there is a geodesic ray in  $\mathcal{G}(e)$  each of whose elements have representatives of length at most  $L_0 = L_0(\xi)$  in  $E$ , and which go out the end  $e$ . In Section 3.1, we defined  $\mathbf{a}(e) \subseteq \partial\mathcal{G}(e)$  as the set of endpoints of such geodesic rays. Proposition 3.1.3 tells us that the vertices of any tight geodesic which tends to a point of  $\mathbf{a}(e)$  also have bounded length representatives in  $M$ .

We can continue, as in Section 2.14, to construct an annulus system,  $W = \bigcup \mathcal{W}$ , in  $\Sigma \times [0, \infty)$ . In particular, the curves corresponding to  $\Omega$  are all a bounded distance (in fact 1) from a geodesic ray,  $\pi$ , in  $\mathcal{G}(e)$ . This can be assumed to converge on any given  $a \in \mathbf{a}(e)$ . Moreover, for all  $\Omega \in \mathcal{W}$ , we have  $\Omega^* \subseteq E$  and  $\text{length}(\Omega^*) \leq L = L(\xi)$ . This follows from Proposition 3.5.8, exactly as in

the incompressible case. (In fact, as observed in the proof of Proposition 3.5.8, provided  $\Phi$  is sufficiently deep in the end, it must be incompressible. However, we will not formally need that.) In fact, we can also assume that  $D(\Omega^*) = \rho_E(\Omega^*, \partial E) \geq t$  for an arbitrarily large constant  $t \geq 0$  (chosen as below).

Now let  $\tilde{\Psi}(P_e)$  be the universal cover of  $\Psi(P_e)$ . There is a cover  $\hat{E}$  of  $E$ , that naturally embedded in  $\mathbb{H}^3$ , namely a component of the preimage of  $E$  under the covering map  $\mathbb{H}^3 \rightarrow M$ . Let  $\hat{\Psi}(P_e)$  be the corresponding cover of  $\Psi(P_e)$ . That is,  $\tilde{\Psi}(P_e)$  is triangulated by the truncated complex,  $R(\Pi)$ . We will construct an equivariant map  $\hat{f}_e : \hat{\Psi}(P_e) \rightarrow \hat{E}$  and then project to give us the map  $f_e : \Psi(P_e) \rightarrow \Psi(M_e)$ . For this we use the simplicial complex  $\Pi$  described as in Section 2.7, associated to the universal cover of  $\Psi(P_e)$ . That is,  $\tilde{\Psi}(P_e)$  is triangulated by the truncated complex,  $R(\Pi)$ . There is a quotient,  $\hat{\Pi}$  corresponding to  $\hat{\Psi}(P_e)$ . This satisfies the conditions (C1)–(C7) laid out in Section 2.7, so we can construct  $\hat{f}_e : \hat{\Psi}(P_e) \rightarrow M$  as in Section 2.8. In fact, this maps to the preimage of  $\Psi(M)$  in  $\mathbb{H}^3$ . Its image is a bounded distance from the union of the preimages of closed geodesics  $\bar{\Omega}$  and Margulis tubes  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}$ . Thus, by taking  $t$  large in relation to this bound, we will have  $\hat{f}_e(\hat{\Psi}(P_e)) \subseteq \hat{E}$ . Projecting, we get a proper map,  $f_e : \Psi(P_e) \rightarrow E$ . Let  $E_0 \subseteq \Psi(M_e)$  be a neighbourhood of the end, homeomorphic to  $\Sigma(e) \times [0, \infty)$ , contained in  $f_e(\Psi(P_e))$ .

Now, by Proposition 2.3.7, as in Section 2.14, the collection of Margulis tubes in  $E$  is unlinked. Since this includes all tubes with sufficiently short core curves, we note as a consequence:

**Proposition 3.7.1.** *There is some  $\eta(\xi) > 0$  depending only on  $\xi = \xi(\Sigma)$  such that if  $e \in \mathcal{E}_D$  is any degenerate end of  $\Psi(M)$ , then there is a neighbourhood,  $E$ , of  $e$  in  $\Psi(M)$ , with  $E \cong \Sigma(e) \times [0, \infty)$  such that the set of all closed geodesics in  $M$  of length at most  $\eta(\xi)$  and lying in  $E$  are unlinked in  $E$ .*

We can go on to show that the end invariant of  $e$  is well defined:

**Proposition 3.7.2.** *If  $e \in \mathcal{E}_D$ , then  $\mathbf{a}(e)$  is a singleton.*

*Proof.* Let  $a \in \mathbf{a}(e)$ . Let  $\Psi(P_e)$  be a model based on an annulus system  $W = \bigcup \mathcal{W}$  in  $\Sigma(e) \times [0, \infty)$ , where the core curves lie a bounded distance from a geodesic ray,  $\pi \subseteq \mathcal{G}(e)$  tending to  $a \in \partial \mathcal{G}(e)$ . As with Proposition 3.7.1, we see that every closed curve of length at most  $\eta(\xi)$  in  $E$  corresponds to a Margulis tube in  $\Psi(P_e)$ , and so lies a bounded distance from  $\pi$  in  $\mathcal{G}(e)$ .

If  $a' \in \mathbf{a}(e)$ , then we can similarly construct  $\pi'$ ,  $W' = \bigcup \mathcal{W}'$ , a model  $\Psi(P'_e)$ , and a map  $f'_e : \Psi(P'_e) \rightarrow \Psi(M_e)$ . We split into two cases.

Suppose there is a sequence of closed geodesics of length at most  $\eta(\xi)$  going out the end  $e$  of  $\Psi(M)$ . The curves must lie a bounded distance from both  $\pi$  and  $\pi'$ . It follows that  $\pi$  and  $\pi'$  are parallel, and so  $a = a'$ .

Suppose there is no such sequence. Then we can assume that  $E$  contains no such closed geodesic. In other words, it has bounded geometry. Now every point  $x$  of  $E$

lies a bounded distance from a closed geodesic of the form  $\bar{\Omega}$  for  $\Omega \in \mathcal{W}$ . It also lies a bounded distance from another  $\bar{\Omega}'$  for  $\Omega'$  in  $\mathcal{W}'$ . We can now apply Lemma 2.9.3, with  $\Theta = E$ ,  $\Sigma = \Sigma(e)$ , and  $B \subseteq \Theta$  a band of the form  $\Sigma \times [0, h] \subseteq \Sigma \times [0, \infty) \equiv E$ . This tells us that the curves corresponding to  $\Omega$  and  $\Omega'$  are a bounded distance apart in  $\mathcal{G}(e)$ . Also, by the construction of the models, such curves lie a bounded distance respectively from  $\pi$  and  $\pi'$  in  $\mathcal{G}(e)$ . Letting  $x$  tend out the end  $e$ , we see again that  $\pi$  and  $\pi'$  are parallel in  $\mathcal{G}(e)$  and so  $a = a'$ .  $\square$

We can now define  $a(e)$  by setting  $\mathbf{a}(e) = \{a(e)\}$ .

Here is another, more natural way of describing the end invariant:

**Proposition 3.7.3.** *Let  $\pi$  be any geodesic ray in  $\mathcal{G}(e)$  tending to  $a(e)$  in  $\partial\mathcal{G}(e)$ . Given any  $l \geq 0$ , there is some  $r$  depending only on  $\xi(\Sigma(e))$  and  $l$ , and a neighbourhood  $E(l) \cong \Sigma \times [0, \infty)$ , of  $e$  in  $\Psi(M)$ , with the following property. If  $\gamma \in \mathbf{C}(\Sigma)$  with  $\gamma_M^* \subseteq E(l)$  and  $\text{length}(\gamma_M^*) \leq l$ , then  $d_{\mathcal{G}}(\pi, \gamma) \leq r$*

It follows that any sequence of simple curves of bounded length in  $\Psi(M)$  going out the end  $e$  must converge on  $a(e)$  in  $\mathcal{G}(e)$ . This therefore gives another characterisation of  $a(e)$ .

Note also that  $d_{\mathcal{G}}(\gamma, \text{proj } \gamma)$  is bounded in terms of  $\xi(\Sigma)$ . Thus, in Proposition 3.7.3, there is no loss of generality in setting  $l = l_0$ , in which case,  $r$  will depend only on  $\xi$ .

*Proof of Proposition 3.7.3 :* This elaborates on the proof of Proposition 3.7.2.

We can take  $\pi$  to be any geodesic converging to  $a(e)$ . In particular, we can take it to be the geodesic featuring in the construction of the annulus system  $W = \bigcup \mathcal{W}$ , and the resulting model,  $\Psi(P_e)$  described above. Thus,  $d_{\mathcal{G}}(\pi_V(\Omega), \pi)$  is bounded above in terms of  $\xi(\Sigma)$  for all  $\Omega \in \mathcal{W}$ . We construct  $f_e : \Psi(P_e) \rightarrow \Psi(M)$  as before. We have neighbourhoods  $E, E_0 \cong \Sigma \times [0, \infty)$  with  $E_0 \subseteq f_e(\Psi(P_e)) \subseteq E$ .

Let  $\rho_E$  be the electric pseudometric on  $E$  (as defined in Section 3.4), and let  $D : E \rightarrow [0, \infty)$  be the depth function. Let  $\gamma^* \subseteq E$  be a closed geodesic in  $M$ , of length at most  $l_0$ , with  $\gamma = [\gamma^*] \in \mathcal{G}(e) \equiv \mathcal{G}(\Sigma)$ , and with  $D(\gamma^*)$  sufficiently large depending on  $l$ . We want to show that  $d_{\mathcal{G}}(\gamma, \pi)$  is bounded in terms of  $\xi(\Sigma)$ .

We fix some closed geodesic  $\beta^* \subseteq E_0$  with  $\text{length}(\beta^*) \leq l_0$ , and with  $\beta := [\beta^*] \in \mathbf{C}(\Sigma)$ . We can take  $D_1 := D(\beta^*)$  to be arbitrarily large. We choose some  $D_2 > D_1$ , with  $D_2 - D_1$  sufficiently large, as determined below. We can assume that  $D(\gamma^*) \geq D_2$ .

We first claim that there is a sequence,  $\gamma = \gamma_0, \gamma_1, \dots, \gamma_p$ , in  $\mathbf{C}(\Sigma)$  which is geodesic in  $\mathcal{G}(\Sigma)$ , and with  $\text{length}(\gamma_i^*) \leq l_1$ ,  $\gamma_i^* \subseteq E$ , and  $D(\gamma_i^*) \geq D_1$  for all  $i \in \{0, \dots, p-1\}$ , and with  $D(\gamma_p) \leq D_1$ . Here  $l_1$  depends only on  $\xi(\Sigma)$ . Note that, necessarily  $\rho_E(\gamma_i^*, \gamma_{i+1}^*)$  is bounded above in terms of  $\xi(\Sigma)$ . In particular,  $D_1 - D(\gamma_p)$  is bounded.

To prove the claim, first note that we can find a sequence  $\gamma = \beta_0, \beta_1, \dots, \beta_q$  with  $\text{length}(\beta_j^*) \leq l_0$ ,  $d_{\mathcal{G}}(\beta_j, \beta_{j+1}) \leq k_0$  for all  $j$ ,  $D(\beta_j^*) > D_1$  for  $j < q$ , and  $D(\beta_q) \leq D_1$ . For this, we apply the argument of Proposition 3.6.2. We now

connect  $\beta_0$  to  $\beta_j$  by a tight geodesic  $\pi_j$  in  $\mathcal{G}(\Sigma)$ . We note that consecutive  $\pi_j$  remain a bounded distance apart in  $\mathcal{G}(\Sigma)$ . As in the proof of Proposition 3.1.2, we can obtain  $(\gamma_i)_{i=0}^p$  as an initial segment of  $\pi_j$ , for some  $j$ . The bound on  $\text{length}(\gamma_i^*)$  is a consequence of the A-priori Bounds Theorem (see Theorem 3.6.4).

We now proceed as in Lemma 2.4.1 to construct an annulus system,  $W_0 = \bigcup \mathcal{W}_0$  in  $\Sigma \times [-1, 1]$  with  $\gamma = \gamma_0 \subseteq \mathbf{C}_-(\mathcal{W}_0)$  and  $\gamma_p \subseteq \mathbf{C}_+(\mathcal{W}_0)$ . (Here  $\mathbf{C}_\pm(\mathcal{W}_0) = \pi_V(W \cap (\Sigma \times \{\pm 1\}))$ .) From the construction, we can assume that  $\gamma_i = \pi_V(\Omega_i)$  for some  $\Omega_i \in \mathcal{W}_0$ . From the A-priori Bounds Theorem (Theorem 3.6.4), we have  $\text{length}(\Omega_i^*) \leq l'$  for all  $\Omega \in \mathcal{W}_0$ , where  $l' = l'(\xi(\Sigma))$  depends only on  $\xi(\Sigma)$ . Also, taking  $D_1$  large enough, we can assume that  $D(\Omega^*)$  is greater than some arbitrarily large constant for  $\Omega \in \mathcal{W}_0$ . We now use  $\mathcal{W}$  to construct a model,  $\Psi(P)$ , where each annulus  $\Omega$  is replaced by a Margulis tube  $T(\Omega)$ , or an annulus  $\Delta(\Omega)$ , where  $\Omega$  meets  $\partial_H \Psi(P)$ .

Given that the curves  $\Omega^*$  all have bounded length and arbitrary depth in  $E$ , we can construct a Lipschitz homotopy equivalence  $f : \Psi(P) \rightarrow E$  as with  $\Psi(P_e)$ , using the construction of Section 2.8. Moreover, we can assume that  $f(\Psi(P)) \subseteq E_0$ . The map  $f$  sends the boundary curves  $\gamma_p$  and  $\gamma$  to  $\gamma_p^*$  and  $\gamma^*$  respectively.

Since  $D(\gamma^*) - D(\gamma_p^*) \leq D_2$  is arbitrarily large,  $\rho_E(\gamma_p^*, \gamma^*)$  is arbitrarily large. Now  $d_{\mathcal{G}}(\gamma_p, \gamma)$  is linearly bounded below in terms of  $\rho_E(\gamma_p^*, \gamma^*)$  and so we can assume that  $p = d_{\mathcal{G}}(\gamma_p, \gamma)$  to be as large as we want. We fix  $m \in \mathbb{N}$ , as determined below. We can assume that  $p \geq m$ , so that (after reparametrising the second coordinate)  $\Sigma \times \{0\}$  meets  $W$  in a curve,  $\alpha$ , with  $d_{\mathcal{G}}(\alpha, \gamma) = m$ . Let  $\Psi_0 \subseteq \Psi(P)$  be the subset of the model corresponding to  $\Sigma \times [-1, 1]$ . Thus  $\gamma$  and  $\alpha$  are boundary curves in  $\partial_H \Psi_0$ . Note that  $f(\alpha) = \alpha^*$ .

We distinguish two cases.

Suppose first that  $f(\Psi_0)$  is not a subset of the thick part,  $\Theta(M)$ . In other words, it meets some Margulis tube lying in  $E_0 \subseteq f_e(\Psi(P_e))$ . This Margulis tube must feature in both  $\Psi_0$  and  $\Psi(P_e)$ . In particular, its core curve corresponds to an annulus in both  $\mathcal{W}$  and  $\mathcal{W}_0$ . It is therefore a bounded distance from both  $\gamma$  and  $\pi$ . Thus,  $d_{\mathcal{G}}(\gamma, \pi)$  is bounded above in terms of  $m$  and hence in terms of  $\xi(\Sigma)$ .

We can therefore assume that  $f(\Psi_0) \subseteq \Theta(M)$ . This implies that the injectivity radius in  $\Psi_0$  is also bounded below. The idea now is to find a sufficiently wide band  $B$  in  $E$ , with base surface  $\Sigma$ , lying in  $f(\Psi_0)$  and  $f(\Psi(P))$ . This will then contain bounded-length (in fact closed geodesic) realisations of curves in the respective annulus systems. (As usual, by a ‘‘band’’,  $B$ , we mean a subset of  $E$  bounded by two disjoint horizontal fibres,  $\partial_+ B$  and  $\partial_- B$ .) We can then apply Lemma 2.9.3 to tell us that these curves are a bounded distance apart in  $\mathcal{G}(\Sigma)$ .

To this end, we know that  $f : \Psi(P) \rightarrow E$  is a quasi-isometry with respect to the electric pseudometrics. Thus, if  $m$  is sufficiently large, we have  $\rho_E(\partial_- \Psi_0, \partial_+ \Psi_0)$  large. In particular (using Theorem 1.6.1, as in the proof of (Q3) of Lemma 2.10.2), we can assume that  $f(\Psi_0)$  contains a band  $B$  with

$\partial_{\pm}B$  of bounded diameter, and with  $d_M(\partial_-B, \partial_+B)$  large. Thus,  $B$  will be  $k$ -convex, where  $k$  depends only on the diameters of  $\partial_{\pm}B$ . Moreover, since  $B \subseteq f(\Psi_0) \cap f_e(\Psi(P_e))$ , we can assume that it contains closed geodesics,  $\Omega^*$  and  $\Omega_0^*$ , where  $\Omega \in \mathcal{W}$  and  $\Omega_0 \in \mathcal{W}_0^*$ . Writing  $\delta = [\partial_V\Omega]$ ,  $\delta_0 = [\partial_V\Omega_0] \in \mathbf{C}(\Sigma)$ , Lemma 2.9.3, tells us that  $d_{\mathcal{G}}(\delta, \delta_0)$  is bounded. In the above, the bounds depend only on  $\xi(\Sigma)$ . We can therefore fix some  $m$  sufficiently large in relation to  $\xi(\Sigma)$  to make the argument work. Now,  $d_{\mathcal{G}}(\delta, \pi)$  and  $d_{\mathcal{G}}(\delta_0, \gamma)$  are bounded in terms of  $\xi(\Sigma)$ . It follows that  $d_{\mathcal{G}}(\gamma, \pi)$  is bounded, as required.  $\square$

As noted in Section 1.5, equivalent characterisations can be formulated in terms of pleated surfaces or non-realizability of laminations.

### 3.8. PROOF OF THE ENDING LAMINATION THEOREM IN THE GENERAL CASE.

In this section, we explain how the model space is constructed and give a proof of Proposition 1.7.1 in the general case. We go on to explain how this implies the Ending Lamination Theorem.

We recall from Section 2.6, the notions of (universally) sesquipschitz maps.

Suppose  $\Psi$  is a topologically finite 3-manifold such that  $\partial\Psi$  is a disjoint union of tori and cylinders. Suppose we have a decomposition of its set of ends as  $\mathcal{E} = \mathcal{E}_F \sqcup \mathcal{E}_D$ , such that no base surface of any end is a disc, annulus, sphere or torus, and no base surface of an end of  $\mathcal{E}_D$  is a three-holed sphere. Suppose that to each end  $e \in \mathcal{E}_D$ , we have associated some  $a(e) \in \partial\mathcal{G}(e)$ . From this data, we will construct a riemannian manifold,  $P$ , without boundary, with a submanifold,  $\Psi(P)$ , which is homeomorphic to  $\Psi$ . We show that it satisfies the conclusion of Proposition 1.7.1.

The construction of  $P$  will be essentially the same for the irreducible case. If  $e \in \mathcal{E}_F$ , we take the same model  $\Psi(P_e)$  as before. If  $e \in \mathcal{E}_D$ , we construct a model  $\Psi(P_e)$  as in Section 3.7. (If  $\xi(\Sigma(e)) = 1$ , then the end is irreducible, and we construct  $P_e$  as in the indecomposable case.) We now construct  $\Psi(P)$  by attaching the  $\Psi(P_e)$  to a core with an arbitrary riemannian metric extending that of the  $\Psi(P_e)$ , and such that the components of  $\partial\Psi(P)$  are all intrinsically euclidean cylinders or unit square tori. Finally, we construct  $P$  by gluing in cusps to these boundary components exactly as in the incompressible case.

We construct a map  $f : P \rightarrow M$  in stages. First, we define  $f_e : \Psi(P_e) \rightarrow \Psi(M)$  for each  $e \in \mathcal{E}$  and then extend arbitrarily over the compact core to give  $f : \Psi(P) \rightarrow \Psi(M)$ , and then over the cusps in the obvious way as before.

Suppose the  $e \in \mathcal{E}_F$ . Here the construction is as in the incompressible case. We made use of multiplicative bounds of distance distortion when projecting to neighbourhoods of the convex core, but the same bounds remain valid in this case.

Suppose  $e \in \mathcal{E}_D$ . We defined  $f_e : \Psi(P_e) \rightarrow \Psi(M)$  in Section 3.7. We can now proceed, exactly as in Section 2.14 to show that  $f_e : \Psi(P_e) \rightarrow E_e$  is universally sesquipschitz, where  $E_e = E$  is the neighbourhood defined in Section 3.7.

The extension to the core,  $\Psi_0$ , and then to the cusps is essentially elementary. This gives us a Lipschitz map  $f : P \rightarrow M$ , and by construction  $f|_{\Psi(P)}$  is proper, and each end of  $\Psi(P)$  goes to the corresponding end of  $\Psi(M)$ .

Since for all  $e \in \mathcal{E}$ , the map  $f_e : \Psi(P_e) \rightarrow \Psi(E_e)$  is universally sesquilipschitz, hence sesquilipschitz, the map  $\hat{f}_e : \hat{\Psi}(P_e) \rightarrow \hat{E}$  is sesquilipschitz. Note that  $\hat{\Psi}(P_e)$  and  $\hat{E}_e$  are subsets of the universal covers of  $\Psi(P)$  and  $\Psi(M)$  respectively. This is what we use when gluing the pieces together, to see, as in Section 2.14, that the lift between universal covers is a quasi-isometry.

This proves Proposition 1.7.1 in the general case.

Now we can use the same model space for two homeomorphic hyperbolic 3-manifolds with the same degenerate end invariants to deduce:

**Proposition 3.8.1.** *Suppose that  $M$  and  $M'$  are complete hyperbolic 3-manifolds and that there is a homeomorphism from  $M$  to  $M'$  that sends each cusp of  $M$  into a cusp of  $M'$  and conversely. Suppose that the induced map between the non-cuspidal parts sends each geometrically finite end to a geometrically finite end and each degenerate end to a degenerate end. Suppose that (under the induced homeomorphisms of base surfaces) the end invariants of corresponding pairs of degenerate ends are equal. Then there is an equivariant quasi-isometry between the universal covers of  $M$  and  $M'$ .*

This statement is identical to Proposition 2.14.5, except with the hypothesis of “indecomposable” omitted. The argument is the same, given Proposition 1.7.1.

In particular, we get an equivariant quasiconformal extension,  $f : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$ . In the case where the geometrically finite end invariants are also equal, we can find another equivariant map,  $g : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$ , which agrees with  $f$  on the limit set, and is conformal on the discontinuity domain. We need to verify that  $g$  is also quasiconformal.

Let  $U$  be a component of the domain of discontinuity of  $\Gamma \equiv \pi_1(M)$  acting on  $\partial\mathbb{H}^3$ . As in Section 2.14, we see that  $g^{-1} \circ f|_U$  moves every point a bounded distance in the Poincaré metric. To show that  $g^{-1} \circ f$ , and hence  $g$ , is quasiconformal, it is enough to give a suitable bound on the Euclidean metric,  $d_e$ , in terms of the Poincaré metric (using an identification of  $\partial\mathbb{H}^3$  with  $\mathbb{C} \cup \{\infty\}$ ). In Section 2.14, the formula we used was:  $d_e(z, w) \leq (e^{2k} - 1) \max\{d_e(z, \partial U), d_e(w, \partial U)\}$ , under the assumption that  $U$  was simply connected. We want a variant of this when  $U$  might not be simply connected. Let  $\Delta$  be the universal cover of  $U$ , which we identify with the Poincaré disc. We have a group,  $G$ , acting on  $\Delta$ , and a normal subgroup,  $H \triangleleft G$ , such that  $U = \Delta/H$  and  $R = \Delta/G$  is a Riemann surface of finite type — here the corresponding geometrically finite end invariant. Note that  $H$  has no parabolic elements (since the limit set has no isolated points).

**Lemma 3.8.2.** *If  $a, b \in U$  are a distance at most  $k$  apart in the Poincaré metric on  $U$ , then  $d_e(a, b) \leq (e^{\mu k} - 1) \max\{d_e(a, \partial U), d_e(b, \partial U)\}$ , where  $\mu > 0$  is a constant depending only on the Riemann surface  $R$ .*



*Proof.* Choose  $t > 0$  so that the shortest closed geodesic on  $R$  has length greater than  $2t$  in the Poincaré metric. Since  $H$  has no parabolics, any  $t$ -disc in  $R$  lifts to an embedded disc in  $U$ . Put another way, if  $z \in U$ , then the  $t$ -disc,  $D$ , about  $z$  in the Poincaré metric is embedded in  $U$ . We can lift this to the Poincaré disc,  $\Delta$ , centred at  $z$ . Here  $D$  will have euclidean radius  $\tanh t$ . Now consider the map  $D \rightarrow U$  with respect to the euclidean metric on both  $D$  and  $U$ . By the Koebe Quarter Theorem, the norm of the derivative at  $z$  is at most  $\frac{r}{4 \tanh t}$  at the centre, where  $r = d_e(z, \partial U)$ . But a euclidean unit vector at the origin has Poincaré norm 2, and so we deduce that  $|ds| \geq \frac{1}{\mu r} |dz|$ , where  $\mu = 2/\tanh t$ , and where  $|ds|$  and  $|dz|$  are the infinitesimal Poincaré and euclidean metrics. We can now integrate, as in Lemma 2.14.7, to derive the required inequality.  $\square$

Now since there are only finitely many geometrically finite ends we can take the same constant  $\mu$  for all components of the discontinuity domain. Thus, the inequality of Lemma 3.8.2 applies equally well when  $U$  is replaced by the whole discontinuity domain. This is sufficient to bound the metric quasiconformal distortion of  $g^{-1} \circ f$  on the limit set, showing that  $g^{-1} \circ f$  and hence  $f$  is quasiconformal by Lemma 2.14.8.

The remainder of the argument is now standard. We see using the result of Sullivan, stated here as Theorem 1.6.8, that  $f$  must be conformal. It therefore gives rise to an isometric conjugacy between the actions of  $\pi_1(M)$  and  $\pi_1(M')$  on  $\mathbb{H}^3$ , showing that  $M$  and  $M'$  are isometric. The fact that this isometry satisfies the conditions laid out in Theorem 1.5.4 is now elementary.

This finally proves the Ending Lamination Theorem, stated here as Theorem 1.5.4.

#### 4. PROMOTION TO A BILIPSCHITZ MAP

In this chapter, we will describe how the universally sesquipschitz map between the model space and the hyperbolic 3-manifold can be promoted to a bilipschitz homeomorphism. It is not essential to the proof of the Ending Lamination Theorem as we have presented it, but has a number of other applications.

##### 4.1. SUBSURFACES IN TRIANGULATED 3-MANIFOLDS.

In this section, we prove a technical lemma about realising surfaces in a 3-manifold with respect to a given triangulation. Clearly one can easily push any surface into the 2-skeleton. The statement here is that, if an embedded surface is not too “dense”, then after subdividing the triangulation a bounded number of times, one can isotope the surface a small distance, so that it ends up as an embedded surface in the 2-skeleton.

Since triangulations and cubulations are easily interchangeable via subdivision, one can equivalently phrase the result for cubulations. (That is a subdivision into

3-cubes meeting along their faces.) Since the proof is easier to describe in that setting, we formulate it in that way first (Proposition 4.1.1). We reformulate it for triangulations at the end (Proposition 4.1.2). The result will be used in Section 4.4, but is otherwise independent from the rest of the paper.

Let  $X$  be a cube complex, with induced path metric,  $d_X$ , where each cube is given the structure of a unit euclidean cube. (We are not placing any curvature restrictions on the cubulation.) Write  $\mathcal{Q}^n = \mathcal{Q}^n(X)$  for the set of (closed)  $n$ -cells, and  $X^n = \bigcup_{i \leq n} \mathcal{Q}^i(X)$  for the  $n$ -skeleton. Write  $\mathcal{Q} = \bigcup_i \mathcal{Q}^i$  for the set of all cells. Given a subset  $A \subseteq X$ , write  $N(A) = \bigcup \{Q \in \mathcal{Q} \mid A \cap Q \neq \emptyset\}$ . We define the **local complexity** of the cubulation to be the maximal number of  $n$ -cells containing any given vertex. (We will always be assuming this to be finite.)

Write  $X_p$  for the  $p$ -ary subdivision of  $X$ . That is, we divide each  $n$ -cell into  $p^n$  equal cubes. We shall take these to have side length  $1/p$ , so that the induced path metric agrees with  $d_X$ . (In practice,  $p$  will be controlled, so rescaling to unit cubes will give the same result up to uniform bilipschitz constants.)

Now suppose that  $X$  is (homeomorphic to) a 3-manifold, with (possibly empty) boundary,  $\partial X$  (necessarily a subcomplex). Let  $S \subseteq X$  be any embedded subsurface (possibly disconnected) with  $\partial S = S \cap \partial X$ . Let  $\Delta(S) = \sup_Q (\text{area}(S \cap Q))$ , where  $Q$  varies over all 3-cells of the cubulation. Here area is always measured with respect to euclidean Lebesgue measure arising from the cubulation. We refer to  $\Delta(S)$  as the **density** of  $S$ .

We say that  $S$  is **incompressible** if each component is  $\pi_1$ -injective. (See Section 1.5.) Using Dehn's Lemma, this is equivalent to saying that if  $D \subseteq X$  is an embedded disc with  $D \cap \partial S = \partial D$ , then  $\partial D$  also bounds a disc in  $S$ . We say that  $S$  is **boundary incompressible** if whenever  $D \subseteq X$  is a disc with  $\partial D = \alpha \cup \beta$  and  $\alpha = D \cap S$  and  $\beta = D \cap \partial X$  are both arcs, then  $\alpha$  cuts off a disc,  $D'$ , in  $S$ .

**Proposition 4.1.1.** *Given any  $\Delta, C \geq 0$  there some  $p \geq 0$  with the following property. Suppose that  $X$  is a cubulated aspherical 3-manifold of local complexity at most  $C$ , with incompressible (possibly empty) boundary. Suppose that  $S \subseteq X$  is a (possibly disconnected) incompressible and boundary incompressible subsurface with density,  $\Delta(S) \leq \Delta$ . Then there is a surface  $S' \subseteq X_p^2$  (that is contained in the 2-skeleton of the  $p$ -ary subdivision) isotopic to  $S$  with  $S' \subseteq N(S)$ .*

(One can refine the statement in various ways. For example, it holds for all sufficiently large  $p$  in relation to  $\Delta$ . One could also remove the restriction on local complexity and instead place a bound on the area of the surface intersected with the link of any vertex. With some elaboration, our arguments would prove this assertion, but they are not needed for applications.)

To simplify the exposition, we first assume that  $\partial \Sigma \neq \emptyset$ . We explain at the end how one deals with the general case.

First we explain how the cellulation gives rise to a handle decomposition of  $X$ , using the ternary subdivision,  $X_3$ , of  $X$ .

Note that each  $Q \in \mathcal{Q}^3(X)$  is cut into 27 subcubes in  $\mathcal{Q}^3(X_3)$ , namely 1 “centre cube”, 6 “side cubes”, 12 “edge cubes” and 8 “corner cubes”, depending on whether the minimal dimension of a face of  $Q$  which it meets is 3, 2, 1 or 0. In fact, any  $Q \in \mathcal{Q}(X)$  has associated with it a “centre cube”, denoted  $\text{cent}(Q) \in \mathcal{Q}(X_3)$ , of the same dimension.

Given  $Q \in \mathcal{Q}(X)$ , let  $\mathcal{S}(Q) = \{P \in \mathcal{Q}^3(X_3) \mid \text{cent}(Q) \subseteq P\}$ , and let  $H = \bigcup \mathcal{S}'(Q)$ . This is always a topological 3-ball. We write  $\partial H$  for its boundary, and  $\text{int}(H) = H \setminus \partial H$  for its interior.

Write  $\mathcal{H}^n \{\bigcup \mathcal{S}(Q) \mid Q \in \mathcal{Q}^{3-n}(X)\}$ , and  $\mathcal{H} = \bigcup_{n=0}^3 \mathcal{H}^n$ . We refer to an element of  $\mathcal{H}^n$  as an  $n$ -**handle**. The interiors of the handles are all disjoint, and  $X = \bigcup \mathcal{H}$ . Note that if  $Q \in \mathcal{Q}^3(X)$ , then  $H(Q) = \text{cent}(Q)$ . In other words, the 0-handles are precisely the centre 3-cubes.

Given  $Q \in \mathcal{Q}(X)$  and  $H = H(Q)$ , write  $\mathcal{L}(Q) \subseteq \mathcal{Q}^2(X_3)$  for the set of 2-cells of  $X_3$  which are contained in  $H$  but which do not meet  $\text{cent}(Q)$ . We write  $L = L(H) = \bigcup \mathcal{L}(Q)$ . Thus,  $L \subseteq \partial H$ . We write  $L^C = L^C(H)$  for the closure of  $H \setminus L(H)$ .

If  $H$  is a 0-handle, then  $L = \emptyset$ . If  $H$  is a 1-handle (the union of two side cubes meeting in a centre 2-cube) then  $L$  is the disjoint union of two discs, where  $H$  attaches to the 0-handles. If  $H$  is a 2-handle, then  $L$  is an annulus, where it attaches to the union of the 0- and 1-handles. If  $H$  is a 3-handle, then  $L = \partial H$ .

Now suppose that  $S \subseteq X$  is an incompressible surface, which we assume initially to be piecewise linear and in general position with respect to the cubulation. We will first isotope it so that it intersects each 2-cell in a bounded number of arcs, and each 3-cell in a bounded number of discs. We will achieve this in a series of four steps dealing with 0, 1, 2 and 3 handles in turn.

Suppose that  $\gamma \subseteq S$  is a multicurve. Given a homotopically trivial component  $\alpha$  of  $\gamma$ , write  $D_S(\alpha) \subseteq S$  for the disc bounded by  $\alpha$ . We say that  $\alpha$  is  $S$ -**outermost** if  $D_S(\alpha)$  is not properly contained in  $D_S(\beta)$  for any other component,  $\beta$ , of  $\gamma$ .

Step 0:

Let  $Q \in \mathcal{Q}^3(X)$ , and let  $H = H(Q) = \text{cent } Q$  be the corresponding 0-handle.

Define  $\phi : Q \rightarrow [0, 1/3]$  by setting  $\phi(x)$  to be the  $l^\infty$ -distance to from  $x$  to  $H$  (where we take coordinates parallel to the 1-faces of  $Q$ ). Thus,  $\phi$  is 1-lipschitz (with respect to  $d_X$ ). Let  $Q_t = \phi^{-1}[0, t]$ . Thus,  $Q_t \subseteq Q$  is a cube containing  $H$ . In fact,  $Q_0 = H$  and  $Q_{1/3} = Q$ .

Putting  $S$  into general position, we can assume that  $\gamma_t = S \cap \partial Q_t = S \cap \phi^{-1}(t)$  is a multicurve for all but finitely many  $t$ . For generic  $t$ , let  $l_t = \text{length}(\gamma_t)$ . By the coarea formula,  $\int_0^{1/3} l_t dt \leq \text{area}(S \cap (Q \setminus H)) \leq \Delta$ . It follows that there is some  $t \in (0, 1/3)$  with  $l_t \leq 3\Delta$ . Write  $B = Q_t$ . Since  $S$  is incompressible, all components of  $\gamma_t$  are homotopically trivial in  $S$ . Let  $\beta \subseteq \gamma_t$  be the set of  $S$ -outermost components of  $\gamma_t$ .

If  $\alpha$  is component of  $\beta$  then  $\alpha$  cuts  $B$  into two discs, and we let  $D(\alpha) \subseteq \partial B$  be the disc of smaller area. (We choose arbitrarily if  $\alpha$  cuts  $\partial B$  into two discs of equal area.) One checks easily that  $\text{area}(D(\alpha)) \leq \frac{3}{4} \text{length}(\alpha)$ , and so the collection of all such  $D(\alpha)$  (considered disjoint) has total area at most  $\frac{3}{4}(3\Delta) = \frac{9}{4}\Delta$ . Moreover, the discs  $D(\alpha)$  are nested in  $\partial B$  (that is, any pair of them are either disjoint, or one is contained in the other). Therefore, we can push their interiors slightly into  $B$  while fixing their boundary curves. We can suppose that they still do not meet  $H$ , and that their areas increase by an arbitrarily small amount. In particular their total area remains less than  $3\Delta$ , say.

We now replace each disc  $D_S(\alpha)$  with  $D(\alpha)$  for each component  $\alpha$  of  $\beta$ . This gives us a new surface, still denoted  $S$ , and given that  $X$  is aspherical, it is easily seen that this is isotopic to the original. In the process we have increased  $\Delta$  by at most a factor of 4. So, in particular, the new density of  $S$  is at most  $4\Delta$ .

Step 1:

The next step is to put  $S$  into nice position with respect to the 1-handles. At this point we will only need to control the combinatorics of  $S$ , that is the way in which it intersects the cubes composing in the handle. We will no longer need to explicit control of the area.

Let  $H = H(Q)$  be a 1-handle. Thus  $H = P_1 \cup P_2$ , where  $P_1, P_2 \in \mathcal{Q}^3(X_3)$ , and  $P_1 \cap P_2 = \text{cent}(Q)$ . Let  $F_i \in \mathcal{Q}^2(X_3)$  be the face of  $P_i$  opposite  $\text{cent}(Q)$ . Then  $L = L(H) = F_1 \cup F_2$ . The annulus  $L^C = L^C(H)$  is a union of eight 2-faces of  $X_3$ , four in each of  $P_1$  and  $P_2$ .

Now  $S \cap \partial H \subseteq L^C$  is a disjoint union of closed curves. Each such curve is either homotopically trivial in  $L^C$ , or homotopic to a core curve of  $L^C$ . Let  $\gamma \subseteq S \cap \partial H$  be the union of curves of the latter type. We claim that there are at most  $60\Delta$   $S$ -outermost such core curves.

To see this, let  $R$  be the union of the two 3-cubes of  $X$  which meet along  $Q$ , so  $H \subseteq R$ . We claim that if  $\alpha$  is core curve, then  $\text{area}(D_S(\alpha)) \geq \frac{1}{60}$ . Define  $\phi : R \rightarrow [0, 1/3]$  by letting  $\phi(x)$  be the  $l^\infty$ -distance from  $x$  to  $H$ , and let  $R_t = \phi^{-1}R_t$ . Thus,  $R_0 = H$ . Let  $\delta_t = D_S(\alpha) \cap \partial R_t$ , which we can take to be a (possibly empty) multicurve, and let  $l_t = \text{length}(\delta_t)$ . Thus,  $\int_0^{1/3} l_t dt \leq \text{area}(D_S(\alpha) \cap R)$ . If  $l_t \geq \frac{1}{20}$  for all  $t$ , then  $\int_0^{1/3} l_t dt \geq \frac{1}{60}$ , and we have proven the claim in this case. If  $l_t < \frac{1}{20}$  for some  $t$ , then we can fill in each  $S$ -outermost curve,  $\epsilon$ , in  $\delta_t$  a disc in  $\partial R_t$  so that their total area of most  $l_t < \frac{1}{20}$ . These discs might not be disjoint, but replacing each  $D_S(\epsilon)$  by  $D(\epsilon)$  gives rise to a singular disc,  $D'$ , in  $R_t \subseteq R$ , of area at most  $\text{area}(D_S(\alpha) \cap R) + \frac{1}{20}$ . We now project  $D'$  to  $\text{cent}(Q)$  by (euclidian) nearest point projection. Since  $\alpha$  is essential in  $L^C$ , this must be surjective. Since nearest point projection is 1-lipschitz, we have  $\text{area}(D') \geq \frac{1}{9}$  and so  $\text{area}(D_S(\alpha) \cap R) \geq \frac{1}{9} - \frac{1}{20} > \frac{1}{60}$ , proving the claim in this case too.

Now  $\text{area}(S \cap R) \leq \Delta$ . Let  $\beta$  be the union of  $S$ -outermost components of  $\gamma$ . The discs  $D_S(\alpha)$  for components  $\alpha$  of  $\beta$  are all disjoint. It therefore follows that there are at most  $60\Delta$  such curves as claimed.

Up until now, we have not altered  $S$ . We next surger  $S$  to simplify its intersection with  $H$ .

Suppose that  $\alpha$  is a trivial curve in  $S \cap \partial H$ . Then it bounds a disc,  $D(\alpha) \subseteq L^C$  disjoint from  $\gamma$ . It also bounds a disc,  $D_S(\alpha) \subseteq S$ . If  $\alpha$  is innermost, then  $D(\alpha) \cap S = \alpha$ . We surger  $S$ , replacing  $D_S(\alpha)$  with  $D(\alpha)$ , and push  $D(\alpha)$  slightly off  $H$ . This reduces the number of components of  $S \cap L^C$ , and so after a finite number of steps we can assume that  $S \cap L^C = \gamma$  consists only of core curves.

The discs in  $S$  bounded by the  $S$ -outermost curves are disjoint, and we can replace them by a disjoint collection of discs in  $H$ . We can isotope them so that are transverse discs on one of the cubes  $P_1$  or  $P_2$ : that is to say parallel to the face  $\text{cent}(Q)$ .

In summary, after completing this step,  $S \cap H$  consists of at most  $60\Delta$  such transverse discs. Also,  $S$  remains disjoint from the 0-handles.

Step 2:

To avoid repetition in what follows, we will say that  $S$  is “efficient” with respect to a 2-cell (respectively a 3-cell) of  $X_3$  if its intersection with that cell consists of a bounded number of arcs (respectively discs). Here “bounded” means that the number of components is bounded by some fixed function of  $\Delta$  and  $C$ . This function may depend on the stage of the argument. It could be explicitly computed, though we won’t bother to do this.

Note that  $S$  is already in efficient position with respect to all cells contained in any 0-handle or 1-handle. We next deal with 2-handles.

Let  $H = H(Q)$  be a 2-handle. This consists of at most  $C$  3-cubes of  $X_3$  meeting along a common 1-cell, namely  $\text{cent}(Q)$ . Also  $L(H)$  consists of at most  $2C$  2-cells of  $\mathcal{Q}(X_3)$ , cyclically arranged so as to form an annulus, and  $L^C(H)$  is a disjoint union of two discs each with at most  $C$  2-cells. Now each 2-cell,  $P$ , of  $L(H)$  lies in a 1-handle, and so  $S \cap P$  consists of at most  $60\Delta$  arcs, each connecting the two boundary components of  $L(H)$ . By an innermost disc argument (similarly as with Step 1) we can eliminate all closed curve components of  $S \cap L^C(H)$ . It follows that  $S \cap L^C(H)$  consists of at most  $60\Delta C$  arcs, and so  $S \cap \partial H$  consists of at most  $30\Delta C$  closed curves. Again by a simple surgery, we can suppose that each such curve bounds a disc in  $S \cap H$ . In other words,  $S \cap H$  consists of at most  $30\Delta C$  discs. We can now isotope  $S$  in the interior of  $H$  so that it intersects every cell of  $H$  efficiently.

Step 3:

Let  $H = H(Q)$  be a 3-handle. This is the union of at most  $C$  3-cells meeting at the vertex,  $\text{cent}(Q)$ . The boundary,  $\partial H$ , is the union of at most  $3C$  2-cells. We

know that  $S$  is efficient with respect to each of these 2-cells. Therefore,  $S \cap \partial H$  consists of a bounded number of closed curves. After surgery, we can assume that each of these curves bounds a disc in  $S \cap H$ . In other words,  $S \cap H$  is the union of a bounded number of disjoint discs. We can now isotope  $S$  on the interior of  $H$  so that it meets each cell of  $H$  efficiently.

After Steps 0–3,  $S$  is now in efficient position with respect to each cell of  $X_3$ .

We now subdivide the cubing  $X_3$  a bounded number of times with the aim of pushing  $S$  into the 2-skeleton of the subdivision. To begin,  $S$  is already disjoint from the 0-skeleton of  $X_3$ . Next note that  $S$  meets each 1-cell of  $X_3$  in a bounded number of points. Therefore after a bounded subdivision and isotoping  $S$  slightly, we can assume that  $S$  meets the 1-cell in the 0-skeleton of the subdivision. Next, consider a 2-cell,  $P$ , of  $X_3$ . Since  $S \cap P$  consists of a bounded number of arcs, we can subdivide further, and isotope  $S$  so that  $S \cap P$  lies in the 1-skeleton of the subdivision. Finally consider a 3-cell,  $Q$ , of  $X_3$ . It intersects  $S$  in a bounded number of 2-discs, and after further subdivision we can isotope  $S$  on the interior of  $Q$ , so that it lies in the 2-skeleton of the subdivision.

Note that (since there are only boundedly many combinatorial possibilities) the number of subdivisions required is uniformly bounded in terms of  $\Delta$  and  $C$ . At the end of the day,  $S$  lies in the 2-skeleton. We also note that at each stage, any 3-cell of  $\mathcal{Q}(X)$  which meets the final surface also meets the original.

This therefore proves Proposition 4.1.1 in the case where  $\partial S = \emptyset$ .

The general case only requires slight modification. We define handles in the same way. Step 0 remains unchanged. For Step 1, we need to consider the case where  $Q \subseteq \partial X$ . Then  $H = H(Q)$  consists of a single 3-cube,  $P$  in  $X_3$  with  $\text{cent}(Q)$  as one of its 2-faces. The relative boundary of  $H$  in  $X$  consists of the opposite face,  $L(H)$  and an annulus  $L^C(H)$  comprising the remaining four faces. Again  $S \cap L(C) = \emptyset$ . A similar argument shows that  $S \cap L^C(H)$  consists of a bounded number of  $S$ -outermost core curves, together with trivial closed curves, together also with arcs with both endpoints in  $L^C(H) \cap \partial X$ . We can proceed as before.

Steps 2 and 3 are similar. Again we have to consider the case where  $H$  meets  $\partial X$ , so that the relative boundary of  $H$  is a disc rather than a 2-sphere, and it intersects  $S$  in a number of arcs as well as closed curves, but the argument remains essentially the same.

This proves Proposition 4.1.1.

We finally restate this result in terms of triangulations. We define the **local complexity** of a triangulation, and the **density** of a subsurface in exactly the same way as for a cubulated 3-manifold, except that the cells are now simplices rather than cubes.

Here is the main result:

**Proposition 4.1.2.** *Given any  $\Delta, C \geq 0$  there some  $K \geq 0$  with the following property. Suppose that  $X$  is a triangulated aspherical 3-manifold of local complexity at most  $C$ , with incompressible (possibly empty) boundary. Let  $S \subseteq X$  be a (possibly disconnected) incompressible and boundary incompressible subsurface. Let  $N(S)$  be the union of all simplices meeting  $S$ . Suppose that  $\text{area}(S \cap P) \leq \Delta$  for all 3-simplices  $P$ . Then there is a subdivision of the triangulation, with each simplex subdivided into at most  $K$  simplices, and a surface  $S'$  contained in the 2-skeleton of the subdivision, isotopic to  $S$ , and with  $S' \subseteq N(S)$ .*

Here one can interpret area with respect to the euclidean path-metric induced on  $X$  where each simplex is euclidean-regular with unit side-lengths.

The statement can be reduced to Proposition 4.1.1 on noting that one can subdivide a simplicial complex into a cube complex, and vice versa, by a simple operation of coning cells over their midpoints.

The statement of Proposition 4.1.2 does not specify how simplices are to be subdivided, though the proof gives an explicit procedure via cube complexes. It is likely that one could also use other procedures, such as barycentric subdivision. For our application, we only care that the subdivision has bounded combinatorics, so we will not pursue these issues here.

## 4.2. A TOPOLOGICAL SURGERY CONSTRUCTION.

In this section, we describe a “cut-and-paste” argument which allows us to replace a singular surface in a 3-manifold with an embedded one in the same homology class while keeping control on the genus. It relies on a tower argument similar to that in [FHS]. The result will be used in Section 4.3

Suppose  $S$  is a compact orientable surface, possibly disconnected. We define the genus of  $S$  to be the sum of the genera of its components. If  $\hat{S}$  is the surface obtained by gluing in a disc to each boundary component, then  $\text{genus}(S) = \text{genus}(\hat{S}) = \frac{1}{2} \dim H_1(\hat{S}, \mathbb{Z}_2)$ . If  $B$  is the image of the (injective) map  $H_1(\partial S, \mathbb{Z}_2) \rightarrow H_1(S, \mathbb{Z}_2)$ , then we can identify  $H_1(\hat{S})$  with the quotient  $H_1(S, \mathbb{Z}_2)/B$ .

Let  $V$  be a 3-manifold with (possibly empty) boundary,  $\partial V$ .

**Lemma 4.2.1.** *Suppose that  $F$  is a compact surface, and  $f : F \rightarrow V$  is a (general position) map with  $f^{-1}(\partial V) = \partial F$  satisfying:*

- (1)  $f|_{\partial F}$  is injective,
- (2)  $f$  is  $\pi_1$ -injective, and
- (3) the kernel of the induced map  $H_1(F, \mathbb{Z}_2) \rightarrow H_1(V, \mathbb{Z}_2)$  lies in the image of  $H_1(\partial F, \mathbb{Z}_2) \rightarrow H_1(F, \mathbb{Z}_2)$ .

*Then if  $U \subseteq V$  is any open set containing  $f(F)$ , there is a (possibly disconnected) surface  $G \subseteq V$ , with  $G \subseteq U$ ,  $\partial G = G \cap \partial V = f(\partial V)$ ,  $\text{genus}(G) \leq \text{genus}(F)$ , and*

with the pair  $G, \partial G$  representing the same element of  $H_2(V, \partial V, \mathbb{Z}_2)$  as the image of  $F, \partial F$  under  $f$ . Moreover,  $G$  is obtained by surgery on  $f(F)$ .

To explain the last statement, note that the image  $f(F) \subseteq V$  is a 2-complex made out of a 1-dimensional singular set,  $K$ , with surfaces attached. Since  $f|_{\partial F}$  is assumed injective,  $K \cap \partial V = \emptyset$ . Let  $L$  be an (arbitrarily small) regular neighbourhood of  $K$  in  $U \setminus \partial V$ . Then we can arrange that  $G \subseteq L \cup f(F)$ , and that  $G \cap L$  is a disjoint union of annuli. (In the riemannian set-up where we apply this, we can consequently take  $\text{area}(G) \leq \text{area}(f(F)) + \epsilon$  for arbitrarily small  $\epsilon > 0$ . We will only really need some bound on  $\text{area}(G)$ .)

The proof of Lemma 4.2.1 will be an adaptation of the proof of Lemma 2.1 of [FHS] — though the hypotheses are somewhat different. A related discussion is given in the proof of Theorem 2.6.2 of [Bow6]. We refer to those arguments for any missing details below.

*Proof.* Throughout the proof,  $H_1(\cdot)$  and  $H_2(\cdot)$  will denote homology with  $\mathbb{Z}_2$  coefficients.

Let  $\hat{F}$  be the closed surface obtained by gluing a disc to each boundary component of  $F$ . Let  $\hat{V} \supseteq V$  be the 3-manifold obtained by gluing a disc to each component of  $f(\partial F) \subseteq \partial V$  and thickening it up slightly. We can naturally extend  $f$  to a map  $\hat{f} : \hat{F} \rightarrow \hat{V}$  with  $\hat{f}^{-1}V = F$  and with  $\hat{f}(\hat{F}) \cap \partial \hat{V} = \emptyset$ .

Note that hypothesis (3) implies that the induced map  $H_1(\hat{F}) \rightarrow H_1(\hat{V})$  is injective. (Note that the kernel of the map  $H_1(V) \rightarrow H_1(\hat{V})$  included in the image of  $H_1(\partial F)$  under the map  $H_1(\partial F) \rightarrow H_1(F)$ . Using hypothesis (3), it follows that kernel of the composition,  $H_1(F) \rightarrow H_1(\hat{F}) \rightarrow H_1(\hat{V}) \cong H_1(F) \rightarrow H_1(V) \rightarrow H_1(\hat{V})$  is also supported on  $H_1(\partial F)$ . Therefore, the kernel of  $H_1(\hat{F}) \rightarrow H_1(\hat{V})$  is trivial.)

Let  $\hat{U}$  be a neighbourhood of  $\hat{f}(\hat{F})$  with  $\hat{U} \cap V \subseteq U$ .

We now construct a tower as in [FHS]. To begin, let  $N_0$  be a regular neighbourhood of  $\hat{f}(\hat{F})$  in  $\hat{V}$ . If  $H_1(\hat{F}) \rightarrow H_1(N_0)$  is surjective, we stop. If not, let  $N'_0$  be a double cover of  $N$  such that  $\hat{f}$  lifts to a map  $\hat{f}_1 : \hat{F} \rightarrow N'_0$ . Let  $N_1 \subseteq N'_0$  be a regular neighbourhood of  $\hat{f}_1(\hat{F})$  in  $N'_0$ . If the induced map  $H_1(\hat{F}) \rightarrow H_1(N_1)$  is surjective, we stop. If not, we pass to a double cover and continue inductively. The process must terminate (since the singular set in  $\hat{F}$  is getting smaller at each stage).

At the top of the tower, we arrive at a maps,  $h : \hat{F} \rightarrow \hat{N}$  and  $\lambda : \hat{N} \rightarrow \hat{V}$ , with  $\hat{f} = \lambda \circ h$ . Here  $\hat{N}$  is a 3-manifold (a regular neighbourhood of  $\lambda(\hat{N})$ ),  $\lambda$  is locally injective, and  $h$  induces a surjection, hence an isomorphism,  $H_1(\hat{F}) \rightarrow H_1(\hat{N})$ . Let  $N = \lambda^{-1}V \subseteq \hat{N}$ . We can assume that  $\lambda|_{(\hat{N} \setminus N)}$  is a homeomorphism to  $\hat{f}(\hat{F}) = \hat{f}(\hat{F}) \cap V$ . (This is because  $f|_{(\hat{F} \setminus F)}$  is already a homeomorphism to its range and this remains unaltered in the above process.) As with any 3-manifold,



$\dim H_1(\partial\hat{N}) \leq 2 \dim H_1(\hat{N})$ . Now,  $\dim H_1(\hat{N}) = \dim H_1(\hat{F})$ , and it follows that  $\text{genus}(\partial\hat{N}) \leq 2 \text{genus}(\hat{F}) = 2 \text{genus}(F)$ .

There is an intersection pairing of  $H_2(\hat{N})$  with  $H_1(\hat{N}, \partial\hat{N})$ . By construction,  $h(\hat{F})$  is non-trivial in this pairing. As in [FHS], we write  $\partial\hat{N} = \hat{A} \sqcup \hat{B}$ , where  $\hat{A}, \hat{B}$  are each a non-empty union of components of  $\partial\hat{N}$ , such that any arc in  $\hat{N}$  with endpoints in  $\partial\hat{N}$  intersects  $h(\hat{F})$   $\mathbb{Z}_2$ -homologically non-trivially if and only if it has one endpoint in  $\hat{A}$  and the other endpoint in  $\hat{B}$ . Note that  $\hat{A}, \hat{B}$  and  $h(\hat{F})$  all represent the same element of  $H_2(\hat{N})$ . We can assume that  $\text{genus}(\hat{A}) \leq \text{genus}(F)$ . (In [FHS] it was assumed that the map corresponding to  $\hat{f}$  was an immersion and homotopic to an embedding. However, as noted in [Bow6], only general position and injectivity on  $H_1(\cdot)$  were required at this point.)

Let  $W \subseteq N$  be a regular neighbourhood of the singular locus of  $h(\hat{F})$  in  $\hat{N}$ . We can suppose that  $\lambda(W) \subseteq L \subseteq \hat{V}$ . Pushing  $\hat{A}$  into  $\hat{N}$  we get a parallel surface,  $\hat{C} \subseteq \hat{N}$ , with  $\hat{C} \subseteq h(\hat{F}) \cup W$ . Note that  $\text{genus}(\hat{C}) \leq \text{genus}(F)$ . (However  $\hat{C}$  might not be connected.)

As in [FHS], we now map back down the tower, one step at a time. At each step, two curves get identified, and we perform surgery to remove the intersection. This involved cutting the surface along the singular curve and regluing the pieces. There is a choice (of two alternatives) as to how we reglue. However, it can always be done in such a way so that the number of components of the surface never increases, and it follows that the genus can never increase. Moreover, (however we do the surgeries) the homology class of the surface (projected to  $\hat{V}$ ) does not change.

At the bottom of the tower, we obtain an embedded surface,  $\hat{G}$ , with  $\hat{G} \setminus V = \hat{f} \setminus V$ , and with  $\hat{G}$  equal to  $\hat{f}(\hat{F})$  in  $H_2(V)$ . Now let  $G = V \cap \hat{G}$ . We see that this has the properties stated.  $\square$

### 4.3. SUBFIBRES IN PRODUCT SPACES.

In this section, we elaborate on some of the ideas in Section 2.3. In particular, we give a description of a ‘‘subfibre’’ in a product space,  $\Psi = \Sigma \times \mathbb{R}$ , which can be thought of as an unknotted subsurface. The main result will be used in Section 4.5. We begin with a purely 2-dimensional observation.

**Lemma 4.3.1.** *Let  $\Phi$  be a connected subsurface of  $\Sigma$ . Let  $F$  be a (possibly disconnected) compact orientable surface. Let  $f : F \rightarrow \Sigma$  be a (general position) map with  $f|_{\partial F} : \partial F \rightarrow \partial\Phi$  a homeomorphism, with  $f$  of non-zero degree on  $\Phi$ . Then  $\text{genus}(F) \geq \text{genus}(\Phi)$ . Moreover, if  $\text{genus}(F) = \text{genus}(\Phi)$ , then  $F$  is connected, and  $F$  is homotopic, fixing  $\partial F$ , to an embedding with image  $\Phi$ .*

Here  $\partial\Phi$  denotes the intrinsic boundary of  $\Phi$  as a surface. Its relative boundary is  $\partial\Phi \setminus \partial\Sigma$ .

*Proof.* We will assume that  $\Phi$  is not an annulus, as that case is elementary.

First, we assume that  $F$  is connected. For this part, it is convenient to realise a compact surface  $\Sigma$  as a hyperbolic surface with geodesic boundary,  $\partial\Sigma$ . Then  $\text{area}(\Sigma) = 2\pi(2 \text{genus}(\Sigma) + p(\Sigma) - 2)$ , where  $p(\Sigma)$  denotes the number of boundary components.

We realise  $\Sigma$  in this way, and realise  $\Phi$  in  $\Sigma$  with geodesic boundary. (We can still imagine  $\Phi$  to be embedded even if two boundary components get identified in  $\Sigma$ . We can think of the complementary annulus as contributing nothing to the area.)

We can homotope  $f$ , fixing its boundary, so that it is 1-lipschitz with respect to a hyperbolic structure on  $F$ , with concave boundary. In fact, we can take each component of  $\partial F$  to be an intrinsically polygonal curve with all interior angles at least  $\pi$ . (This is a standard construction similar to that used for constructing pleated surfaces. Take any triangulation of  $F$  with all vertices in  $\partial F$ . Homotope  $f$  so that the edges get mapped to geodesic segments, and simplices get mapped injectively. Then take the induced hyperbolic metric on each simplex.) In this case  $\text{area}(F) \leq 2\pi(2 \text{genus}(F) + p(F) - 2)$ , with equality if and only if the boundary is geodesic (all angles equal to  $\pi$ ).

Since  $f$  has non-zero degree on  $\Phi$ , we see that  $\text{area}(F) \geq \text{area}(\Phi)$ . Since  $p(F) = p(\Phi)$  we get  $\text{genus}(F) \geq \text{genus}(\Phi)$ . If  $\text{genus}(F) = \text{genus}(\Phi)$ , then we must have equality everywhere. In particular,  $\text{area}(F) = \text{area}(\Phi)$ , and the map  $F \rightarrow \Phi$  must be an isometry. This proves the lemma when  $F$  is connected.

For the general case, note that at least one component, say  $F_0$ , of  $F$  must map to  $\Phi$  with non-zero degree. Now  $f(\partial F_0) \subseteq \partial\Phi$ . Let  $\Phi_0$  be the closure of the component  $\Sigma \setminus f(\partial F_0)$  containing  $\Phi$ . Thus,  $f|_{\partial F_0} : \partial F_0 \rightarrow \partial\Phi_0$  is a homeomorphism. Also,  $f$  maps  $F_0$  to  $\Phi_0$  with non-zero degree. Clearly  $\text{genus}(\Phi_0) \geq \text{genus}(\Phi)$  and  $\text{genus}(F_0) \leq \text{genus}(F)$ . By the first part,  $\text{genus}(F_0) \geq \text{genus}(\Phi_0)$ , and so  $\text{genus}(F) \geq \text{genus}(\Phi)$ , proving the first statement.

Moreover, if  $\text{genus}(F) = \text{genus}(\Phi)$ , then  $\text{genus}(F_0) = \text{genus}(\Phi_0)$ , and, again by the first part, we can assume that  $f$  maps  $F_0$  homeomorphically to  $\Phi_0$ . Now  $f|_{\partial F} : \partial F \rightarrow \partial\Phi$  is also a homeomorphism, and it follows that  $\partial\Phi_0 \cap \partial\Sigma \subseteq \partial\Phi$ . In other words, no component of  $\Phi_0 \setminus \Phi$  can contain a component of  $\partial\Sigma$ . Since  $\text{genus}(\Phi) = \text{genus}(\Phi_0)$  it follows that  $\Phi = \Phi_0$ , and so  $\partial\Phi = \partial\Phi_0$  and so  $F = F_0$ . Thus,  $f$  maps  $F$  homeomorphically to  $\Phi$ .  $\square$

Suppose now that  $f : F \rightarrow \Sigma$  is a map with  $f|_{\partial F} : \partial F \rightarrow \partial\Phi$  a homeomorphism. Then gluing  $F$  to  $\Phi$  via this homeomorphism gives a closed (orientable) surface. Now  $f|_F$  combined with the inclusion of  $\Phi$  gives rise to an element of  $H_2(\Sigma, \mathbb{Z}_2)$ . If this class is zero, then the hypotheses of Lemma 4.3.1 apply. Equivalently, we could suppose that the images of  $F, \partial F$  and  $\Phi, \partial\Phi$  represent the same element of  $H_2(\Sigma, \partial\Sigma, \mathbb{Z}_2)$ .

Now let  $\Psi = \Sigma \times \mathbb{R}$ . We write  $\pi_\Sigma : \Psi \rightarrow \Sigma$  for the projection map. We recall some notions from Section 2.3. By a **subsurface** in  $\Psi$ , we mean an embedded compact orientable subsurface,  $F$ , such that  $F \cap \partial\Psi$  is a union of intrinsic boundary components of  $F$ . We denote the remainder of the boundary by  $\partial_0 F = \partial F \setminus \partial\Psi$ . We will assume  $F$  to be connected, unless we specify otherwise. A subsurface,  $F$ , is **proper** if it is  $\pi_1$ -injective, and  $\partial_0 F = \emptyset$ .

Given  $t \in \mathbb{R}$  let  $\Sigma_t = \Sigma \times \{t\}$  be the ‘‘horizontal fibre’’. A ‘‘fibre’’ is a proper subsurface whose inclusion into  $\Psi$  is a homotopy equivalence. By Brown’s theorem (given as Theorem 1.6.3 here), any fibre is isotopic to a horizontal fibre.

Note that  $\Psi$  is aspherical and atoroidal. In fact, any embedded torus in  $\Psi$  which is not homotopic to a point bounds a solid torus in  $\Psi$ . (This is a simple consequence of Dehn’s Lemma.)

We recall some terminology and notation from Section 2.3. Recall that a collection,  $\mathcal{L}$ , of disjoint curves in  $\Psi$  is ‘‘unlinked’’ if it is contained in a disjoint union of fibres. (Here we will allow two elements of  $\mathcal{L}$  to be homotopic provided they lie in the same fibre, and so bound an annulus in  $\Psi$  not meeting any other curve of  $\mathcal{L}$ .)

We can take a disjoint collection,  $T(\gamma)$ , of regular neighbourhoods of elements,  $\gamma \in \mathcal{L}$ . Note that  $\partial T(\gamma)$  has a standard meridian, no power of which is trivial in  $H_1(\Psi \setminus \text{int } T(\gamma))$ . A **longitude** is a curve which intersects the meridian exactly once. We can think of this as equivalent to a framing of  $\gamma$ . There is a preferred longitude, determined by any fibre containing  $\gamma$ .

**Definition.** A **subfibre** in  $\Psi$  is an embedded subsurface,  $F \subseteq S$ , of a fibre,  $S \subseteq \Psi$ .

Note that  $\partial_0 F$  is a collection of unlinked curves in  $\Psi$ .

From the corresponding fact for fibres, we see that any subfibre,  $F$ , is isotopic to a ‘‘horizontal subfibre’’, that is, a subset of the form  $\Phi_t = \Phi \times \{t\}$ , where  $\Phi$  is a subsurface of  $\Sigma$ . Here  $\Phi$  is well defined up to isotopy, and we refer to it as the **base subsurface** of  $F$ .

**Lemma 4.3.2.** *Suppose that  $F \subseteq \Psi$  is a subsurface with  $\partial_0 F$  and unlinked in  $\Psi$ , and suppose that  $F$  is homotopic, via  $\pi_\Sigma$ , to a subsurface of  $\Sigma$ . Then  $F$  is a subfibre.*

*Proof.* We first assume that no two components of  $\partial_0 F$  are homotopic in  $\Psi$ .

We can suppose that  $\partial_0 F \subseteq \Sigma_0 = \Sigma \times \{0\}$ . Let  $\Phi \subseteq \Sigma$  be the base surface homotopic to  $F$ . So  $\Phi_0 = \Phi \times \{0\}$  is a horizontal fibre with  $\partial_0 \Phi_0 = \partial_0 F$ . Let  $H$  be the closure of  $\Sigma_0 \setminus \Phi_0$ . This is a (possibly disconnected) horizontal surface, with  $\partial_0 H = \partial_0 F$ . We assume  $F$  in general position with respect to  $H$ , so that  $F \cap H$  is a union of arcs and curves.

In fact, we can assume there are no arcs. This is because the framing of any boundary curve,  $\gamma$ , of  $\partial_0 F$  as determined by  $F$  is the standard one, determined by  $\Phi_0$ . To see this, note that the intersections of  $F$  and  $\Phi_0$  with  $\partial T(\gamma)$  together

bound a singular orientable surface in  $\Psi \setminus \text{int } T(\gamma)$ . No two curves in  $\partial_0 F$  are homotopic in  $\Psi$ , and so no component of  $\Sigma_0 \setminus \Phi_0$  is an annulus. It follows that  $F \cap \partial T(\gamma)$  and  $\Phi_0 \cap \partial T(\gamma)$  are both longitudes of  $\partial T(\gamma)$ . The kernel of the map  $H_1(\partial T(\gamma)) \rightarrow H_1(\Phi \setminus \text{int } T(\gamma))$  is generated by the meridian of  $\partial T(\gamma)$ . Therefore these two curves are homologous, hence homotopic, in  $\partial T(\gamma)$ . We can therefore isotope  $F$  so that they are disjoint.

Now each component of  $F \cap H$  is either trivial or peripheral in both  $F$  and  $H$ .

We first remove all trivial curves as follows. Suppose that  $\alpha \subseteq F \cap H$  is trivial. It bounds discs  $D_F \subseteq F$  and  $D_H \subseteq H$ . Choosing  $\alpha$  innermost in  $F$ , we can suppose that  $D_F \cap H = D_F \cap D_H = \alpha$ . Now,  $D_F \cup D_H$  is a 2-sphere, bounding a ball in  $\Psi$ . Thus, we can isotope  $H$ , replacing  $D_H$  by  $D_F$  and pushing it slightly off  $F$  so as to eliminate the intersection  $\alpha$ . Continuing in this way, we isotope  $H$  so that all remaining components of  $F \cap H$  are peripheral.

Let  $\beta \subseteq F \cap H$  be such a curve. Then  $\beta$  bounds peripheral annuli  $A_F \subseteq F$  and  $A_H \subseteq H$ . By choosing  $\beta$  outermost in  $F$ , we have  $A_F \cap H = A_F \cap A_H = \beta \cup \gamma$ , where  $\gamma \subseteq \partial F = \partial H$ . Now  $A_F \cup A_H$  is an embedded torus, and so bounds a solid torus,  $T \subseteq \Psi$  with  $T \cap F = A_F$ . Since  $\gamma$  is primitive in  $\Psi$ , hence in  $T$ , it follows that  $A_H$  is isotopic to  $A_F$  in  $T$ , fixing  $\beta \cup \gamma$ . We can therefore isotope  $H$ , fixing  $\partial H$ , by replacing  $A_H$  by  $A_F$  and pushing slightly off  $F$  away from  $\gamma$ . This eliminates the intersection  $\gamma$ .

Continuing in this way, we isotope  $H$  fixing  $\partial H$  so that  $F \cap H = \partial F = \partial H$ . Now  $F \cup H$  is a fibre of  $\Psi$ , and so  $F$  is a subfibre. This proves the lemma in the case where no two curves of  $\partial_0 F$  are homotopic.

Finally, suppose two components of  $\partial_0 F \subseteq \Sigma_0$  are homotopic in  $\Psi$ . Then they bound an embedded horizontal annulus,  $A \subseteq \Sigma_0 \subseteq \Phi$ . Now  $F \cap A$  consists only of closed curves. We can push  $F$  off  $A$ , by the procedure described above. We can now glue in  $A$ , and apply a similar construction to all other such annuli. We can then apply the above to the resulting surface.  $\square$

**Lemma 4.3.3.** *Suppose that  $F \subseteq \Psi$  is a subfibre and  $G \subseteq \Psi$  is a (possibly disconnected) subsurface, with  $\partial F = \partial G$  and  $\text{genus}(G) \leq \text{genus}(F)$ . Suppose that  $\gamma$  is a curve in  $\Psi$  with  $\gamma \cap F = \gamma \cap G = \emptyset$ , and with  $\gamma \cup \partial F$  unlinked in  $\Psi$ . Suppose that the singular cycle  $F \cup G$  is trivial in  $H_2(\Psi \setminus \gamma, \mathbb{Z}_2)$ . Then  $G$  is isotopic in  $\Sigma \setminus \gamma$  to  $F$ , fixing  $\partial G$ . (So that, in retrospect,  $G$  must be connected.)*

(If  $\gamma$  is not homotopic into  $F$ , then we could weaken the homological hypothesis to assume only that  $F \cup G$  is trivial in  $H_2(\Psi, \mathbb{Z}_2) \cong H_2(\Sigma, \mathbb{Z}_2)$ .)

*Proof.* We can suppose that  $F \subseteq \Sigma_0$  and  $\gamma \subseteq \Sigma_1$ . Thus,  $F = \Phi_0$ , where  $\Phi$  is the base surface of  $F$ . Now  $\pi_\Sigma(F \cup G)$  is trivial in  $H_2(\Sigma, \mathbb{Z}_2)$ , so by Lemma 4.3.1,  $\pi_\Sigma G$  is homotopic to  $\pi_\Sigma F = \Phi$  in  $\Sigma$ .

Let  $A = (\pi_\Sigma \gamma) \times [1, \infty)$ . Thus,  $A \subseteq \Psi$  is a properly embedded annulus, disjoint from  $F$ , with  $\partial A = \gamma$ . Each component of  $A \cap G$  is either trivial or peripheral in both  $G$  and  $A$ . Moreover, since  $F \cup G$  is trivial in  $H_2(\Psi \setminus \gamma, \mathbb{Z}_2)$ , there must

be an even number of peripheral curves. (If  $\gamma$  is not homotopic into  $F$ , then the homology assumption is not needed at this point, and all intersections are trivial.) We claim that we can isotope  $G$  off  $A$ , fixing  $\partial G$ .

First, we can eliminate trivial intersections, pushing  $G$  off discs in  $A$ , similarly as in the proof of Lemma 4.3.2.

Now  $A \cap G$  consists of an even number of core curves. The first two of these (nearest  $\partial A$ ) bound an annulus  $A_0 \subseteq A$ , with  $A \cap F = \partial A_0$ . Now  $\partial A_0$  also bounds an annulus,  $A_1 \subseteq F$ . So  $A_0 \cup A_1$  is an embedded torus, and bounds a solid torus,  $T \subseteq \Psi$  with  $A \cap T = A_0$ . We see that  $\gamma \cap T = \emptyset$  (otherwise  $A$  would have to intersect  $T$  outside  $A_0$ ). Since  $\gamma$  is primitive in  $\Psi$ , hence in  $T$ , we can isotope  $A_0$  to  $A_1$  in  $T$  fixing  $\partial A_0 = \partial A_1$ . We can therefore isotope  $G$  to remove these intersections. Continuing in this manner we isotope  $G$  so that  $G \cap A = \emptyset$ .

Now we isotope  $G$  vertically downwards until  $G \subseteq \Sigma \times (-\infty, 1) \cong \Sigma \times \mathbb{R}$ , and then apply Lemma 4.3.2.  $\square$

We can immediately generalise this as follows.

**Lemma 4.3.4.** *Suppose that  $F \subseteq \Psi$  is a subfibre, and  $G \subseteq \Psi$  is an (a-priori possibly disconnected) subsurface, with  $\partial F = \partial G$  and  $\text{genus}(G) \leq \text{genus}(F)$ . Suppose that  $\mathcal{L}$  is a collection of curves in  $\Psi$ , which, together with the components of  $\partial F$  is unlinked in  $\Psi$ . Suppose that the singular cycle  $F \cup G$  is trivial in  $H_2(\Psi \setminus \bigcup \mathcal{L}, \mathbb{Z}_2)$ . Then  $G$  is isotopic in  $\Sigma \setminus \bigcup \mathcal{L}$  to  $F$ , fixing  $\partial G$ .*

*Proof.* We can suppose that  $F \subseteq \Sigma_0$ . We can index the elements of  $\mathcal{L}$  as  $(\gamma_i)_{i \in I}$ , where  $I \subseteq \mathbb{Z} \setminus \{0\}$ , and  $\gamma_i \subseteq \Sigma_i$ .

Now  $G$  lies in  $\Sigma \times (-n, n) \subseteq \Psi$  for some  $n \in \mathbb{N}$ . As in Lemma 4.3.3, with  $\gamma = \gamma_{n-1}$  (if such exists), we can isotope  $G$  into  $\Sigma \times (-n, n-1)$ , in  $\Sigma \setminus \bigcup \mathcal{L}$ . Continuing in this manner, we isotope it into  $\Sigma \times (-n, 1)$ . Starting at the other end, we similarly isotope it into  $\Sigma \times (-1, 1)$ . We conclude as before.  $\square$

We can rephrase this in terms of,  $\Lambda = \Lambda(\mathcal{L}) = \Psi \setminus \bigcup_{\gamma \in \mathcal{L}} \text{int } T(\gamma)$ , for an unlinked collection of curves as defined in Section 2.3. By a **subfibre** in  $\Lambda$  we mean a proper subsurface,  $F \subseteq \Lambda$  (with  $\partial F = F \cap \partial \Lambda$ ) which lies inside a fibre of  $\Psi$ . (It is therefore a subfibre of  $\Psi$  in the sense already defined.)

In these terms, Lemma 4.3.4 becomes:

**Lemma 4.3.5.** *Let  $\mathcal{L}$  be an unlinked collection of curves in  $\Psi$ . Let  $F \subseteq \Lambda = \Lambda(\mathcal{L})$  be a subfibre, and let  $G \subseteq \Lambda$  be an (a-priori possibly disconnected) subsurface with  $\partial F = \partial G = F \cap \partial \Lambda = G \cap \partial \Lambda$ . Suppose that the singular cycle  $F \cup G$  is trivial in  $H_2(\Lambda, \mathbb{Z}_2)$ , and that  $\text{genus}(G) \leq \text{genus}(F)$ . Then  $G$  is isotopic to  $F$  in  $\Lambda$  fixing  $\partial G$ .*

(Here the curves corresponding to  $\partial F = \partial G$  have been included in  $\mathcal{L}$ .)

Note that it's enough that the pairs  $F, \partial F$  and  $G, \partial G$  represent the same element in  $H_2(\Lambda, \partial \Lambda, \mathbb{Z}_2)$ . In retrospect, of course, it follows that  $G$  must be connected.

**Lemma 4.3.6.** *Let  $\mathcal{L}$  and  $\mathcal{L}'$  be unlinked collections of curves in  $\Psi$ . Let  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}')$  be a proper homotopy equivalence, with  $f^{-1}(\partial\Lambda(\mathcal{L}')) = \partial\Lambda(\mathcal{L})$ . Let  $F \subseteq \Lambda(\mathcal{L})$  be a subfibre. Suppose that  $f|_F$  is in general position, and that  $f|\partial F$  is injective. Then there is a subfibre,  $F' \subseteq \Lambda(\mathcal{L}')$ , obtained by surgery on  $f(F)$  (fixing  $\partial F$ ), which is homotopic to  $f(F)$  fixing  $\partial F$ .*

We will again require the fact (Proposition 2.3.2) that  $f$  is homotopic to a homeomorphism.

*Proof.* Note that  $f|_F$  satisfies the hypotheses of Lemma 4.2.1 with  $V' = \Lambda(\mathcal{L}')$ . Let  $F'$  be the surface constructed by surgery on  $f(F)$  as given by Lemma 4.2.1. In particular, the pair  $F', \partial F'$  represents the same element of  $H_2(\Lambda(\mathcal{L}'), \partial\Lambda(\mathcal{L}'), \mathbb{Z}_2)$ , as the image of  $F, \partial F$  under  $f$ . Therefore the cocycle  $F' \cup f(F)$  is trivial in  $H_2(\Lambda(\mathcal{L}'), \mathbb{Z}_2)$ . Moreover,  $\text{genus}(F') \leq \text{genus}(F)$ . Since  $f$  is homotopic to a homeomorphism, we know that  $f(F)$  is homotopic to a subfibre,  $F'' \subseteq \Lambda(\mathcal{L}')$ , and we can assume that  $f(\partial F) = \partial F''$ . Now  $F' \cup F''$  is also trivial in  $H_2(\Lambda(\mathcal{L}'), \mathbb{Z}_2)$ . Therefore, by Lemma 4.3.5 (with  $G = F'$ ),  $F'$  is isotopic to  $F''$ , and so  $F'$  is itself a subfibre.  $\square$

#### 4.4. CONSTRUCTING A BILIPSCHITZ MAP.

In this section, we describe some conditions under which a sesquilateral map,  $f : \Theta \rightarrow \Theta'$ , between two bounded geometry 3-manifolds can be promoted to a bilipschitz map. These are formulated by Proposition 4.4.2. In our applications,  $\Theta$  and  $\Theta'$  will be the thick parts of the model space and of the hyperbolic 3-manifold respectively. (It will subsequently be easy to extend over the thin parts.) To focus on the main issues, we express it more general terms, describing the key properties that we need.

We will use the bounded geometry hypothesis to construct a bilipschitz equivalent triangulation of the manifold. A version of this (for manifolds without boundary) can be found in [BoiDG]. A slightly different approach (including the case with boundary) can be found in [Bow10]. We can state what we need as:

**Theorem 4.4.1.** *Let  $\Theta$  be a bounded geometry riemannian  $n$ -manifold with (possibly empty) boundary. Then  $\Theta$  admits a bilipschitz equivalent smooth triangulation, where the bilipschitz constants depend only on  $n$  and the parameters of bounded geometry.*

In our case,  $n = 3$ . We can interpret “bounded geometry” as defined in Section 2.9. A triangulation is a homeomorphism from a simplicial complex, where each simplex is given the structure of a regular euclidean simplex with unit side-lengths. We refer to this as the *standard* (euclidean) metric. The triangulation

is “smooth” if its restriction to each simplex is smooth (though that need not concern us here).

The bilipschitz constants of the conclusion depend only on (the dimension and) the bounded-geometry constants. Note that there is necessarily a bound on the local complexity of the triangulation as defined in Section 4.1 (that is the number of 3-simplices in the link of any simplex). This will allow us, up to linear bounds, to interpret length and volume combinatorially.

Given a triangulation, we say that a subset (typically a submanifold) is **simplicial** if it is a subcomplex of the triangulation. Note that  $\partial\Theta$  is necessarily simplicial.

Recall that a **proper subsurface** of a 3-manifold is a compact connected  $\pi_1$ -injective orientable embedded subsurface,  $F \subseteq \Theta$ , with  $F \cap \partial\Theta = \partial F$ .

Let  $\Theta$  and  $\Theta'$  be orientable irreducible anannular riemannian 3-manifolds with (possibly empty) boundaries,  $\partial\Theta$  and  $\partial\Theta'$ . (Recall that “anannular” means that any properly embedded annulus in  $\Theta$  can be homotoped into  $\partial\Theta$ .) Let  $\mathcal{F}$  be a collection of disjoint ( $\pi_1$ -injective) proper subsurfaces. We refer to the completion of a component of  $\Theta \setminus \bigcup \mathcal{F}$  as a **complementary region**. Let  $f : \Theta \rightarrow \Theta'$  be a continuous proper map with  $f^{-1}(\partial\Theta') = \partial\Theta$ . We assume that  $f$  is a proper homotopy equivalence. In particular, it lifts to a proper map  $\tilde{f} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$  between universal covers. We also assume:

- (F1)  $\Theta, \Theta'$  have bounded geometry.
- (F2)  $f$  is lipschitz.
- (F3) The lift,  $\tilde{f} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$  is a quasi-isometry.
- (F4) The areas (and topological complexities) of the elements of  $\mathcal{F}$  are bounded above.
- (F5)  $\bigcup \mathcal{F}$  is cobounded in  $\Theta$ .
- (F6) Each complementary region  $\mathcal{F}$  meets boundedly many elements of  $\mathcal{F}$ .
- (F7) For each  $F \in \mathcal{F}$ ,  $f(F)$  is homotopic, fixing  $f(\partial F) \subseteq \partial\Theta'$ , to an embedded surface  $F'$ , obtained by surgery on  $f(F)$ .
- (F8) There is some constant,  $K \geq 0$ , and an equivalence relation,  $\sim$ , on  $\mathcal{F}$ , such that if  $F_1, F_2 \in \mathcal{F}$  and there are non-peripheral essential simple closed curves,  $\alpha_1 \subseteq F_1, \alpha_2 \subseteq F_2$  with  $\alpha_1$  freely homotopic to  $\alpha_2$  in  $\Theta$ , then  $F_1 \not\sim F_2$ . If  $F_1 \not\sim F_2$ , then  $d_\Theta(F_1, F_2) \geq K$ . Moreover each  $\sim$ -class has bounded cardinality.

We will show:

**Proposition 4.4.2.** *Suppose that the constant  $K$  of (F8) is sufficiently large in relation to the quasi-isometry constants of (F3), and that  $\Theta, \Theta', f, \mathcal{F}$  satisfy the hypotheses (F1)–(F8) above. Then there is a bilipschitz homeomorphism,  $g : \Theta \rightarrow \Theta'$ , properly homotopic to  $f$ , and such that the lifts,  $\tilde{f}, \tilde{g} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$ , are*

a bounded distance apart. Moreover, the constants of the conclusion depend only on those of the hypotheses (F1), (F2), (F3), (F4), (F5), (F6) and (F8).

Remarks:

(1) In our application to the product manifold case,  $\Theta$  (and  $\Theta'$ ) will be the complement of an unlinked collection of solid tori in  $\Sigma \times \mathbb{R}$ , and  $\mathcal{F}$  will be a collection of subfibres as defined in Section 4.3. The elements of any  $\sim$ -class will be homotopic in  $\Sigma \times \mathbb{R}$ , to disjoint subsurfaces of  $\Sigma$ . (Typically, these will be subfibres of a given fibre.)

(2) In fact, from our construction, it will be the case that no closed curve in  $\Theta$  crosses  $\bigcup \mathcal{F}$  transversely in a single point. This means that the completion of each component of  $\Theta \setminus \bigcup \mathcal{F}$  is the same as its closure in  $\Theta$ . This is not essential, but makes things a bit easier to describe.

(3) Note that (F7) is the conclusion of Lemma 4.2.1, where the phrase “obtained by surgery” is defined. In particular,  $f(F)$  is contained in an arbitrarily small neighbourhood of  $f(F)$ , and  $\text{area}(F') - \text{area}(f(F))$  is also arbitrarily small. Property (F7) will be justified for our application using Lemma 4.3.6. (It is simpler to state directly, rather than list the relevant hypotheses in a general setting.)

(4) The complexity bound, as alluded to in (F4), is a consequence of the other hypotheses. In any case, this particular fact will be immediate from the construction of the model manifold,  $\Theta$ , in our application. Given that  $\Theta$  has bounded geometry, this places a lower bound on the injectivity radius of any  $F \in \mathcal{F}$ , and hence an upper bound on its diameter.

(5) Note that  $f : \Theta \rightarrow \Theta'$  is also a quasi-isometry. We will assume that the constant  $K$  in (F8) is large enough (in relation to those of (F3) and (F4)) so that if  $F_1, F_2 \in \mathcal{F}$  with  $f(F_1) \cap f(F_2) \neq \emptyset$  then  $F_1 \sim F_2$ , and so no essential non-peripheral curve in  $F_1$  can be homotopic to one in  $F_2$ . In particular,  $F'_1 \cap F'_2$  can consist only of trivial or peripheral curves. (This is not the only restriction we will want to impose on  $K$ : see also Lemma 4.4.5.)

(6) As observed above, we can assume  $\Theta$  and  $\Theta'$  to have bilipschitz triangulations of bounded local complexity. Moreover, in view of Proposition 4.1.2, we can assume (after subdividing the triangulation) that each element of  $\mathcal{F}$  is simplicial. In this case, the area bound translates into a bound on the number of 2-simplices it is made out of.

(7) Each complementary region will be haken. In particular, the theorem of Waldhausen, [Wal], stated here as Theorem 1.6.5 applies: any relative homotopy equivalence to another manifold with boundary is homotopic to a homeomorphism.

Before beginning the proof of Proposition 4.4.2, we give a general discussion of maps between triangulated manifolds.



Suppose that  $V$  is a compact triangulated  $n$ -manifold (not necessarily connected) with (possibly empty) boundary  $\partial V$  (necessarily a subcomplex). We write  $|V|$  for the number of  $n$ -simplices: the combinatorial volume of  $V$ . We equip  $V$  with the standard euclidean metric (that is, so that all simplices are regular euclidian and all 1-simplices have unit length). We write  $V = \tilde{V}/G$ , where  $\tilde{V}$  is the universal cover, and where  $G \cong \pi_1(V)$ . We will assume  $\tilde{V}$  is contractible.

Suppose that  $V' = \tilde{V}'/G'$  is another such manifold. If  $f : V \rightarrow V'$  is any homotopy equivalence, we have a lift  $\tilde{f} : \tilde{V} \rightarrow \tilde{V}'$ . This is equivariant with respect to the actions of  $G$  and  $G'$ , via an isomorphism from  $G$  to  $G'$ . Indeed this isomorphism determines  $f$  up to homotopy. The map  $\tilde{f}$  is necessarily a quasi-isometry.

If  $f$  is  $\mu$ -bilipschitz, then so is  $\tilde{f}$ . In particular, the quasi-isometry constants of  $\tilde{f}$  depend only on  $\mu$ . Note that given any quasi-isometry constants, there are only finitely many possibilities for the isomorphism from  $G$  to  $G'$ , hence only finitely many possibilities for  $f$  up to homotopy.

In fact, we can refine this statement somewhat. Let  $\theta : G \rightarrow G'$  be an isomorphism. Given a function,  $\omega : [0, \infty) \rightarrow [0, \infty)$ , we say that  $\theta$  has **expansion at most**  $\omega$  if there exist  $x \in V$  and  $y \in V'$  such that for all  $g \in G$ ,  $d(y, \theta(g)y) \leq \omega(d(x, gx))$ . If  $f : V \rightarrow V'$  is any homotopy equivalence, we say that  $f$  has **expansion at most**  $\omega$  if the induced isomorphism  $\theta$  does. If  $\tilde{f}$  is a quasi-isometry, then we can take  $\omega$  to be linear, given directly by the parameters of the quasi-isometry. Also, for any given bound on expansion, there are only finitely many possibilities for the homotopy class of  $f$ .

If  $n \leq 3$ , then we can take  $V$  and  $V'$  to be smooth. Any homeomorphism is homotopic to a diffeomorphism, and in particular can be assumed bilipschitz. For any given homotopy class of maps with  $n \leq 2$ , we arbitrarily choose one such smooth representative, we refer to it as the “standard” representative.

Note, in particular, given a bound on  $|V|$  and  $|V'|$  together with a particular expansion function, there are only finitely many possibilities for the map  $f$  up to homotopy. In particular, there is a bound on the bilipschitz constants on their standard representatives for  $n \leq 2$ .

For  $n = 3$ , we make the following stronger statement:

**Lemma 4.4.3.** *Suppose that  $V, V'$  are triangulated aspherical 3-manifolds, with boundaries,  $\partial V$  and  $\partial V'$ . Suppose we have (possibly disconnected) simplicial sub-surfaces,  $S \subseteq \partial V$  and  $S' \subseteq \partial V'$ . Suppose that  $f : V \rightarrow V'$  is a homeomorphism, with  $f(S) = S'$  and with  $f|_S : S \rightarrow S'$  the standard representative of its homotopy class on each component of  $S$ . Then  $f$  is homotopic relative to  $S$  to a  $\mu$ -bilipschitz homeomorphism,  $f' : V \rightarrow V'$ , with  $f|_S = f'|_S$ , where  $\mu$  is bounded in terms of  $|V|, |V'|$  and expansion of  $f$ .*

*Proof.* For any given  $f$ , we can extend  $f|_S$  to a smooth map from  $V$  to  $V'$ . Given bounds on  $|V|, |V'|$  and the quasi-isometry constants, there are only finitely many

possibilities for  $V$ ,  $V'$  and the maps  $f$  and  $f|S$  up to homotopy, and therefore only finitely many possibilities for the standard representatives  $f|S$ . We therefore need only to consider boundedly many diffeomorphisms from  $V$  to  $V'$ . This gives a bound on  $\mu$ , simply by taking the maximum bilipschitz constant among all possibilities.  $\square$

We now move on to the proof of Proposition 4.4.2. Suppose that  $\Theta, \Theta', \mathcal{F}, f$  are as in the hypotheses (F1)–(F8). We refer to the constants arising as the “parameters” of the hypotheses.

We write  $\Theta = \tilde{\Theta}/\Gamma$  and  $\Theta' = \tilde{\Theta}'/\Gamma'$ , where  $\Gamma \cong \pi_1(\Theta)$  and  $\Gamma' \cong \pi_1(\Theta')$ . We fix a particular lift,  $\tilde{f} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$ . This determines an isomorphism,  $\theta : \Gamma \rightarrow \Gamma'$ .

As noted earlier, by bounded geometry (F1), we can suppose that  $\Theta$  and  $\Theta'$  admit bilipschitz triangulations of bounded local complexity, and that each element of  $\mathcal{F}$  is simplicial with boundedly many simplices. Note that all the hypotheses remain valid, up to modifying the relevant parameters by a controlled amount.

As in Section 2.11, let  $\mathcal{R}$  be the set of complementary regions of  $\mathcal{F}$ . These are again simplicial. In general each element of  $\mathcal{R}$  maps to  $\Theta$  by a  $\pi_1$ -injective map, possibly identifying boundary components (though in our application, it will simply be the closure of a component of  $\Theta \setminus \bigcup \mathcal{F}$ ). Also, by (F5), each region  $R \in \mathcal{R}$  is compact. In fact, by (F5) and (F6), there is a uniform bound on the combinatorial volume,  $|R|$ .

Suppose that  $F \in \mathcal{F}$  and  $R \in \mathcal{R}$  with  $F \subseteq R$ . That is,  $F$  is a relative boundary component of  $R$ . We lift to  $\tilde{F} \in \tilde{\mathcal{F}}$  and  $\tilde{R} \in \tilde{\mathcal{R}}$  so that  $\tilde{F} \subseteq \tilde{R}$ . Since  $|R|$  is bounded, the inclusion of  $\tilde{F}$  into  $\tilde{R}$  is a uniform quasi-isometric embedding: that is, the constants depend only on the parameters of the hypotheses.

Note that, since  $F$  and  $\Theta$  are orientable,  $F$  is 2-sided in  $\Theta$ . Therefore,  $\tilde{F}$  separates  $\tilde{\Theta}$  into two components.

**Lemma 4.4.4.** *The elements of  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{R}}$  are uniformly embedded in  $\tilde{\Theta}$ .*

In other words, if  $\tilde{F} \in \tilde{\mathcal{F}}$ , and  $x, y \in \tilde{F}$ , then  $d_{\tilde{F}}(x, y)$  is bounded above by some fixed function of  $d_{\tilde{\Theta}}(x, y)$  which depends only on the parameters. Similarly for any  $\tilde{R} \in \tilde{\mathcal{R}}$ .

*Proof.* This is a fairly standard argument. First note that we can construct a dual graph,  $\tau$ , where the vertices correspond to elements of  $\tilde{\mathcal{R}}$ , the edges correspond to elements of  $\tilde{\mathcal{F}}$ , and where incidence is given by inclusion. Since each element of  $\mathcal{F}$  separates,  $\tau$  is a tree.

Suppose that  $\tilde{R} \in \tilde{\mathcal{R}}$ , and that  $x, y \in \tilde{R}$ . For simplicity we can assume that  $x, y$  lie in the 0-skeleton of the triangulation. Connect  $x, y$  by a geodesic  $\beta$  in the 1-skeleton of  $\tilde{\Theta}$ . Its length is (linearly) bounded above in terms of  $d_{\tilde{R}}(x, y)$ . It determines a path in  $\tau$ , of bounded length, with both endpoints at the vertex corresponding to  $\tilde{R}$ . If this path is non-trivial (i.e. not a point), then it must double back on itself. In other words,  $\beta$  must enter and leave some  $\tilde{R}_0 \in \tilde{\mathcal{R}}$  by

the same relative boundary component,  $\tilde{F}_0 \in \tilde{\mathcal{F}}$ . Since  $\tilde{F}_0 \leftrightarrow \tilde{R}_0$  is a uniform (quasi-isometric) embedding, we can replace the segment of  $\beta$  in  $\tilde{R}_0$  by a path of bounded length in  $\tilde{F}_0$ . This eliminates this backtrack in  $\tau$ . After a bounded number of steps, we obtain a path from  $x$  to  $y$  in  $\tilde{R}$ , whose length is bounded above in terms of that of  $\beta$ .

The statement for elements of  $\tilde{\mathcal{F}}$  follows by essentially the same argument.  $\square$

Given  $F \in \mathcal{F}$ , let  $F'$  be the embedded surface in  $\Theta'$  as given by (F7). Now  $F$  has bounded area (F4),  $f|F$  is lipschitz by (F2), and  $F'$  is obtained by surgery on  $f(F)$ , by (F7). Therefore the area of  $F'$  is bounded (in fact by that of  $f(F)$  plus an arbitrarily small constant). Since  $\sim$ -classes have bounded cardinality (F8), it follows that the total area, and diameter, of any  $\sim$ -class is bounded. Moreover, since  $f$  is lipschitz, and  $\bigcup \mathcal{F}$  is cobounded in  $\Theta$ , the union of all the  $F'$  is cobounded in  $\Theta'$ .

We now modify the surfaces,  $F'$ , so that they become disjoint. It's enough to do this for any  $\sim$ -class. This can be done by performing an elementary surgery on each curve of intersection. These surgeries can all be done at once, but it's easier to imagine it sequentially.

To this end suppose that  $F_1, F_2 \in \mathcal{F}$  with  $F'_1 \cap F'_2 \neq \emptyset$ . As observed earlier, we can choose  $K$  large enough so that this implies that  $F_1 \sim F_2$ . In particular, each component of  $F'_1 \cap F'_2$  is either trivial or peripheral in both  $F'_1$  and  $F'_2$ .

If there is a trivial curve in  $F'_1 \cap F'_2$ , then by a standard innermost disc argument, one can find embedded discs,  $D_1 \subseteq F'_1$  and  $D_2 \subseteq F'_2$  with  $D_1 \cap F'_2 = D_2 \cap F'_1 = \partial D_1 = \partial D_2$ . Now  $D_1 \cup D_2$  is an embedded sphere which bounds an embedded ball in  $\Theta'$ . We can therefore perform surgery on the common boundary curve to swap  $D_1$  and  $D_2$ . This eliminates one component of  $F'_1 \cap F'_2$ . Continuing in this manner, we arrange that there are no trivial curves of intersection.

One can deal similarly with peripheral curves. If  $F'_1 \cap F'_2 = \emptyset$ , then we can find annuli  $A_1 \subseteq F'_1$  and  $A_2 \subseteq F'_2$ , with  $A_1 \cap F_2 = A_2 \cap F_1 \subseteq \partial A_1 \cap \partial A_2$ , and with either  $A_1 \cap A_2 = \partial A_1 = \partial A_2$ , or else  $A_1 \cap A_2$  consisting of a single curve,  $\gamma$ , and with the other boundary curves of  $A_1$  and  $A_2$  lying in  $\partial\Theta$ . In the former case  $A_1 \cup A_2$  bounds a solid torus. In the latter case (since  $\Theta'$  is anannular) there is an annulus  $A \subseteq \partial\Theta$  such that  $A_1 \cup A_2 \cup A$  bounds a solid torus. In either case, we can swap  $A_1$  and  $A_2$  so as to reduce the number of components of  $F'_1 \cap F'_2$ . We eventually obtain that  $F'_1 \cap F'_2 = \emptyset$ .

Note that these surgeries can increase the total area of any  $\sim$ -class only by an arbitrarily small amount. In particular, the area of each  $F'$  remains bounded. Moreover the union of all the  $F'$  remains cobounded in  $\Theta$ . We also have  $F' \cap f(F) \neq \emptyset$  for all  $F$ . (All that is essential for us is that  $d_{\Theta'}(F', f(F))$  remains bounded.)

We now apply Proposition 4.1.2, so that after subdividing the triangulation of  $\Theta'$  a bounded number of times, we can assume each  $F'$  to be simplicial in  $\Theta'$

(i.e. a union of 2-simplices). We write  $\mathcal{F}' = \{F' \mid F \in \mathcal{F}\}$  for the collection of subsurfaces arising. Again (by Proposition 4.1.2) the areas remain bounded.

In summary, we now have a collection of bounded-area simplicial subsurfaces,  $\mathcal{F}'$  in  $\Theta'$ , and a bijection,  $[F \mapsto F'] : \mathcal{F} \longrightarrow \mathcal{F}'$ , such that  $F'$  lies in a bounded neighbourhood of  $f(F)$ , and such that the singular surface  $f|F$  is homotopic to  $F'$  in  $\Theta$  sliding  $f(\partial F)$  in  $\partial\Theta'$ . Note that  $\bigcup \mathcal{F}'$  is cobounded in  $\Theta'$ .

**Lemma 4.4.5.** *The map,  $f : \Theta \longrightarrow \Theta'$ , the collections  $\mathcal{F}, \mathcal{F}'$  and the bijection  $[F \mapsto F']$  satisfy the hypotheses of Lemma 2.11.6.*

*Proof.* All that remains to be checked is that the map  $[F \mapsto F'] : \mathcal{F} \longrightarrow \mathcal{F}'$  preserves the order on any parallel pair of surfaces. To this end, suppose that  $F_1, F_2 \in \mathcal{F}$  are parallel. Then  $F'_1, F'_2 \in \mathcal{F}'$  are parallel. Let  $P$  and  $P'$  be respectively the product regions of  $V$  and  $V'$  bounded by these surfaces.

Now  $F_1 \not\sim F_2$ , and so (by (F8))  $d_\Theta(F_1, F_2) > K$ . Note that  $F'_1 \cap f(F_1)$  and  $F'_2 \cap f(F_2)$  are both non-empty. We can suppose that  $K$  is large enough so that  $f(F_1) \cap f(F_2) = \emptyset$ . Let  $\alpha \subseteq P'$  be a path from  $f(F_1)$  to  $f(F_2)$  in  $P'$  meeting these sets precisely at its endpoints. Since  $f|P$  maps to  $P'$  with degree  $\pm 1$ , some component,  $\beta$ , of the preimage  $f^{-1}\alpha$  connects  $F_1$  to  $F_2$  in  $P$ . Let  $x \in \beta$  be a point with  $d_\Theta(x, F_1) = d_\Theta(x, F_2) = K/2$ . Now  $d_\Theta(x, V \setminus P) = K/2$ , so if  $K$  is sufficiently large in relation to the quasi-isometry constants of  $f$ , then  $f(x) \notin f(V \setminus P)$ . Also the homotopies from  $f(F_i)$  to  $F'_i$  both have degree 0 to  $f(x)$ . We can therefore homotope  $f$  to a map  $f'$  such that  $f'|F_i$  is such a homeomorphism to  $F'_i$ , and with  $f'|V \setminus P$  also of degree 0 to  $f(x)$ . Therefore, by Lemma 2.11.4 and the subsequent remark,  $f$  respects the order on  $F_1, F_2$  as claimed.  $\square$

It follows by Lemma 2.11.6 that  $f$  is properly homotopic to a homeomorphism,  $f_0 : \Theta \longrightarrow \Theta'$ , such that  $f_0(F) = F'$  for all  $F \in \mathcal{F}$ . Given  $R \in \mathcal{R}$ , write  $R' = f_0(F) \subseteq V'$ . Then  $R'$  is a complementary region of  $\mathcal{F}'$  in  $V'$ . Let  $\mathcal{R}'$  be the set of such regions.

We have already observed that that  $\mathcal{F}'$  is cobounded in  $\Theta'$ . It follows that for each  $R \in \mathcal{R}$ , the relative boundary,  $\partial_0 R'$ , of  $R'$  is uniformly cobounded in  $R'$ . Moreover,  $\partial_0 R'$  has a bounded number of components, each of bounded area. In particular, we get:

**Lemma 4.4.6.**  *$|R'|$  is bounded for each  $R \in \mathcal{R}$ .*

From this, we in turn deduce:

**Lemma 4.4.7.** *The elements of  $\tilde{\mathcal{F}}'$  and  $\tilde{\mathcal{R}}'$  are uniformly embedded in  $\tilde{\Theta}'$ .*

*Proof.* This follows exactly as in Lemma 4.4.4. Its proof only used the fact that the volumes of the complementary components are uniformly bounded. In this case this is given by Lemma 4.4.6.  $\square$

In summary, we have a homeomorphism  $f_0 : \Theta \longrightarrow \Theta'$ , with  $f_0(F) = F'$  for all  $F \in \mathcal{F}$ , and which is properly homotopic to the original map,  $f$ . We now construct

such a map,  $g$ , which is bilipschitz, where the bilipschitz constants depend only on the parameters of the hypotheses. We first define  $g$  on  $\bigcup \mathcal{F}$ .

Let  $\tilde{f}_0$  be the lift of  $f_0$  which induces the same isomorphism  $\theta : \Gamma \rightarrow \Gamma'$  as  $\tilde{f}$ . This can be obtained by lifting the homotopy from  $f$  to  $f_0$  to the universal covers, so as to give an equivariant homotopy from  $\tilde{f}$  to  $\tilde{f}_0$ .

Given  $F \in \mathcal{F}$ , we choose a particular lift,  $\tilde{F} \in \mathcal{F}$ . Thus  $F = \tilde{F}/G$ , where  $G \leq \Gamma$  is the setwise stabiliser of  $\tilde{F}$ , so that  $G \cong \pi_1(F)$ . Let  $\tilde{F}' = \tilde{f}_0(\tilde{F})$  and  $G' = \theta(G) \subseteq \Gamma'$ . Thus,  $\tilde{F}'$  is  $G'$ -invariant and  $F' = \tilde{F}'/G'$ . Recall that, by definition, the expansion of  $f_0|F : F \rightarrow F'$  depends only on the isomorphism  $\theta|G : G \rightarrow G'$  (and the metrics on  $d_{\tilde{F}}$  and  $d_{\tilde{F}'}$ ).

**Lemma 4.4.8.** *The expansion of  $f_0|F$  is uniformly bounded.*

In other words, the expansion is bounded by some fixed function,  $\omega$ , depending only on the original parameters.

*Proof.* Recall that  $F'$  lies in a bounded neighbourhood of  $f(F)$ , and so  $\tilde{F}'$  lies in a bounded neighbourhood of  $\tilde{f}(\tilde{F})$ . Choose any  $y \in \tilde{F}'$ . There is some  $x \in \tilde{F}$  with  $d_{\tilde{\Theta}'}(y, \tilde{f}(x))$  bounded. By equivariance, we have  $\tilde{f}(gx) = \theta(g)\tilde{f}(x)$ . Since  $\tilde{f}$  is a quasi-isometry, we know that  $d_{\tilde{\Theta}'}(\tilde{f}(x), \theta(g)\tilde{f}(x))$  is linearly bounded in terms of  $d_{\tilde{\Theta}}(x, gx)$  for all  $g \in G$ . It follows that  $d_{\tilde{\Theta}'}(y, \theta(g)y)$  is also linearly bounded. By Lemma 4.4.7,  $F'$  is uniformly embedded in  $\Theta'$ , and so this bounds  $d_{\tilde{F}'}(y, \theta(g)y)$  in terms of  $d_{\tilde{\Theta}}(x, gx) \leq d_{\tilde{F}}(x, gx)$ . This bounds the expansions of  $\theta|G$  with respect to the metrics  $d_F$  and  $d_{F'}$ . By definition, this is the expansion of  $f_0|F$ .  $\square$

Now let  $g : F \rightarrow F'$  be the standard bilipschitz representative of the homotopy class of  $f_0|F$ . Since the combinatorial areas  $|F|$  and  $|F'|$ , as well as the expansion of  $f_0|F$  are all bounded, so are the bilipschitz constants of  $g$ .

We have defined  $g$  on  $\bigcup \mathcal{F}$ . We need to extend over complementary regions.

Let  $R \in \mathcal{R}$ . We have an induced homotopy equivalence,  $(f_0)_R : R \rightarrow R'$ .

**Lemma 4.4.9.** *The expansion of  $(f_0)_R : R \rightarrow R'$  is uniformly bounded.*

*Proof.* This follows by essentially the same argument as Lemma 4.4.8. Write  $R = \tilde{R}/H$  and  $R' = \tilde{R}'/H'$ . Now,  $\theta$  restricts to an isomorphism of  $H$  with  $H'$  induced by the lift  $(\tilde{f}_0)_R : \tilde{R} \rightarrow \tilde{R}'$ . We can find  $x \in R$  and  $y \in R'$  with  $d_{\tilde{\Theta}'}(y, (\tilde{f}_0)_R x)$  bounded. The statement now follows as in Lemma 4.4.8.  $\square$

The volumes,  $|R|$  and  $|R'|$  are also bounded. Therefore, by Lemma 4.4.3, we can extend  $g$  to a uniformly bilipschitz map from  $R$  to  $R'$ .

Doing this for all  $R \in \mathcal{R}$ , gives us a bilipschitz map,  $g : \Theta \rightarrow \Theta'$ , properly homotopic to  $f$ .

This proves Proposition 4.4.2.

## 4.5. THE BILIPSCHITZ THEOREM.

In this section, we finally prove that there is a bilipschitz map from the model space to the hyperbolic 3-manifold. We describe the bilipschitz theorem for doubly degenerate surface groups first, and then describe how it can be generalised.

Let  $\Sigma$  be a compact surface. Let  $\Gamma \cong \pi_1(\Sigma)$  be a doubly degenerate Kleinian group with quotient,  $M = \mathbb{H}^3/\Gamma$ , and non-cuspidal part thereof,  $\Psi(M) \cong \Sigma \times \mathbb{R}$ . Let  $a(e^\pm)$  be the end invariants. Let  $P$  be a model space constructed from these invariants (as described in Sections 2.4 and 2.5). Write  $\Psi(P) \cong \Sigma \times \mathbb{R}$  for its non-cuspidal part. By Theorem 2.13.9, there is a universally sesquilipschitz map,  $\Psi(P) \rightarrow \Psi(M)$ , which extends to a map  $P \rightarrow M$ . We aim to promote this to a bilipschitz map. Specifically, we claim:

**Theorem 4.5.1.** *There is a bilipschitz homeomorphism from  $P$  to  $M$  which sends  $\Psi(P)$  to  $\Psi(M)$ , and which is properly homotopic to the identity on  $\Sigma \times \mathbb{R}$ . The bilipschitz constant depends only on the complexity of  $\Sigma$ .*

In particular, it follows that the model space,  $P$ , is unique up to bilipschitz equivalence, even though there were significant combinatorial choices involved in its construction.

One can interpret “non-cuspidal part” as being defined with respect to a fixed sufficiently small Margulis constant. In any case, it is easily seen that choosing a different constant (or allowing the Margulis constant to vary within bounds) will only change the thick parts to within uniform bilipschitz equivalence, so the statement is quite robust.

Our argument will not give computable bounds on the bilipschitz constants (neither for the lower bound, nor the upper bound).

We briefly recall the construction of the model space,  $P$ .

The end invariants,  $a(e^\pm)$  give rise to a complete annulus system,  $W = \bigcup \mathcal{W} \subseteq \Psi = \Sigma \times \mathbb{R}$ . Let  $\Lambda = \Lambda(\mathcal{W})$  be the space obtained by splitting open each annulus,  $\Omega \in \mathcal{W}$ , so as to give us a toroidal boundary component,  $\Delta(\Omega)$ , of  $\Lambda$ . Let  $\Upsilon$  be the space obtained by gluing in a solid torus,  $T(\Omega)$ , to each  $\Delta(\Omega)$ . Finally,  $P$  is obtained by gluing a cusp to each boundary component of  $\Upsilon$  (so that  $\Upsilon = \Psi(P)$ ). Thus,  $\Lambda \subseteq \Upsilon \subseteq P$ . Note that there is a homeomorphism from  $\Upsilon$  to  $\Psi$ . In fact, we can choose this homeomorphism so that the inclusion of  $\Lambda$  into  $\Upsilon$ , postcomposed with the homeomorphism, in turn postcomposed with projection to the second coordinate is the same as vertical projection in  $\Lambda$  (i.e. that obtained from the projection of  $\Psi = \Sigma \times \mathbb{R}$  to  $\mathbb{R}$ ). We equip  $P$  with its riemannian metric as described in Section 2.8. The main result in the doubly degenerate case (Theorem 2.13.9) gives us a lipschitz map  $P \rightarrow M$  which lifts to a quasi-isometry,  $\tilde{P} \rightarrow \tilde{M}$ , of universal covers. The constant only depends on the topological complexity of  $\Sigma$ .

To construct our bilipschitz map, we need to go back to a previous stage, where we had a map on defined on the thick parts (which we could retrospectively

assume to the restriction of the map from  $P$  to  $M$ ). Recall that Theorem 2.8.2 gives a partition,  $\mathcal{W} = \mathcal{W}_0 \sqcup \mathcal{W}_1$ . (Loosely speaking, we can think of  $\mathcal{W}_0$  as annuli which correspond to tubes of bounded depth.) Let  $\Theta = \Theta(P) = \Lambda(\mathcal{W}, \mathcal{W}_0)$  be the space obtained by gluing solid tori only to those  $\Delta(\Omega)$  for which  $\Omega \in \mathcal{W}_0$ . Thus,  $\Lambda \subseteq \Theta \subseteq \Upsilon \subseteq P$ . The metric on  $\Theta$  has bounded geometry.

We can restrict the quotient map  $\Upsilon \rightarrow \Psi$  to a map  $p : \Theta \rightarrow \Psi$ . This is a homeomorphism on the complement of  $\bigcup \mathcal{W}$ . If  $\Omega \in \mathcal{W}_0$ , then  $p|_{\Delta(\Omega)}$  folds the torus  $\Delta(\Omega)$  onto the annulus  $\Omega$ . If  $\Omega \in \mathcal{W}_1$ , then  $p|_{T(\Omega)}$  collapses the tube  $T(\Omega)$  to the annulus  $\Omega$ . The set of preimages of horizontal curves in  $\Omega \setminus \partial_H \Omega$  defines a foliation of  $T(\Omega)$  by annuli. Since  $T(\Omega)$  has bounded depth, we can assume that these annuli all have bounded area.

By Proposition 2.12.9, there is a lipschitz proper homotopy equivalence  $f : \Theta(P) \rightarrow \Theta(M)$ , to the thick part,  $\Theta(M)$ , of  $M$ , which lifts to a quasi-isometry of universal covers,  $\tilde{f} : \tilde{\Theta}(P) \rightarrow \tilde{\Theta}(M)$ . Again, the constants depend only on the complexity of  $\Sigma$ . We can suppose that  $f^{-1}(\partial\Theta(M)) = \partial\Theta(P)$ .

Write  $\Theta = \Theta(P)$  and  $\Theta' = \Theta(M)$ . We aim to show that  $f : \Theta \rightarrow \Theta'$  satisfies the hypotheses of Theorem 4.4.1. This amounts to constructing a family of surfaces in  $\Theta$  satisfying properties (F1)–(F8). It is simpler first to describe a family,  $\mathcal{S}$ , of horizontal surfaces in  $\Psi$ , which are then easily modified to give the required family,  $\mathcal{F}$ , in  $\Theta$ .

We use the definitions relating to bands from Section 2.4. Given a band  $B \subseteq \Psi$ , we write  $H(B)$  for its height. We write  $\Phi(B) \subseteq \Sigma$  for its base surface (defined up to isotopy). Given subsurfaces,  $\Phi, \Phi'$ , of  $\Sigma$ , we write  $\Phi \leq \Phi'$  (respectively  $\Phi < \Phi'$ ) to mean that  $\Phi$  is contained in (respectively strictly contained in)  $\Phi'$ .

We fix some  $h \in \mathbb{N}$ , sufficiently large in relation to the constant  $K$  required of property (F8). We will specify  $h$  later. We let  $\mathcal{B}$  be the set of bands,  $B$ , with  $H(B) \geq 4h + 1$ . Given  $B \in \mathcal{B}$ , we can find subbands  $B'' \subseteq B' \subseteq B$  with the same base surface, of depth  $2h$  and  $h$  in  $B$  respectively. (In other words, the components of  $B \setminus B'$  and of  $B' \setminus B''$  all have height  $h$ .) We can assume  $h$  is large enough so that we necessarily have  $\partial_V B \subseteq \bigcup \mathcal{W}_1$ .

Let  $B \in \mathcal{B}$ . We can find horizontal fibres,  $S_0, S_1, \dots, S_n$ , of  $B$  with  $n \geq 2$ , which bound subbands,  $A_1, \dots, A_n \subseteq B$ , with  $\partial_- A_i = S_{i-1}$ ,  $\partial_+ A_i = S_i$  and  $H(A_i) = h$  for all  $i$ , and with  $B'' \subseteq \bigcup_{i=1}^n A_i \subseteq B'$ . Let  $A = \bigcup_{i=1}^n A_i$ . Thus  $A$  is a subband with  $\partial_- A = S_0$  and  $\partial_+ A = S_n$ . We will take  $S_i$  to be in general position with respect to  $W$ , i.e.  $S \cap \partial_H \Omega = \emptyset$  for all  $\Omega \in \mathcal{W}$ . Write  $\mathcal{A}(B) = \{A_1, \dots, A_n\}$  and  $\mathcal{S}(B) = \{S_0, \dots, S_n\}$ . Let  $\mathcal{B}(B) = \{C \in \mathcal{B} \mid \Phi(C) < \Phi(B)\}$ . We can also suppose that if  $S \in \mathcal{S}(B)$  and  $C \in \mathcal{B}(B)$ , with  $S \cap B \neq \emptyset$ , then  $S \cap C'' \setminus \partial_H C'' \neq \emptyset$  (so that  $S \cap C = S \cap C''$  is a horizontal fibre of  $C''$ ). This can be achieved by reparameterising the vertical coordinate of the band  $C$ . (This step is not strictly necessary, but will simplify the discussion.)

We similarly choose a bi-infinite sequence of horizontal fibres,  $(S_i)_{i \in \mathbb{Z}}$ , of  $\Psi$ , which cut  $\Psi$  into bands  $A_i$  with  $\partial_- A_i = S_{i-1}$  and  $\partial_+ A_i = S_i$ , and with  $H(A_i) =$

$h$  for all  $i$ . We can similarly assume that if  $C \in \mathcal{B}$  and  $S_i \cap C \neq \emptyset$ , then  $S_i \cap C'' \setminus \partial_H C'' \neq \emptyset$ . We write  $\mathcal{S}(\Psi) = \{S_i \mid i \in \mathbb{Z}\}$  and  $\mathcal{A}(\Psi) = \{A_i \mid i \in \mathbb{Z}\}$ . We write  $\mathcal{S} = \mathcal{S}(\Psi) \cup \bigcup_{B \in \mathcal{B}} \mathcal{S}(B)$ , and  $\mathcal{A} = \mathcal{A}(\Psi) \cup \bigcup_{B \in \mathcal{B}} \mathcal{A}(B)$ .

Given  $A \in \mathcal{A}$ , let  $\mathcal{A}(A) = \{C \in \mathcal{A} \mid \Phi(C) < \Phi(A)\} = \bigcup_{\Phi(C) < \Phi(A)} \mathcal{A}(C)$ . Let  $R = R(A)$  be the closure in  $\Psi$  of  $A \setminus \bigcup \mathcal{A}(A)$ . We write  $\partial_V R = (R \cap \partial \Psi) \cup (\partial R \cap W)$ , and write  $\partial_H R$  for the closure of  $R \setminus \partial_V R$ . Thus  $\partial_V R$  is a union of vertical annuli, and  $\partial_H R$  is a union of horizontal subsurfaces. We write  $\text{int } R = R \setminus \partial R$ .

Note that Lemma 2.4.5 tells us that the size of  $R$  (that is, the number of bricks which it meets) is bounded above as a function of  $h$  and the complexity of  $\Sigma$ .

We claim that  $\{R(A) \mid A \in \mathcal{A}\}$  gives a decomposition of  $\Psi$ . By this, we mean that  $\Psi = \bigcup_{A \in \mathcal{A}} R(A)$ , and  $\text{int } R(A) \cap \text{int } R(A') = \emptyset$ , whenever  $A \neq A'$ .

To see this, let  $x \in \Psi$ , and choose  $A \in \mathcal{A}$  with  $x \in A$ , such that the complexity,  $\xi(\Phi(A))$  is minimal. Then  $x \notin C$  for any  $C \in \mathcal{A}(A)$  and so  $x \in R(A)$ . Conversely, if  $x \in \text{int } R(A)$ , then by construction,  $\xi(\Phi(A))$  is minimal among all  $A \in \mathcal{A}$  with  $x \in A$ . Thus if also  $x \in \text{int } R(A')$ , then  $\xi(\Phi(A)) = \xi(\Phi(A'))$ . If one of  $\Phi(A)$  or  $\Phi(A')$  is  $\Sigma$ , then so is the other. If not, let  $A \in \mathcal{A}(B)$  and  $A' \in \mathcal{A}(B')$ . Then  $\text{int } B \cap \text{int } B' \neq \emptyset$ , so by Lemma 2.4.4, either  $\Phi(A) \subseteq \Phi(A')$  or  $\Phi(A') \subseteq \Phi(A)$ . Therefore  $\Phi(A) = \Phi(A')$ , and it follows that  $B = B'$ , so  $A = A'$ .

In other words, we have decomposed  $\Psi$  into pieces,  $\{R(A) \mid A \in \mathcal{A}\}$ , by cutting along a union of horizontal surfaces,  $\bigcup_{A \in \mathcal{A}} \partial_H R(A) \subseteq \bigcup \mathcal{S}$ , together with a union of vertical annuli. We can describe the horizontal surfaces as follows.

Suppose  $S \in \mathcal{S}$ . By definition,  $S \in \mathcal{S}(B)$  for some  $B \in \mathcal{B} \cup \{\Psi\}$ . Let  $S' = S \setminus \text{int } \bigcup \mathcal{B}(C)$ . This is a (possibly empty, possibly disconnected) subsurface. By construction of  $\mathcal{S}$ , if  $S$  meets some  $C \in \mathcal{B}(B)$ , then  $S \cap C \subseteq C''$ , and so  $S \cap C$  is a horizontal fibre for some band in  $\mathcal{A}(C)$ . Let  $\mathcal{S}' = \{S' \mid S \in \mathcal{S}\}$ . If  $A \in \mathcal{A}$ , then  $\partial_H R(A) \subseteq \partial_- A \cup \partial_+ A \cup \bigcup \{\partial_H R(C) \mid \Phi(C) < \Phi(A)\} = (\partial_- A)' \cup (\partial_+ A)' \cup \bigcup \{\partial_H R(C) \mid \Phi(C) < \Phi(A)\}$ . By induction on complexity, we see that  $\partial_H R(A) \subseteq \bigcup \mathcal{S}'$ . In fact, we get  $\bigcup_{A \in \mathcal{A}} \partial_H R(A) = \bigcup \mathcal{S}'$ .

To apply Proposition 4.4.2, we need a family of surfaces in  $\Theta$ , rather than in  $\Psi$ . We therefore need to modify the surfaces  $\mathcal{S}'$  as follows.

Recall that we have a quotient map,  $p : \Theta \rightarrow \Psi$ , which collapses the tubes  $T(\Omega)$  for  $\Omega \in \mathcal{W}_0$ . Given  $S'' = p^{-1}S \subseteq \Theta$ . Note that if  $S \cap \Omega \neq \emptyset$  for some  $\Omega \in \mathcal{W}_0$ , then  $S$  does not meet any band in  $\mathcal{B}$ . Therefore,  $S \cap \Omega$  is a horizontal curve in  $\text{int } S'$ , and  $p^{-1}(S \cap \Omega)$  is a bounded-area annulus in  $T(\Omega)$ . Therefore,  $S''$  is obtained from  $S''$  by cutting along curves in  $S' \cap \Omega$  for  $\Omega \in \mathcal{W}$ , and gluing in an annulus whenever  $\Omega \in \mathcal{W}_0$ . In particular, note that  $S'' \cap \partial \Theta = \partial S''$ . We write  $\mathcal{F}$  for the set of components of each  $S''$  for  $S \in \mathcal{S}$ . Thus,  $\mathcal{F}$  is a family of disjoint proper subsurfaces of  $\Theta$ .

Given  $A \in \mathcal{A}$ , let  $R''(A) = p^{-1}(R(A)) \subseteq \Theta$ . We have  $\partial_H R''(A) \subseteq \bigcup \mathcal{F}$ . Let  $\mathcal{R} = \{R''(A) \mid A \in \mathcal{A}\}$ . Then  $\mathcal{R}$  is the set of closures of components of  $\Theta \setminus \bigcup \mathcal{F}$ .



We need to verify that  $\mathcal{F}$  satisfies the properties (F4)–(F8) of Proposition 4.4.2. (Properties (F1)–(F3) are immediate.) For (F8) we can take  $K$  to be as large as we require in order to apply Lemma 4.4.2.

Recall that, in the metric  $d_\Theta$  defined on  $\Theta$ , all building blocks have bounded volume and diameter. Here, a “building block” is either (the preimage of) a brick of  $\Lambda$ , or a solid torus,  $T(\Omega)$ , for  $\Omega \in \mathcal{W}_0$ . In addition, we can suppose that each horizontal fibre meets each brick in a bounded-area surface (either a 3HS or a 1HT). If  $F \in \mathcal{F}$ , then  $F$  meets a bounded number of bricks, and meets a bounded number of solid tori each in an annulus of bounded area. Therefore the total area of  $F$  is bounded (as is its complexity). This is Property (F4).

Let  $R \in \mathcal{R}$ . Thus,  $R = R''(A)$  for some  $A \in \mathcal{A}$ . We have seen that the size of  $R(A)$  is bounded: that is it is contained in a bounded number of bricks. From this it follows that  $R''(A)$  lies in a bounded number of building blocks, hence has bounded volume and diameter. In particular, it has a bounded number of relative boundary components in  $\mathcal{F}$ , and lies in a bounded neighbourhood of these components. From this, we obtain (F5) and (F6).

For (F7), note that the tubes  $T(\Omega)$  for  $W \in \mathcal{W}_1$  form a family of unlinked tori in  $\Upsilon \cong \Sigma \times \mathbb{R}$ . Now  $\Theta$  is the complement of the interiors of these tubes. Moreover, each  $F \in \mathcal{F}$ , is a subfibre, as defined in Section 4.3. Similarly,  $\Theta'$  is the complement of a family of unlinked tubes in  $\Psi(M) \cong \Sigma \times \mathbb{R}$ , and  $f : \Theta \rightarrow \Theta'$  is a homotopy equivalence. Therefore, by Lemma 4.3.6, we can perform surgery on  $f(F)$  in  $\Theta'$  to obtain a homotopic embedded surface,  $F' \subseteq \Theta'$ . This is property (F7).

For (F8), suppose that  $S_0, S_1 \in \mathcal{S}$  are distinct. Then  $S_0 \in \mathcal{S}(B_0)$  and  $S_1 \in \mathcal{S}(B_1)$  for some  $B_0, B_1 \in \mathcal{B} \cup \{\Psi\}$ . Let  $\alpha \subseteq \Theta$  be a path of minimal length connecting  $S_0$  and  $S_1$ . Let  $\beta = p\alpha \subseteq \Psi$  be its projection to  $\Psi$ . We claim that  $\beta$  crosses at least  $h$  bricks of  $\Psi$ . To see this, suppose first that  $B_0 = B_1$ . Then  $\beta$  crosses at least one of the bands in  $\mathcal{A}(B_0)$ . But these bands all have height  $h$ , so the statement follows. On the other hand, if  $B_0 \neq B_1$ , then we can assume that  $\Phi(B_0)$  is not contained  $\Phi(B_1)$ . In this case,  $\beta$  must cross either  $\partial_- B_1$  or  $\partial_+ B_1$ . Since the depth of  $S_1$  in  $B_1$  is at least  $h$ , it follows that  $\beta$  must cross at least  $h$  bricks of  $B_1$ . This proves the claim. Now we can choose  $h$  large enough in relation to the predetermined constant  $K$  so that the length of  $\alpha$  is at least  $K$ . This shows that  $d_\Theta(S_0'', S_1'') \geq K$ .

Now suppose that  $F_0, F_1 \in \mathcal{F}$ . By definition,  $F_i$  is a connected component of  $S_i''$  for some  $S_i \in \mathcal{S}$ . If  $S_0 \neq S_1$ , then by the above  $d_\Theta(F_0, F_1) \geq d_\Theta(S_0'', S_1'') \geq K$ . If  $S_0 = S_1$ , then  $F_0$  and  $F_1$  project to disjoint subsurfaces in  $\Sigma$ . In particular, if a curve in  $F_0$  is homotopic to a curve in  $F_1$ , then they must both be either trivial or peripheral in these surfaces. Property (F8) now follows by taking  $F_0 \sim F_1$  to mean that  $S_0 = S_1$ .

We have now verified the hypotheses of Proposition 4.4.2. We deduce that there is a uniformly bilipschitz map  $g : \Theta \rightarrow \Theta'$  properly homotopic to  $f$ .

As a final step, we extend this to a bilipschitz map  $g : P \rightarrow M$ . This is done exactly as before in Sections 2.6 and 2.13. To extend over Margulis tubes, we use Lemma 2.6.8. Extending over cusps is an elementary construction: just send geodesic rays isometrically to geodesic rays.

This finally proves Theorem 4.5.1.

To finish, we give a statement of bilipschitz equivalence in the general case.

Let  $\Gamma$  be any finitely kleinian generated group, with quotient  $M = \mathbb{H}^3/\Gamma$ , and non-cuspidal part thereof  $\Psi(M)$ . Let  $P$  be the model space as constructed in Sections 2.14 and 3.8 (respectively for the indecomposable and general cases). Let  $\Psi(P)$  be the thick part. The construction gives us a preferred proper homotopy class of homeomorphism from  $\Psi(P)$  to  $\Psi(M)$  (and also from  $P$  to  $M$ ).

**Theorem 4.5.2.** *There is a bilipschitz homeomorphism from  $P$  to  $M$  which sends  $\Psi(P)$  to  $\Psi(M)$ , and whose restriction to  $\Psi(P)$  is in the preferred proper homotopy class.*

Note that we no longer claim that the bilipschitz constant is uniform.

Recall that  $P$  is constructed in a series of steps. First to each end,  $e$ , of  $\Psi(M)$  (which will be identified with an end of  $\Psi(P)$ ) we associate a model end,  $\Psi(P_e)$ . The core,  $\Psi_0(P)$ , is compact and the relative boundary components are in bijective correspondence with the ends of  $\Psi(M)$ . If  $e$  is such an end, we can identify its horizontal boundary,  $\Sigma(e)$ , with the corresponding relative boundary of  $\Psi_0(P)$ . We can now arbitrarily extend this to a smooth riemannian metric on  $\Psi_0(P)$  (with the same second fundamental form as that arising from  $\Psi(P_e)$ ). We now glue the model ends to these boundary components, so as to give a riemannian metric on  $\Psi(P)$ . Finally we glue in standard cusps to give a riemannian metric on  $P$ .

Our earlier construction gives us a map of each  $\Psi(P_e)$  to an end,  $E_e = \Psi(M_e) \subseteq \Psi(M)$ , of  $\Psi(M)$ . If  $e$  is geometrically finite, then by Theorem 2.14.2, this is bilipschitz. If we can arrange also that the map is bilipschitz on the degenerate ends, then we can extend arbitrarily to a diffeomorphism from  $\Psi_0(P)$  to the core of  $\Psi(M)$ . This will necessarily be bilipschitz, and so gives us a bilipschitz map from  $\Psi(P)$  to  $\Psi(M)$ . Extending over the cusps is then a simple procedure as described above and in earlier sections.

To conclude, we therefore need to consider a degenerate end,  $e$ . The earlier construction (in Sections 2.14 and 3.8) gave us a lipschitz map from  $\Psi(P_e)$  to  $E_e$ , whose lift to universal covers is a quasi-isometry. It therefore suffices to construct a family,  $\mathcal{F}$ , of proper subsurfaces, satisfying the remaining hypotheses of Proposition 4.4.2. The construction is essentially the same as before, except that now we have  $\Psi \cong \Sigma \times [0, \infty)$ , where  $\Sigma = \Sigma(e)$  is the base surface of the end.

This proves Theorem 4.5.2.

## 5. APPENDIX

## 5.1. THE UNIFORM INJECTIVITY THEOREM.

In this section we give an account of the Uniform Injectivity Theorem applicable in the decomposable case. It implies Lemma 3.2.4. This was in turn used in adapting the argument of [Bow3]: see Proposition 3.5.6.

The original Uniform Injectivity Theorem for pleated surfaces goes back to Thurston [Th1], and there have been several variations since. Most have made some assumption of incompressibility, which is sufficient for the indecomposable case of the Ending Lamination Conjecture, see for example [Mi1]. A variation for the pleating loci of pleated surfaces in handlebodies, is given in [N]. The argument there would apply to more general situations, where the surface is deep in the end of a 3-manifold, without assuming the end is incompressible. However the quantification means that the required depth may be dependent on other constants, and it is not clear that this result can be adapted to the argument given in [Bow3].

In this section, we give a version which depends just on a local incompressibility assumption. As with earlier versions, we argue by contradiction, passing to a limit. This means that the constants involved are not a-priori computable. We will make our statement for laminations, though we only apply it in this paper for multicurves. It will easily be seen to imply Lemma 3.2.4. To simplify notation we only deal with 1-lipschitz pleating surfaces, though the argument will be seen to apply equally well to uniformly lipschitz maps. The basic idea of constructing a partial covering space to derive a contradiction can be found in Thurston's original [Th1], though since we have altered a number of definitions and hypotheses, we work things through from first principles.

Let  $(M, d)$  be a complete hyperbolic 3-manifold, with projectivised tangent bundle,  $\mathbf{E} \rightarrow M$ . We write  $d_{\mathbf{E}}$  for the metric on  $\mathbf{E}$ . The map  $(\mathbf{E}, d_{\mathbf{E}}) \rightarrow (M, d)$  is then 1-lipschitz. In this section, we shall define a **pleating surface** as a 1-lipschitz map  $\phi : (\Sigma, \sigma) \rightarrow (M, d)$  where  $(\Sigma, \sigma)$  is a compact hyperbolic surface with (possibly empty) horocyclic boundary. By a **lamination**  $\lambda \subseteq \Sigma$ , we mean a geodesic lamination in the usual sense, see for example [CanaEG]. We say that a pleating surface,  $\phi : \Sigma \rightarrow M$  **realises**  $\lambda$  if it sends each leaf of  $\lambda$  locally isometrically to a geodesic in  $M$ . We write  $\psi = \psi_{\phi} : \lambda \rightarrow \mathbf{E}$  for the lift to  $\mathbf{E}$ , and let  $\Lambda = \psi(\lambda)$ . (It will follow from subsequent hypotheses that  $\psi$  will be a homeomorphism to  $\Lambda$ , in which case, the notion coincides with that already defined for multicurves in Section 3.2.) Note that we are not assuming  $\lambda$  to be connected.

Here is our formulation of uniform injectivity:

**Proposition 5.1.1.** *Given positive  $\xi, \eta, \epsilon$ , there is some  $\delta > 0$  with the following property. Suppose that  $\Sigma$  is a compact surface with  $\xi(\Sigma) = \xi$ , and suppose that  $\phi : (\Sigma, \sigma) \rightarrow (M, d)$  is a pleating surface to a complete hyperbolic 3-manifold  $M$ , realising a geodesic lamination  $\lambda \subseteq \Sigma$ . Suppose:*

(U1) *For all  $x \in \lambda$ , the injectivity radius of  $M$  at  $\phi(x)$  is at least  $\eta$ .*

(U2) *There is a map  $\theta : N(\phi(\lambda), \eta) \rightarrow \Sigma$  such that the composition  $\theta \circ \phi|_{N(\lambda, \eta)} \rightarrow \Sigma$  is homotopic to the inclusion of  $N(\lambda, \eta)$  in  $\Sigma$ .*

*Then for all  $x, y \in \lambda$ , if  $d_{\mathbf{E}}(\psi_\phi(x), \psi_\phi(y)) \leq \delta$  then  $\sigma(x, y) \leq \epsilon$ .*

In (U1), we are demanding the  $f(\lambda)$  lie in the thick part of  $M$ . This is equivalent to putting a bound on the diameter of each component of  $f(\lambda)$  (and excluding cores of Margulis tubes, though such components could be easily dealt with explicitly).

In (U2),  $N(Q, r)$  denotes the open  $r$ -neighbourhood of  $Q$ . We are assuming that  $\phi$  is 1-lipschitz, and so  $\phi(N(\lambda, \eta)) \subseteq N(\phi(\lambda), \eta)$ . As remarked earlier, it is not hard to see that the hypotheses imply that  $\psi|_\lambda$  is injective. Thus,  $\psi$  is a homeomorphism to  $\Lambda$ . Note that  $\Lambda$  admits a decomposition into geodesic leaves, being invariant under a local geodesic flow. The map  $\theta$  need only be defined up to homotopy.

Although it is implicit in our earlier definitions that  $\Sigma$  is connected, this is not really required. Indeed for the proof, it is convenient to allow for a disconnected surface. This allows us to cut away the thin part of  $\Sigma$ , so that all of  $\Sigma$  maps into the thick part of  $M$ .

We can assume that the length of each boundary curve of  $\Sigma$  is bounded below by some positive constant. To see this, note that  $\lambda$  cannot cross any horocycle of length 1, we can simply cut away the remainder of  $\Sigma$ . Similarly, we can cut away the thin part of  $\Sigma$  along curves of length bounded below, and constant outward curvature bounded above. Thus, we can assume that the injectivity radius of  $\Sigma$  is bounded below, though at the possible cost of disconnecting the surface.

We finally note that it would be enough to assume that  $\phi$  is  $\mu$ -lipschitz for some  $\mu$ , in which case, of course,  $\delta$  will also be a function of  $\mu$ . The argument is unchanged modulo introducing various factors of  $\mu$  into the proceedings.

Before beginning with the proof, we recall some basic facts and introduce some notation relating to laminations.

Let  $\lambda \subseteq \Sigma$  be a lamination, and write  $\vec{\lambda}$  for the unit tangent bundle to  $\lambda$ . Thus  $\lambda$  is a quotient of  $\vec{\lambda}$  under the involution, denoted  $[\vec{a} \mapsto -\vec{a}]$ , that reverses direction. We write  $a \in \lambda$  for the “basepoint” of  $\vec{a}$  (or of  $-\vec{a}$ ). Given  $\vec{a} \in \vec{\lambda}$ , write  $\vec{a}_t \in \vec{\lambda}$  for the vector obtained by flowing a distance  $t$  in the direction of  $\vec{a}$ . If  $\vec{a}, \vec{b} \in \vec{\lambda}$ , we write  $\vec{a} \approx \vec{b}$  if  $\sigma(a_t, b_t) \rightarrow 0$ . This is an equivalence relation on  $\vec{\lambda}$ . Each equivalence class has at most two elements. (It identifies pairs in a finite set of non-closed directed boundary leaves.) If  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\vec{a} \approx \vec{b}$  and with  $\sigma(a, b) \leq \eta$ , then  $\sigma(a_t, b_t)$  is monotonically decreasing for  $t \geq 0$ . If  $a, b \in \lambda$  lie in the same non-closed leaf,  $l$ , of  $\lambda$ , we write  $[a, b]$  for the interval of  $\lambda$  connecting

them. Similarly, if  $a, b \in \Sigma$  with  $\sigma(a, b) < \eta$ , we write  $[a, b]$  for the unique shortest geodesic connecting them. These notations are consistent.

We write  $\Upsilon(\lambda)$  for the union of  $\lambda$  and all those intervals  $[a, b]$  with  $a, b \in \lambda$ , for which  $\sigma(a, b) \leq \eta/8$  and  $\vec{a} \approx \vec{b}$  for some tangents,  $\vec{a}$  and  $\vec{b}$ . Thus, each component of  $\Upsilon(\lambda)$  is either a closed leaf of  $\lambda$ , or else a closed non-annular subsurface of  $\Sigma$ . Each component of  $\Upsilon(\lambda) \setminus \lambda$  is a “spike” between two asymptotic rays in  $\lambda$ .

We can now begin the proof of Proposition 5.1.1. Let us assume that it fails. In this case, we can find a sequence,  $\sigma_i$ , of hyperbolic metrics on  $\Sigma$ , geodesic laminations,  $\lambda_i \subseteq (\Sigma, \sigma_i)$ , pleating surfaces  $\phi_i : (\Sigma, \sigma_i) \rightarrow (M_i, d_i)$ , maps  $\theta_i : N(\phi_i(\lambda_i), \eta) \rightarrow \Sigma$ , and points,  $x_i, y_i \in \lambda_i$  so that  $\sigma_i(x_i, y_i) \geq \epsilon$ , but  $d_{\mathbf{E}_i}(\psi_i(x_i), \psi_i(y_i)) \rightarrow 0$ , where  $\psi_i = \psi_{\phi_i}$ . For each  $i$ , the hypotheses of Proposition 5.1.1 are satisfied for fixed  $\eta, \epsilon > 0$ .

As observed earlier, we can assume that the lengths of the boundary curves of  $(\Sigma, \sigma_i)$  are all bounded below, and so the structures  $(\Sigma, \sigma_i)$  all lie in a compact subset of moduli space. We can thus pass to a subsequence so that these structures converge on some hyperbolic structure  $(\Sigma, \sigma)$ . This may involve precomposing the  $\phi_i$  and postcomposing the  $\theta_i$  with suitable inverse mapping classes of  $\Sigma$ . Indeed, after applying precomposing  $\phi_i$  suitable self homeomorphisms of  $\Sigma$ , we can suppose that the metrics  $\sigma_i$  converge to  $\sigma$ .

Passing to another subsequence we can assume that  $\lambda_i$  converges to a lamination  $\lambda \subseteq \Sigma$  in the Hausdorff topology. Note that  $N_i = N(\lambda_i, \eta/2)$  converges on  $N = N(\lambda, \eta/2)$ . Let  $O_i = N(\phi_i(\lambda_i), \eta/2)$ . Thus  $\phi_i(N_i) \subseteq O_i$ . We can again pass to a subsequence so that  $(O_i, d_i)$  converges on a space,  $(O, d)$ , in the Gromov-Hausdorff topology. The space  $(O, d)$  is an incomplete hyperbolic 3-manifold, in the sense of being locally isometric to  $\mathbb{H}^3$ . (The metric  $d$  need not be a path metric on  $O$ . Indeed  $O$  need not be connected.) We can also observe that the maps  $\phi_i : N_i \rightarrow O_i$  converge to a 1-lipschitz map  $\phi : N_i \rightarrow O_i$  (in the sense that their graphs converge in the Gromov-Hausdorff topology). Now let  $\mathbf{E}_O \rightarrow O$  be the projectivised tangent bundle to  $O$ , and let  $\psi : \lambda \rightarrow \mathbf{E}_O$  be the lift of  $\phi|_{\lambda}$  to  $\mathbf{E}_O$ . We write  $\Lambda = \psi(\lambda)$ . As before,  $\Lambda$  is partitioned into leaves, which are images of leaves of  $\lambda$ . We write  $\vec{\lambda}$  and  $\vec{\Lambda}$  for the tangent spaces of  $\lambda$  and  $\Lambda$  respectively. Note that if  $p, q \in \Lambda$  with  $d(p, q) \leq \eta/2$ , then there is a geodesic,  $[p, q]$ , of length  $d(p, q)$  connecting  $p$  and  $q$  in  $O$ .

Finally, we can pass to yet another subsequence so that  $x_i \rightarrow x \in \lambda$  and  $y_i \rightarrow y \in \lambda$ . Thus  $\sigma(x, y) > \eta$ , but  $\psi(x) = \psi(y)$ . In particular,  $\psi$  is not injective. (From this point on, we could focus our attention on one component of  $O$  where the restriction of  $\psi$  is not injective, though this is not logically necessary.)

**Lemma 5.1.2.** *There is some  $k \geq 0$  such that if  $\pi$  is a path in  $N$  connecting two points,  $a, b \in \lambda$  with  $d(\phi(a), \phi(b)) \leq \eta/2$  such that  $\phi(\pi) \cup [\phi(a), \phi(b)]$  is homotopically trivial in  $O$ , then  $\pi$  is homotopic relative to  $a, b$  in  $\Sigma$  to a path in  $\Sigma$  of length at most  $k$ .*

*Proof.* We first note that if  $\tau$  is any closed curve in  $N$  such that  $\phi(\tau)$  is homotopically trivial in  $O$ , then  $\tau$  is homotopically trivial in  $\Sigma$ . To see this, note that for sufficiently large  $i$ ,  $\tau$  lies in  $N_i$  and  $\phi_i(\tau)$  is trivial in  $O_i$ . Thus,  $\theta_i \circ \phi_i(\tau)$  is trivial in  $\Sigma$ . But  $\theta_i \circ \phi_i$  is homotopic to the inclusion of  $N_i$  in  $\Sigma$ , and so  $\tau$  is trivial in  $\Sigma$  as claimed.

We now consider a lift  $\tilde{\phi} : \tilde{N} \rightarrow \tilde{O}$  to the universal cover,  $\tilde{O}$ , of  $O$  (or more precisely the appropriate connected component of  $O$ ), where  $\tilde{N}$  is some cover of  $N$ . Let  $\tilde{\pi}$  be a lift of  $\pi$  to  $\tilde{N}$ . The endpoints of  $\tilde{\phi}(\tau(\Pi))$  are  $(\eta/2)$ -close in  $\tilde{O}$ . It now follows (from the discreteness of the covering group on  $O$ , and its coboundedness  $\tilde{N}$ ) that the endpoints of  $\tilde{\pi}$  are connected by a path  $\tilde{\pi}'$  of bounded length in  $\tilde{N}$ . This projects to a path  $\pi'$  in  $N$ , with endpoints  $a, b$ . Now  $\phi(\pi \cup \pi')$  is homotopically trivial in  $O$ . (It lifts to the closed curve  $\tilde{\phi}(\pi) \cup \tilde{\phi}(\tilde{\pi}')$ .) Thus,  $\pi \cup \pi'$  is homotopically trivial in  $\Sigma$ . In other words,  $\pi'$  is homotopic to  $\pi$  relative to  $a, b$ , as claimed.  $\square$

Suppose  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\vec{a} \approx \vec{b}$ . Then  $\sigma(a_t, b_t) \rightarrow 0$  and so  $d(\phi(a_t), \phi(b_t)) \rightarrow 0$ . We have the following converse:

**Lemma 5.1.3.** *Suppose  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\sigma(a, b) \leq \eta/2$  and with  $d(\phi(a_t), \phi(b_t)) \rightarrow 0$ , then  $\vec{a} \approx \vec{b}$ .*

*Proof.* Since  $d(\phi(a_t), \phi(b_t))$  is monotonically decreasing for  $t \geq 0$ , we have  $d(\phi(a_t), \phi(b_t)) \leq \eta/2$  for all  $t \geq 0$ . Let  $\pi_t$  be the path  $[a_t, a] \cup [a, b] \cup [b, b_t]$  from  $a_t$  to  $b_t$  in  $N$ . Now  $\phi(\pi_t) \cup [\phi(a_t), \phi(b_t)]$  is homotopically trivial in  $O$ , since  $\phi([a, b]) \cup [\phi(a), \phi(b)]$  has length less than  $\eta$  and  $\phi([a_t, a]) \cup [\phi(a), \phi(b)] \cup \phi([b, b_t]) \cup [\phi(b_t), \phi(a_t)]$  is spanned by the disc  $\bigcup_{u \in [0, t]} [\phi(a_u), \phi(b_u)]$ . It follows by Lemma 5.1.2 that  $\pi_t$  is homotopic in  $\Sigma$  to a path of bounded length  $k$ . This means that the half-leaves of  $\lambda$ , based at  $\vec{a}$  and  $\vec{b}$  are asymptotic, taking account of homotopy class. In other words, there is some  $s \in \mathbb{R}$  with  $\vec{c} \approx \vec{b}$ , where  $c = a_s$ . Now  $\sigma(a, c) \leq \sigma(a, b) \leq \eta/2$ , so  $|s| \leq \eta/2$ . Note that  $\sigma(b_t, c_t) \rightarrow 0$  and  $d(\phi(a_t), \phi(c_t)) \leq d(\phi(a_t), \phi(b_t)) + d(\phi(b_t), \phi(c_t)) \rightarrow 0$ . Now some subsequence of  $\vec{a}_t$  must converge on some  $\vec{p} \in \vec{\lambda}$ . The corresponding  $\vec{c}_t$  converge on  $\vec{p}_s$ , and we have  $\phi(p) = \phi(p_s)$ . The loop  $\phi([p, p_s])$  in  $O$  has length at most  $\eta/2$  and so must be homotopically trivial. From this it follows that  $s = 0$ , and so  $\vec{c} = \vec{a}$ , so  $\vec{b} \approx \vec{a}$  as required.  $\square$

In particular, we see that if  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\sigma(a, b) \leq \eta/2$ , then  $\vec{a} \approx \vec{b}$  if and only if  $d(\phi(a_t), \phi(b_t)) \rightarrow 0$  and if and only if  $d_{\mathbf{E}}(\psi(a_t), \psi(b_t)) \rightarrow 0$ .

**Lemma 5.1.4.** *If  $a, b \in \lambda$  with  $\psi(a) = \psi(b)$ , then either  $a = b$  or  $\sigma(a, b) > \eta/2$ .*

*Proof.* Suppose  $\sigma(a, b) \leq \eta/2$ . Let  $\vec{a}, \vec{b}$  be unit tangent vectors at  $a, b$  with  $\psi(\vec{a}) = \psi(\vec{b})$ . Then  $\psi(a_t) = \psi(b_t)$  for all  $t \in \mathbb{R}$ . Applying Lemma 5.1.3 for  $t \geq 0$  we get  $\vec{a} \approx \vec{b}$ , and for  $t \leq 0$ , we get  $-\vec{a} \approx -\vec{b}$ . It follows that  $a = b$ .  $\square$

Now consider the map  $\psi : \lambda \rightarrow \Lambda = \psi(\lambda)$ . Given  $n \in \mathbb{N}$ , let  $\Lambda(n) = \{c \in \Lambda \mid |\psi^{-1}(c)| \geq n\}$ , and let  $\lambda(n) = \psi^{-1}(\Lambda(n))$ . In view of Lemma 5.1.4,  $\Lambda(n)$  and hence  $\lambda(n)$  is closed. Moreover, these sets are invariant under flow along leaves. In particular,  $\lambda(n)$  is a sublamination of  $\lambda$ .

Now choose  $n$  maximal so that  $\lambda(n) \neq \emptyset$ . Since  $\psi$  is not injective,  $n \geq 2$ . Let  $\xi = \lambda(n)$  and let  $\Xi = \Lambda(n)$ . The map  $\psi : \xi \rightarrow \Xi$  is now everywhere  $n$  to 1. This is also the case for the lifted map  $\psi : \vec{\xi} \rightarrow \vec{\Xi}$ .

We define the closed equivalence relation,  $\sim$ , on  $\xi$  by writing  $a \sim b$  if  $\psi(a) = \psi(b)$ . We can thus identify  $\xi$  with  $\Xi/\sim$ . We similarly define  $\sim$  on  $\vec{\xi}$ , so that  $\vec{\Xi}$  gets identified with  $\vec{\xi}/\sim$ . We write  $[a]$  for the equivalence class of  $a$ .

**Lemma 5.1.5.** *For all  $h \leq \eta/4$ , there is some  $h'$  such that if  $a, b, x \in \lambda$  with  $a \sim x$ ,  $\sigma(a, b) \leq h'$  and  $a \neq b$ . Then there is some  $y \in \lambda \setminus \{x\}$  with  $y \sim b$  and  $\sigma(x, y) < h$ .*

*Proof.* If this fails, there is a sequence  $b_i \rightarrow a \in \lambda$  with  $b_i \neq a$  and such that no point of  $N(x, h) \setminus \{x\}$  is equivalent to  $b_i$ . Now by Lemma 5.1.4, as  $z$  varies over  $[a] = [x]$ , the neighbourhoods  $N(z, h)$  are disjoint, and each can contain at most one element of  $[b_i]$ . Since all these classes have  $n$  elements, the pigeon-hole principle tells us that there is some  $c_i \sim b_i$  with  $\sigma(c_i, [a]) \geq h$ . Passing to a subsequence, we get  $c_i \rightarrow c \neq [a]$ . But  $b_i \rightarrow a$ , contradicting the fact that  $\sim$  is closed.  $\square$

**Lemma 5.1.6.** *Suppose  $\vec{a}, \vec{b}, \vec{x}, \vec{y} \in \vec{\xi}$  with  $\sigma(a, b) < \eta/2$ ,  $\sigma(x, y) < \eta/2$ ,  $\vec{a} \sim \vec{x}$  and  $\vec{b} \sim \vec{y}$ . If  $\vec{a} \approx \vec{b}$ , then  $\vec{x} \approx \vec{y}$ .*

*Proof.* As observed before Lemma 5.1.3, we have  $d_{\mathbf{E}}(\psi(a_t), \psi(b_t)) \rightarrow 0$ . But  $\psi(a_t) = \psi(x_t)$  and  $\psi(b_t) = \psi(y_t)$  for all  $t$ , and so  $d_{\mathbf{E}}(\psi(x_t), \psi(y_t)) \rightarrow 0$ . By Lemma 5.1.3,  $\vec{x} \approx \vec{y}$  as claimed.  $\square$

**Lemma 5.1.7.** *Suppose  $\vec{a}, \vec{b}, \vec{x}, \vec{y} \in \vec{\xi}$  with  $\sigma(a, b) < \eta/2$ ,  $\sigma(x, y) < \eta/2$ ,  $\vec{a} \approx \vec{b}$  and  $\vec{x} \approx \vec{y}$ . If  $\vec{a} \sim \vec{x}$ , then  $\vec{b} \sim \vec{y}$ .*

*Proof.* Let  $h' > 0$  be the constant of Lemma 5.1.5 given  $h = \eta/4$ . We can assume that  $h' \leq \eta/2$ . Choose  $t \geq 0$  so that  $\sigma(a_t, b_t) < h'$ . By Lemma 5.1.5, applied to  $a_t, b_t, x_t$ , there is some  $z \in \lambda \setminus \{x_t\}$  with  $z \sim b_t$  and  $\sigma(x_t, z) < \eta/2$ . Let  $\vec{z} \in \vec{\lambda}$  be the vector with  $\vec{z} \sim \vec{b}_t$ . Now  $\sigma(a_t, b_t) < \eta/2$  and  $\vec{a}_t \sim \vec{x}_t$ . Thus, applying Lemma 5.1.6 with  $\vec{a}_t, \vec{b}_t, \vec{x}_t, \vec{z}$ , we get  $\vec{x}_t \approx \vec{z}$ . But  $\vec{x}_t \approx \vec{y}_t$ , and  $\vec{z} \neq \vec{x}_t$ . Thus, since any  $\approx$ -class has at most two elements, it follows that  $\vec{z} = \vec{y}_t$ . In other words,  $\vec{b}_t \sim \vec{y}_t$ , and it follows that  $\vec{b} \sim \vec{y}$  as claimed.  $\square$

Lemmas 5.1.5, 5.1.6 and 5.1.7 are effectively telling us that the map  $\psi : \xi \rightarrow \Xi$  is a covering space. This can be made precise as follows. Let  $\Phi = \Upsilon(\xi) \subseteq \Sigma$  be the space obtained by filling in the spikes of  $\Sigma \setminus \xi$  as described earlier. Note that  $\Phi \subseteq N$ . We can extend  $\sim$  to a closed equivalence relation on  $\Phi$  as follows.

Suppose  $p \in \Phi \setminus \xi$ . Then  $p$  lies on a geodesic  $[a, b]$ , with  $a, b \in \xi$ ,  $\vec{a} \approx \vec{b}$  and  $\sigma(a, b) \leq \eta/8$ . Suppose  $q \in \Phi \setminus \xi$  similarly lies in  $[x, y]$ . Then  $\vec{a} \sim \vec{x}$  if and only if  $\vec{b} \sim \vec{y}$ . In this case, we write  $p \sim q$  if and only if  $p$  cuts  $[a, b]$  in the same ratio that  $q$  cuts  $[x, y]$ . It is now readily checked that  $\sim$  is a closed equivalence relation with  $n$  points in each class. Let  $\Omega = \Phi/\sim$ . The quotient map  $\omega : \Phi \rightarrow \Omega$  is an  $n$ -fold covering. Each component of  $\Omega$  is a circle or a closed surface.

Suppose  $p, q, a, b, x, y \in \Phi$  are as above. By assumption  $\sigma(a, b) \leq \eta/8$  and  $\sigma(x, y) \leq \eta/8$ . Since  $\phi : N \rightarrow O$  is 1-lipschitz and  $\phi(a) = \phi(x)$  and  $\phi(b) = \phi(y)$ , we see that  $d(\phi(a), \phi(b)) \leq \eta/8$ . Put another way, if  $z \in \Omega$ , then  $\text{diam } \phi(\omega^{-1}(z)) \leq \eta/8$ . Note also that  $\phi(\Phi) \subseteq N(\phi(\lambda), \eta/8)$ .

Recall that  $\phi : N \rightarrow O$  is a limit of the maps  $\phi_i : N_i \rightarrow O_i$ . Moreover, by construction,  $\Phi \subseteq N_i$  for all sufficiently large  $i$ . We see that for all large  $i$ ,  $\text{diam } \phi(\omega^{-1}(z)) < \eta/4$ , say, for all  $z \in \Omega$ , and that  $\phi(\Phi) \subseteq N(\phi_i(\lambda_i), \eta/4)$ . Note that, by the condition of injectivity radius of  $\phi_i(\lambda_i)$  in  $O_i$ , we see that  $\phi_i(\omega^{-1}(z))$  lies in a hyperbolic  $(\eta/2)$ -ball embedded in  $O_i$ .

We now fix some such  $i$ , and set  $\mu(z) \in O_i$  to be the centre of  $\phi_i(\omega^{-1}(z))$ , in other words, the point so that  $\phi_i(\omega^{-1}(z))$  lies in the closed  $r$ -ball about  $\mu(z)$  for  $r$  minimal. Here  $r \leq \eta/8$  and the point is uniquely defined, given the fact that  $\phi_i(\omega^{-1}(z))$  lies in a hyperbolic  $(\eta/2)$ -ball embedded in  $O_i$ . Moreover, it gives us a continuous map,  $\mu : \Omega \rightarrow O_i$ . Also, for each  $x \in \Phi$ ,  $d(\phi_i(x), \mu(\omega(x))) < \eta/4$ . So again by the condition of injectivity radius, we see that the maps  $\phi_i : \Phi \rightarrow O_i$  and  $\mu \circ \omega : \Phi \rightarrow O_i$  are homotopic (by linear homotopy along short geodesics).

By hypothesis,  $\theta_i \circ \phi_i : \Phi \rightarrow \Sigma$  (being a restriction of  $\theta_i \circ \phi_i : N_i \rightarrow \Sigma$ ) is homotopic to inclusion, and so therefore is  $\theta_i \circ \mu \circ \omega : \Phi \rightarrow \Sigma$ . Writing  $f = \theta_i \circ \mu$ , we can summarise this as follows:

**Lemma 5.1.8.** *We have an  $n$ -to-1 covering map  $\omega : \Phi \rightarrow \Omega$ , with  $n \geq 2$ , and a map  $f : \Omega \rightarrow \Sigma$  such that  $f \circ \omega : \Phi \rightarrow \Sigma$  is homotopic to inclusion.*

In order to get a contradiction, we make the following purely topological observation.

**Lemma 5.1.9.** *Suppose  $\Phi$  is a (not necessarily connected) subsurface of the compact surface  $\Sigma$ . Suppose that  $\omega : \Phi \rightarrow \Omega$  is a  $n$ -fold covering map to a (not necessarily connected) surface  $\Omega$ . Suppose that there is a map  $f : \Omega \rightarrow \Sigma$  such that  $f \circ \omega : \Phi \rightarrow \Sigma$  is homotopic to inclusion. Then either  $n = 1$  or  $\Phi$  is homotopic into a 1-dimensional submanifold of  $\Sigma$  (a multicurve union the boundary of  $\Sigma$ ).*

*Proof.* It is clearly sufficient to prove the result when  $\Sigma$  is connected. We first note that we can also reduce to the case where  $\Phi$  is connected. For if  $\Phi_0$  is a component of  $\Phi$ , then  $\omega|_{\Phi_0}$  is an  $n_0$ -fold cover of a component,  $\Omega_0$ , of  $\Omega$ . Together with  $f|_{\Omega_0}$ , this satisfies the hypotheses of the lemma. Suppose that  $\Omega$  is not homotopic into a closed curve. The lemma then tells us that  $n_0 = 1$ . If  $n > 1$ , then some other component,  $\Phi_1$ , of  $\Phi$  also gets mapped homeomorphically to  $\Omega_0$ , so the inclusions



of  $\Omega_0$  and  $\Omega_1$  into  $\Sigma$  are homotopic. In other words,  $\Omega_0$  can be homotoped to be disjoint from itself in  $\Sigma$ . But this is impossible since we are assuming that it cannot be homotoped into a curve. We conclude that all components of  $\Omega$  are homotopic into curves. Moreover, it is easily seen that we can take these curves to be disjoint in  $\Sigma$  giving the result.

Let us therefore suppose that  $\Phi$  is connected. Suppose first that each (intrinsic) boundary component of  $\Phi$  is homotopically trivial in  $\Sigma$ . If  $\Phi$  is not homotopic to a point in  $\Sigma$ , then  $\Sigma$  is closed, and each component of  $\Sigma \setminus \Phi$  is a disc. Let  $\Omega'$  be the closed surface obtained by gluing a disc to each boundary component of  $\Omega$ . We can now extend  $\omega$  to an  $n$ -fold branched cover  $\omega' : \Sigma \rightarrow \Omega'$ , and extend  $f$  to a map  $f' : \Omega \rightarrow \Sigma$ . The composition  $f' \circ \omega' : \Sigma \rightarrow \Sigma$  is homotopic to the identity, and it follows that  $n = 1$ .

Suppose that  $\alpha$  is a boundary curve of  $\Phi$  that is homotopically non-trivial in  $\Sigma$ . Its inclusion into  $\Sigma$  factors through the boundary curve,  $\omega(\alpha)$ , of  $\Omega$ . It follows that  $\omega|_{\alpha}$  is injective. If  $n > 1$ , then some other boundary curve,  $\beta$ , of  $\Phi$  also gets mapped homeomorphically to  $\omega(\alpha)$ . Now  $\alpha$  and  $\beta$  are homotopic in  $\Sigma$ , and hence bound an annulus. Unless this annulus contains  $\Phi$ , it must be a component of  $\Sigma \setminus \Phi$ . No other boundary component of  $\Phi$  can be homotopic to this annulus. We see that  $n = 2$ , and that the homotopically non-trivial boundary components of  $\Phi$  occur in pairs that bound annular components of  $\Sigma \setminus \Phi$ . It follows that  $\Sigma$  is closed and that each component of  $\Sigma \setminus \Phi$  is either such an annulus or a disc. Let  $\Omega''$  be the surface obtained by gluing a disc to each boundary curve of  $\Omega$  whose  $f$ -image is trivial in  $\Sigma$ . We now extend  $\omega$  to a map  $\omega'' : \Sigma \rightarrow \Omega''$  by collapsing the annular components of  $\Sigma \setminus \Phi$  to boundary curves, without twisting, and extending (anyhow) over the disc components. Let  $f'' : \Omega'' \rightarrow \Sigma$  be any extension of  $f$ . Thus,  $f'' \circ \omega'' : \Sigma \rightarrow \Sigma$  is homotopic to the identity. But it factors through the non-closed surface  $\Omega''$ , giving a contradiction.  $\square$

We can now apply this to the situation described by Lemma 5.1.8. Certain components of  $\Phi$  may be circles, as we have defined it, but these can be thickened up to annuli, so that makes no essential difference. From the construction, no two distinct components of  $\Omega$  can be homotoped into the same closed curve. Lemma 5.1.9 now gives the contradiction that  $n = 1$ , finally proving Proposition 5.1.1.

## REFERENCES

- [Ag] I. Agol, *Tameness and hyperbolic 3-manifolds* : preprint, 2004, posted at arXiv:0405568.
- [Ah] L.V. Ahlfors, *Finitely generated Kleinian groups* : Amer. J. Math. **86** (1964) 413–429, and **87** (1965) 759.
- [Ao] T. Aougab, *Uniform hyperbolicity of the graphs of curves* : Geom. Topol. **17** (2013) 2855–2875.
- [Bea] A.F. Beardon, *The geometry of discrete groups* : Graduate Texts in Mathematics, Vol. 91, Springer-Verlag (1983).
- [BerK] L. Bers, I. Kra (eds.), *A crash course on Kleinian groups* : Lecture notes in Mathematics No. 400, Springer Verlag (1974).

- [BesB] M.Bestvina, K.Bromberg, *On the asymptotic dimension of the curve complex* : Geom. Topol. **19** (2019) 477–489.
- [BoiDG] J.-D.Boissonnat, R.Dyer, A.Ghosh, *Delaunay triangulation of manifolds* : Found. Comput. Math. **18** (2018) 399–431.
- [Bon] F.Bonahon, *Bouts des variétés hyperboliques de dimension 3* : Ann. of Math. **124** (1986) 71–158.
- [Bow1] B.H.Bowditch, *Hyperbolic 3-manifolds and the geometry of the curve complex* : in “European Congress of Mathematics, Stockholm, June 27 – July 2, 2004” (ed. A.Laptev) European Mathematical Society Publishing House (2005) 103–115.
- [Bow2] B.H.Bowditch, *Systems of bands in hyperbolic 3-manifolds* : Pacific J. Math. **232** (2007) 1–42.
- [Bow3] B.H.Bowditch, *Length bounds on curves arising from tight geodesics* : Geom. Funct. Anal. **17** (2007) 1001–1042.
- [Bow4] B.H.Bowditch, *Tight geodesics in the curve complex* : Invent. Math. **171** (2008) 281–300.
- [Bow5] B.H.Bowditch, *Atoroidal surface bundles over surfaces* : Geom. Funct. Anal. **19** (2009) 943–988.
- [Bow6] B.H.Bowditch, *Notes on tameness* : Enseign. Math. **56** (2010) 229–285.
- [Bow7] B.H.Bowditch, *Stacks of hyperbolic spaces and ends of hyperbolic 3-manifolds* : in “Geometry and Topology Down Under”, Contemporary Mathematics Vol. 597, (eds. C.D.Hodgson, W.H.Jaco, M.G.Scharlemann, S.Tillman) American Mathematical Society (2013) 65–138.
- [Bow8] B.H.Bowditch, *Uniform hyperbolicity of the curve graphs* : Pacific J. Math. **269** (2014) 269–280.
- [Bow9] B.H.Bowditch, *Coarse hyperbolic models for 3-manifolds* : preprint, Warwick 2010.
- [Bow10] B.H.Bowditch, *Bilipschitz triangulations of riemannian manifolds* : preprint, Warwick 2020.
- [BridH] M.R.Bridson, A.Haefliger, *Metric spaces of non-positive curvature* : Grundlehren der Math. Wiss. No. 319, Springer (1999).
- [BrinT] M.G.Brin, T.L.Thickstun, *3-manifolds which are end 1-movable* : Memoirs Amer. Math. Soc. No. 81 (1989).
- [Brocb] J.F.Brock, K.Bromberg, *Geometric inflexibility and 3-manifolds that fiber over the circle* : J. Topol. **4** (2011) 1–38.
- [Brocm] J.F.Brock, R.D.Canary, Y.N.Minsky, *Classification of Kleinian surface groups II: The ending lamination conjecture* : Ann. of Math. **176** (2012) 1–149.
- [Brow] E.M.Brown, *Unknotting in  $M^2 \times I$*  : Trans. Amer. Math. Soc. **123** (1966) 480–505.
- [CalG] D.Calegari, D.Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds* : J. Amer. Math. Soc. **19** (2006) 385–446.
- [Cana] R.D.Canary, *Ends of hyperbolic 3-manifolds* : J. Amer. Math. Soc. **6** (1993) 1–35.
- [CanaEG] R.D.Canary, D.B.A.Epstein, P.Green, *Notes on notes of Thurston* : in “Analytic and geometric aspects of hyperbolic space”, London Math. Soc. Lecture Notes Series No. 111, (ed. D.B.A.Epstein) Cambridge University Press (1987) 3–92.
- [CannT] J.W.Cannon, W.P.Thurston, *Group invariant Peano curves* : Geom. Topol. **11** (2007) 1315–1355.
- [CasB] A.J.Casson, S.A.Bleiler, *Automorphisms of surfaces after Nielsen and Thurston* : London Math. Soc. Student Texts No. 9, Cambridge University Press (1988).
- [CLRS] M.Clay, K.Rafi, S.Schleimer, *Uniform hyperbolicity of the curve graph via surgery sequences* : Algebr. Geom. Topol. **14** (2014) 3325–3344.

- [FHS] M.Freedman, J.Hass, P.Scott, *Least area incompressible surfaces in 3-manifolds* : Invent. Math. **71** (1983) 609–642.
- [Ga] D.Gabai, *On the topology of ending lamination space* : Geom. Topol. **18** (2014) 2683–2745.
- [GhH] E.Ghys, P.de la Harpe, *Sur les groupes hyperboliques d’après Mikhael Gromov* : Progress in Maths. No. 83, Birkhäuser (1990).
- [Gr1] M.Gromov, *Hyperbolic groups* : in “Essays in Group Theory” (ed. S.M.Gersten) M.S.R.I. Publications No. 8, Springer-Verlag (1987) 75–263.
- [Gr2] M.Gromov (edited by J.LaFontaine and P.Pansu), *Metric structures for Riemannian and non-Riemannian spaces* : Progress in Mathematics No. 152, Birkhäuser (1998).
- [Ham] U.Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves* : in “Spaces of Kleinian groups” (eds. Y.Minsky, M.Sakuma, C.Series) London Math. Soc. Lecture Notes No. 329 (2005) 187–207.
- [Har] W.J.Harvey, *Boundary structure of the modular group* : in “Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference” (ed. I.Kra, B.Maskit), Ann. of Math. Stud. No. 97, Princeton University Press (1981) 245–251.
- [Hem1] J.Hempel, *3-manifolds* : Ann. of Math. Studies, No. 86, Princeton University Press (1976).
- [Hem2] J.Hempel, *Residual finiteness for 3-manifolds* : in “Combinatorial group theory and topology (Alta, Utah, 1984)” (ed. S.M.Gersten, J.R.Stallings) Ann. of Math. Studies, No. 111, Princeton Univ. Press (1987) 379–396.
- [HenP] S.Hensel, P.Przytycki, *The ending lamination space of the five-punctured sphere is the Nöbeling curve* : J. Lond. Math. Soc. **84** (2011) 103–119.
- [HenPW] S.Hensel, P.Przytycki, R.C.H.Webb, *1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs* : J. Eur. Math. Soc. **17** (2015) 755–762.
- [IT] Y.Imayoshi, M.Taniguchi, *An introduction to Teichmüller spaces* : Springer-Verlag (1992).
- [Ja] W.Jaco, *Lectures on three-manifold topology* : CBMS Regional Conference Series in Mathematics, Vol 24, American Mathematical Society (1980).
- [Jo] T.Jørgensen, *Compact 3-manifolds of constant negative curvature fibering over the circle* : Ann. of Math. **106** (1977) 61–72.
- [Ka] M.Kapovich, *Hyperbolic manifolds and discrete groups* : Progress in Mathematics, No. 183, Birkhäuser (2001).
- [KiLO] I.Kim, C.Lecuire, K.Ohshika, *Convergence of freely decomposable Kleinian groups* : preprint, Seoul/Warwick/Osaka (2004).
- [Kla] E.Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space* : preprint, Michigan (1999).
- [KleS] G.Kleineidam, J.Souto, *Algebraic convergence of function groups* : Comment. Math. Helv. **77** (2002) 244–269.
- [Lea] J.P.Leasure, *Geodesics in the complex of curves of a surface* : thesis, University of Texas at Austin (2006).
- [LehV] O.Lehto, K.I.Virtanen, *Quasiconformal mappings in the plane* (translated by K.W.Lucas) : Grundlehren der mathematischen Wissenschaften, Vol. 126, Springer-Verlag (1973).
- [Mar1] A.Marden, *The geometry of finitely generated kleinian groups* : Ann. of Math. **99** (1974) 383–462.
- [Mar2] A.Marden, *Hyperbolic manifolds: an introduction in 2 and 3 dimensions* : Cambridge University Press (2016).

- [Mas] H.A.Masur, *Measured foliations and handlebodies* : Ergodic Theory Dynam. Systems **6** (1986) 99–116.
- [MasM1] H.A.Masur, Y.N.Minsky, *Geometry of the complex of curves I: hyperbolicity* : Invent. Math. **138** (1999) 103–149.
- [MasM2] H.A.Masur, Y.N.Minsky, *Geometry of the complex of curves II: hierarchical structure* : Geom. Funct. Anal. **10** (2000) 902–974.
- [Mc] D.McCullough, *Compact submanifolds of 3-manifolds with boundary* : Quart. J. Math. **37** (1986) 299–307.
- [Mi1] Y.N.Minsky, *Teichmüller geodesics and ends of hyperbolic 3-manifolds* : Topology **32** (1993) 625–647.
- [Mi2] Y.N.Minsky, *On rigidity, limit sets, and ends of hyperbolic 3-manifolds* : J. Amer. Math. Soc. **7** (1994) 539–588.
- [Mi3] Y.N.Minsky, *The classification of punctured torus groups* : Ann. of Math. **149** (1999) 559–626.
- [Mi4] Y.N.Minsky, *The classification of Kleinian surface groups I : Models and bounds* : Ann. of Math. **171** (2010) 1–107.
- [Moi] E.E.Moise, *Geometric topology in dimensions 2 and 3* : Graduate Texts in Mathematics, Vol. 47, Springer-Verlag (1977).
- [Mosh] L.Mosher, *Stable Teichmüller quasigeodesics and ending laminations* : Geom. Topol. **7** (2003) 33–90.
- [Most] G.D.Mostow, *Quasi-conformal mappings in  $n$ -space and the rigidity of hyperbolic space forms* : Inst. Hautes Études Sci. Publ. Math. **34** (1968) 53–104.
- [N] H.Namaz, *Heegaard splitting and hyperbolic geometry* : thesis, Princeton (2005).
- [NS] H.Namaz, J.Souto, *Non-realizability and ending laminations: proof of the density conjecture* : Acta Math. **209** (2012) 323–395.
- [Oh] K.Ohshika, *Realising end invariants by limits of minimally parabolic, geometrically finite groups* : Geom. Topol. **15** (2011) 827–890.
- [OhS] K.Ohshika, T.Soma, *Geometry and topology of geometric limits I* : preprint, 2010, posted at arXiv:1002.4266.
- [Ot1] J.-P.Otal, *Courants géodésiques et surfaces* : Thesis, Orsay Paris-Sud (1989).
- [Ot2] J.-P.Otal, *Thurston’s hyperbolization of Haken manifolds* : Surveys in differential geometry, Vol. III, 77–194, International Press (1998).
- [Ot3] J.-P.Otal, *Les géodésiques fermées d’une variété hyperbolique en tant que noeuds* : in “Kleinian groups and hyperbolic 3-manifolds” (ed. Y.Komori, V.Markovic, C.Series), London Math. Soc. Lecture Notes Series No. 299, Cambridge University Press (2003) 95–104.
- [Pe] G.Perelman, *The entropy formula for the Ricci flow and its geometric applications* : preprint (2002), posted at arXiv:math/0211159.
- [Pr] G.Prasad, *Strong rigidity of  $Q$ -rank 1 lattices* : Invent. Math. **21** (1973) 255–286.
- [Ra] J.G.Ratcliffe, *Foundations of hyperbolic manifolds* : Graduate Texts in Mathematics, Vol. 149, Springer-Verlag (1994).
- [Re] M.Rees, *The ending laminations theorem direct from Teichmüller geodesics* : preprint, 2004, posted at arXiv:math/0404007.
- [Ri] R.Riley, *Parabolic representations of knot groups. I.* : Proc. London Math. Soc. **24** (1972) 217–242.
- [Sc1] P.Scott, *Finitely generated 3-manifold groups are finitely presented* : J. London Math. Soc. **6** (1973) 437–440.
- [Sc2] P.Scott, *Compact submanifolds of 3-manifolds* : J. London Math. Soc. **7** (1973) 246–250.

- [Sc3] P.Scott, *Subgroups of surface groups are almost geometric* : J. London Math. Soc. **17** (1978) 555–565.
- [Se] J.-P.Serre, *Arbres, Amalgames,  $SL_2$*  : Astérisque Vol. 46, Société Mathématique de France (1977).
- [Sh] K.J.Shackleton, *Tightness and computing distances in the curve complex* : Geom. Dedicata **160** (2012) 243–259.
- [So1] T.Soma, *Existence of ruled wrapping in hyperbolic 3-manifolds* : Geom. Topol. **10** (2006) 1173–1184.
- [So2] T.Soma, *A geometric approach to the ending lamination conjecture* : preprint, Tokyo (2008), posted at arXiv:0801.4236.
- [Su] D.Sullivan, *Quasiconformal homeomorphisms and dynamics II. Structural stability implies hyperbolicity for kleinian groups* : Acta Math. **155** (1985) 243–260.
- [Ta] R.Tang, *The curve complex and covers via hyperbolic 3-manifolds* : Geom. Dedicata **161** (2012) 233–237.
- [Th1] W.P.Thurston, *The geometry and topology of 3-manifolds* : notes, Princeton (1979).
- [Th2] W.P.Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry* : Bull. Amer. Math. Soc. **6** (1982) 357–381.
- [Wal] F.Waldhausen, *On irreducible 3-manifolds which are sufficiently large* : Ann. of Math. **87** (1968) 56–88.
- [Wan] S.Wang, *Non-zero degree maps between 3-manifolds* : in “Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)” (ed. T.Li) Higher Ed. Press (2002), 457–468.
- [We] R.C.H.Webb, *Combinatorics of tight geodesics and stable lengths* : Trans. Amer. Math. Soc. **367** (2015) 7323–7342.

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