LIMIT SETS OF COARSE EMBEDDINGS

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Abstract. We consider the limit set of the image of a coarse embedding of a geodesic space into a hyperbolic space of bounded geometry. Under suitable growth conditions on the domain, we show that such a limit set is perfect.

1. Introduction

Let (N, d_N) be a metric space. Recall that a *geodesic* in N is a path whose length is equal to the distance between its endpoints. We say that N is a *geodesic space* if every pair of points are connected by a geodesic. A very important class are hyperbolic spaces as defined by Gromov [Gr]. To any hyperbolic space, N, we may associate its boundary, ∂N. This is a metrisable topological space (which in the cases of interest here, will be compact). In fact, there is a natural topology on $N \cup \partial N$, inducing the metric topology on N, and with N an open dense subset. Given a subset, $W \subseteq N$, we define its *limit set*, $\Lambda(W) \subseteq \partial N$, to be the set of accumulation points of W in ∂N.

Suppose we have a map, $f : M \longrightarrow N$, where M is a geodesic space, and N is hyperbolic. We will give conditions which imply that the limit set of the image, $\Lambda(f(M))$, is perfect — that is, it contains no isolated points. (This implies that it is uncountable, in particular, not a single point.) Specifically we will assume that M has "fast growth", that N has "bounded geometry" and that f is "bounded-to-one" (for example, a "coarse embedding").

We will postpone the general definition of these terms to Section 3, and begin with a discussion of graphs in Section 2. This contains all the essential points without involving too many technical details. The more general statements can be deduced by similar arguments, or by reducing to this case (see Section 3).

For the moment, we just observe that there are many natural examples of such spaces. For example, (a Cayley graph of) any hyperbolic group has bounded geometry. Moreover, if the group is non-elementary (that is, not virtually cyclic) then it also has fast growth. A Hadamard manifold (i.e. a complete simply connected non-positively curved manifold) of dimension at least 2 is hyperbolic and has fast growth provided the curvature is bounded away from 0. It also has bounded geometry if the curvature is bounded away from $-\infty$. Thus all pinched Hadamard manifolds (including all negatively curved symmetric spaces) satisfy the all above conditions.

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Note that a horoball in (constant curvature) hyperbolic space, despite having exponential volume growth, does not have fast growth in the above sense. Indeed the conclusion fails spectacularly for the identity (or inclusion) map in this case — the limit set is a singleton.

The motivation for this paper comes from work of Pansu [P] where it is shown under slightly different hypotheses that the limit sets of certain coarse embeddings must be non-trivial. It seems that the results here would offer a more direct argument in certain cases. I thank the Pierre Pansu for discussions on this matter, while we were visiting the Newton Institute in Cambridge as part of the programme “Non-positive curvature group actions and cohomology”.

2. Embeddings of graphs

To simplify the exposition we first give a result in the context of graphs. We discuss variations on this for more general spaces later.

Let G_X and G_Y be connected graphs, with combinatorial metrics, d_X and d_Y, respectively assigning each edge unit length. We suppose that G_X is embedded in G_Y, so that $d_Y \leq d_X$. (We assume nothing more about the embedding.) We will work mostly with the respective vertex sets, denoted $X \subseteq Y$, so that d_X and d_Y take values in $\mathbb{N}$. We suppose that Y is hyperbolic and has bounded geometry (that is, there is a finite bound on the degree of vertices in G_Y). Write ∂Y for the Gromov boundary of G_Y, so that $Y \cup \partial Y$ is a compactification of Y. We assume that X satisfies Property (G) as defined below. We write $\Lambda(X) \subseteq \partial Y$ for the limit set of X.

Theorem 2.1. *If Y is hyperbolic with bounded geometry, and X satisfies Property (G) below, then $\Lambda(X)$ is perfect.*

We write $B_X(.,r)$ and $B_Y(.,r)$ respectively for the the r-neighbourhoods with respect to d_X and d_Y. We write $S_X(.,r)$ and $S_Y(.,r)$ for the r-spheres. Note that if $A \subseteq X$, then $B_X(A,r) \subseteq B_Y(A,r)$.

We will assume the following “growth” condition:

(G): $(\exists \theta > 0,\, \lambda > 1)(\forall h, t \geq 0)(\forall x \in X)(\exists A \subseteq S_X(x, t+h))$ such that $|A| \geq \theta\lambda^h$ and $d_X(y,z) > 2t$ for all distinct $y, z \in A$.

This means that we can find at least $\theta\lambda^h$ disjoint t-balls in $B_X(x, h+2t)$ each of which meets $S_X(x,h)$.

Note, for example, that the 3-regular tree satisfies (G): given any vertex p and any $h \in \mathbb{N}$, the complement of the open h-ball about p has 3.2^{h-1} components. We can choose a point in each such component at a distance $h+t$ from p to give us the set A.

A simple consequence of Property (G) is the following.

(A): $(\exists\theta > 0,\ \lambda > 1)(\forall x \in X,\ r > 0)\ |B_X(x,r)| \geq \theta\lambda^r$.

In fact, this holds for $S_X(x,r)$, since applying Property (G) with $t = 0$, gives $A \subseteq S_X(x,r)$.

We assume Y to have bounded geometry: in other words, there is a bound on the degree of each vertex. Among other things, this implies the following:

(B1): $(\exists\phi > 0,\ \mu > 1)(\forall x \in Y,\ r > 0)\ |B_Y(x,r)| \leq \phi\mu^r$.

(B2): $(\exists\psi > 0, q \geq 0)(\forall p, y \in Y)\ |H(p,y,r)| \leq \psi r^q$.

To define $H(p,y,r)$, set $s = d_Y(p,y)$, and let $H(p,y,r)$ be the set of points, $z \in S_Y(p,s)$, such that y and z are connected by a path, γ in Y, with $d_Y(p,\gamma) \geq s$. (Throughout this section, a "path" will be interpreted as a sequence of adjacent points: that is, the vertex set of a path in the graph G_X or G_Y.)

Property (B1) is an immediate consequence of bounded degree (in any connected graph). Property (B2) is a well known consequence of the exponential divergence of geodesic rays outside balls in a hyperbolic space. In particular, if we take w to be a point at distance about $c \log r$ from y along a geodesic in Y from p to y, then any geodesic from p to $z \in H(p,y,r)$ will pass within a bounded distance of w. Here c is a constant, sufficiently large, depending only on the hyperbolicity constant of Y. Therefore all possible such z lie in an $O(\log r)$-ball about y. By (B1), this gives us a uniform cardinality bound which will be polynomial in r.

For notational convenience, we can suppose that $\mu \geq \lambda$, and set $m = \log\lambda/\log\mu$, so that $\mu = \lambda^m$.

We will also use some other general properties of hyperbolic spaces in the course of the proof.

The proof of Theorem 2.1 is based on the following.

Lemma 2.2. *There is some $r \geq 0$ such that if $p \in Y$ and $x \in X$, then there is some $y \in X$ with $d_Y(x,y) \leq r$ and $d_Y(p,y) = d_Y(p,x) + 1$.*

(In fact, we will show $d_X(x,y) \leq r$.) We begin with:

Lemma 2.3. *There exist $n, h_0 \geq 0$ such that the following holds. Suppose $h \geq h_0$ and let $t = nh$. Suppose that $s \geq 0$ and $p, b \in Y$ with $s \leq d_Y(p,b) \leq s + h$. Then there is some $y \in B_X(b,t)$ with either $d_Y(p,y) < s$ or $d_Y(p,y) > s + h$.*

This can be paraphrased by saying that a very large ball in X cannot be contained in a shell in Y of given (sufficiently large) width, and where "very" means a large enough multiple of this width.

Proof. We choose n and h_0 as described below. Suppose, for contradiction, that every point of $B_X(b,t)$ is at distance between s and $s + h$ from p in Y. Let

$z \in B_X(b,t)$. Let b' and z' be, respectively, nearest points to b and z in $B_Y(p,s)$. Thus, $b', z' \in S_X(p,s)$. Clearly $d_Y(b,b') \leq h$ and $d_Y(z,z') \leq h$. Also, b and z are connected by a path of length at most t in X. It follows that b' and z' are connected by a path, γ, in Y, of length at most $t+2h$ and with $d_Y(p,\gamma) \geq s$. In other words, $z' \in H(p,b',t+2h)$. By (B2), there are at most $\psi(t+2h)^q$ possibilities for z'. Since $z \in B_Y(z',h)$ and $|B_Y(z',h)| \leq \phi\mu^h$, there are at most $\psi(t+2h)^q\phi\mu^h = \psi\phi(n+2)^q h^q \lambda^{mh}$ possibilities for z (given that $t = nh$ and $\mu = \lambda^m$). In other words, $|B_X(b,t)| \leq \psi\phi(n+2)^q h^q \lambda^{mh}$.

But, by Property (A), $|B_X(b,t)| \geq \theta\lambda^t = \theta\lambda^{nh}$. It follows that

$$\theta\lambda^{nh} \leq \psi\phi(n+2)^q h^q \lambda^{mh}.$$

Now, we can choose h_0 such that every $h \geq h_0$ satisfies $h^q \leq \lambda^h$. We then get $\theta\lambda^{nh} \leq \psi\phi(n+2)^q \lambda^{(m+1)h}$. We can assume $h \geq 1$, and so

$$\theta\lambda^{n-m-1} \leq \theta\lambda^{(n-m-1)h} \leq \psi\phi(n+2)^q.$$

This is a contradiction, if n is chosen sufficiently large. (Note that n and h_0 depend only on the constants, $\lambda, \mu, \theta, \phi, \psi, q$, of the hypotheses.) □

Proof of Lemma 2.2. Let n and h_0 be as given by Lemma 2.3. Let $h \geq h_0$ be sufficiently large, as chosen below, and let $t = nh$. Set $r = h + 2t = (1+2n)h$. Let $u = d_Y(p,x)$. We suppose, for contradiction, that $B_X(x,r) \subseteq B_Y(p,u)$.

We can suppose that $u \geq h$ (otherwise we could just take y to be any point of X with $d_Y(p,y) = u+1$ and then the conclusion holds with any $r \geq 2h+1$). Set $s = u - h$ (so that $B_X(x,h+2t) \subseteq B_Y(p,s+h)$). Let $x' \in S_Y(p,s)$ be a nearest point to x. (So, $d_Y(x,x') = h$.)

Let $A \subseteq S_X(x,h+t) \subseteq X$ be as given by Property (G). So, $|A| \geq \theta\lambda^h$.

Suppose $a \in A$. Then $B_X(a,t) \subseteq B_X(x,h+2t) \subseteq B_Y(p,s+h)$. By Lemma 2.3, there is some $y \in B_X(a,t)$ with $d_Y(p,y) < s$ (since $d(p,y) \leq s+h$, the second possibility is ruled out). Since $d_X(x,a) = h+t$, any geodesic from x to a in X lies entirely in $B_X(x,h) \cup B_X(a,t)$. Concatenating with a geodesic from a to y, we get a path of length at most $h+2t$ from x to y in $B_X(x,h) \cup B_X(a,t)$. Let $z(a)$ be the first point along this path with $d(p,z(a)) = s$. Now, $z(a) \in S_X(p,s) \cap B_X(a,t)$ and is connected to x by a path, δ, of length at most $h+2t$ with $d_Y(p,\delta) \geq s$. Concatenating this with a geodesic in Y from x to x', we see that x is connected to $z(a)$ in Y by a path, γ, of length at most $h+(h+2t) = 2h+2t$ with $d_Y(p,\gamma) \geq s$. In other words, $z(a) \in H(p,x',2h+2t)$. By (B2) there are at most $\psi(2h+2t)^q$ possibilities for $z(a)$ in total. But the $z(a)$ for $a \in A$ are all distinct (since the balls $B_X(a,t)$ are all disjoint). Therefore, $|A| \leq \psi(2h+2t)^q$. We get

$$\theta\lambda^h \leq \psi(2h+2t)^q = \psi(2+2n)^q h^q,$$

which is a contradiction, if h is sufficiently large (depending only on the constants of the hypotheses). □

We next recall some general facts about a hyperbolic space such as Y.

We first fix any basepoint, $p \in Y$. Given $x, y \in X$, let

$$\langle x, y \rangle = \langle x, y \rangle_p = \frac{1}{2}(d_Y(p, x) + d_Y(p, y) - d_Y(x, y))$$

be the "Gromov product". We extend this for any distinct $x, y \in Y \cup \partial Y$. (For example, take the limsup of Gromov products over all sequences of points in Y which tend respectively to x and y.)

The following is a well known fact about hyperbolic spaces, [GhH].

There is some $\omega > 1$ and some $k \geq 1$, and a metric, ρ on $Y \cup \partial Y$, such that for all $x, y \in Y \cup \partial Y$, we have:

(1): $\rho(x, y) \leq \omega^{-\langle x,y \rangle}$, and

(2): if $x \in \partial Y$, then $\omega^{-\langle x,y \rangle} \leq k\rho(x, y)$.

Given $x \in Y$, write $\eta(x) = \omega^{-d_Y(p,x)}$.

The following is now a corollary of Lemma 2.2.

Lemma 2.4. *There is some $K > 0$ such that if $x \in X$, then there is some $y \in \Lambda(X)$ with $\rho(x, y) \leq K\eta(x)$.*

Proof. Let $d = d_Y(p, x)$, so $\eta(x) = \omega^{-d}$. Applying Lemma 2.2 inductively, starting at $x_0 = x$, we find a sequence, $(x_i)_{i=0}^{\infty}$, with $x_i \in X$, $d_Y(x_i, x_{i+1}) \leq r$ and $d_Y(p, x_i) = d + i$ for all i. Thus, $\langle x_i, x_{i+i} \rangle \geq d_Y(p, x_i) - r \geq d + i - r$, and so

$$\rho(x_i, x_{i+1}) \leq \omega^{-\langle x_i,x_{i+i} \rangle} \leq \omega^{-(d+i-r)} = \omega^{r-d}\omega^{-i}.$$

Therefore, $\rho(x, x_i) \leq \omega^{r-d} \sum_{i=0}^{\infty} \omega^{-i} = \frac{\omega^r}{1-\omega}\omega^{-d} = K\eta(x)$, where $K = \frac{\omega^r}{1-\omega}$.

Now some subsequence of x_i converges to some $y \in \Lambda(X)$ with $\rho(x, y) \leq K\eta(x)$. □

Now let $a \in \partial Y$ and let α be any geodesic ray from p to a. (Such must exist.) The following lemma is a general fact about hyperbolic spaces.

Lemma 2.5. *For all $L \geq 0$, there is some $R = R(L) \geq 0$ such that if $x \in Y$ with $d_Y(x, \alpha) \geq R$, then $\rho(a, x) \geq L\eta(x)$.*

Proof. Let $l = \langle a, x \rangle$. By hyperbolicity, there is some constant $D \geq 0$ such that $\langle a, x \rangle \leq d_Y(p, x) - d_Y(\alpha, x) + D \leq d_Y(p, x) - R + D$, and so $d_Y(p, x) \geq l + R - D$. Now $k\rho(a, x) \geq \omega^{-l}$ and $\eta(x) = \omega^{-d_Y(p,x)} \leq \omega^{-(l+R-D)} \leq k\rho(a, x)\omega^{-(R-D)}$, so $\rho(a, x) \geq \frac{1}{k}\omega^{R-D}\eta(x) \geq L\eta(x)$, provided that R is chosen so that $\omega^{R-D} \geq kL$. □

Lemma 2.6. *For all $R \geq 0$, there is some $T = T(R) \geq 0$ such that if $\alpha \subseteq Y$ is any geodesic, and $x \in X$, then there is some $y \in X \setminus B_Y(\alpha, R)$ with $d_X(x, y) \leq T$.*

Proof. Choose T sufficiently large, as described below. Suppose the conclusion fails; that is, $B_X(x, T) \subseteq B_Y(\alpha, R)$. Now $B_X(x, T) \subseteq B_Y(x, T)$, which projects under nearest-point projection to a segment, β, of α of length at most $2T + D$, where D depends only on the hyperbolicity constant. In particular, $B_X(x,T) \subseteq B_Y(\beta,R)$. Now $|B_Y(\beta,R)| \leq (2T+D+1)\phi\mu^R$. But $|B_X(x,T)| \geq \theta\lambda^T$, and so $\theta\lambda^T \leq (2T+D+1)\phi\mu^R$, which is a contradiction if T is chosen large enough in relation to R. □

Proof of Theorem 2.1. Let $a \in \Lambda(X)$ and let α be a geodesic ray from p to a in Y. There is a sequence $(x_i)_i$ in X with $x_i \to a$. Let K be the constant of Lemma 2.4. Let $L = K+1$, let $R = R(L)$ be as given by Lemma 2.5, and let $T = T(R)$ be as given by Lemma 2.6.

Now we can assume that $d_Y(x_i,\alpha) > T$ for all i (otherwise, by Lemma 2.6, we could replace x_i by some other point of X a bounded distance from x_i, and this new sequence also tends to a).

By Lemma 2.5, $\rho(a,x_i) \geq L\eta(x_i)$. By Lemma 2.4, there is some $y_i \in \Lambda(X)$ with $\rho(x_i,y_i) \leq K\eta(x_i) < \rho(a,x_i)$. In particular, $y_i \neq a$. Also, $\rho(a,y_i) \leq \rho(a,x_i) + \rho(x_i,y_i) \leq (1+K)\rho(a,x_i)$, so $y_i \to a$.

We have shown that a is not isolated in $\Lambda(X)$. In other words, $\Lambda(X)$ is perfect. □

An elaboration on this argument shows that $\Lambda(X)$ is uniformly perfect with respect to the metric, ρ. In fact, if $a \in \Lambda(X)$ and $\nu > 1$ then there is some $\epsilon > 0$ such that if $\delta \leq \epsilon$, then there is some $b \in \Lambda(X)$ with $\delta \leq \rho(a,b) \leq \nu\delta$.

Note that (as a corollary, or by the same argument) one can deduce the same thing for a map $f : X \longrightarrow Y$ which is bounded-to-one, and where $\Lambda(f(X)) \subseteq \partial Y$ is the limit set of $f(X) \subseteq Y$. More precisely, $f : X \longrightarrow Y$ is the restriction of a simplicial map, $f : G_X \longrightarrow G_Y$, and $|f^{-1}(y)|$ is bounded for all $y \in T$.

Note that the closure of any union of perfect subsets of a topological space is perfect. Therefore, it is enough that G_X should be a union of subgraphs, each of which satisfies (G) intrinsically. Indeed it's enough to find a family, $(X_j)_j$, of graphs satisfying (G) together with simplical maps, $g_j : G_{X_j} \longrightarrow X$, such that for all j, $|X \cap G_j^{-1}(y)|$ bounded for all $y \in X$, and with $\bigcup_j g_j G_{X_j}$ cobounded in X. (The parameters and bounds may depend on j.)

3. Consequences for more general spaces

We briefly describe how this result can applied to more general spaces, by reducing them to the case of graphs. We first recall some standard notions.

Let (M,d_M) and (N,d_N) be geodesic metric spaces. A map $f : M \longrightarrow N$ is *coarsely lipschitz* if there is a function $F_1 : [0,\infty) \longrightarrow [0,\infty)$, such that for all $x,y \in M$, $d_N(f(x),f(y)) \leq F_1(d_M(x,y))$. (It is readily checked that one can always take the function F_1 to be linear.) A map $f : M \longrightarrow N$ is a *coarse embedding* (or a *uniform embedding*) if it is coarsely lipschitz, and there is a function $F_2 : [0,\infty) \longrightarrow [0,\infty)$, such that for all $x,y \in M$, $d_M(x,y) \leq F_2(d_N(f(x),f(y)))$. We say that f is a *quasi-isometric embedding* if, in the above, we can take F_2 to be linear. We say that f is a *quasi-isometry* if, in addition, $f(M)$ is cobounded in N. We say that M, N are *quasi-isometric* if there is a quasi-isometry between them. (In the above, we do not assume that f is continuous, nor in general that an "embedding" be injective.)

It is well known, and not hard to see, that any geodesic space, M, is quasi-isometric to a graph. We write $G(M)$ for some such graph. If $f : M \longrightarrow N$, is coarsely lipschitz, then up to bounded distance, it agrees (via the given quasi-isometries) with a map $G(f) : G(M) \longrightarrow G(N)$ which sends vertices to vertices, and each edge to a geodesic of bounded length. In fact, after subdividing the edges of $G(M)$, we may as well take $G(f)$ to be simplicial: that is, it sends every edge of $G(M)$ to a vertex or edge of $G(N)$. If N is hyperbolic, then so is $G(N)$, and we can identify ∂N with $\partial G(N)$. Moreover, $\Lambda(f(M))$ is identified with $\Lambda((G(f)(G(M))))$. In particular, they are homeomorphic.

We have a combinatorial criterion for bounded geometry. One can show that M is quasi-isometric to a locally finite graph if and only if there is some $r > 0$ such that every r-separated subset, $W \subseteq M$, is locally finite. (A subset, W, is *r-separated* if $d_M(x, y) \geq r$ for all distinct $x, y \in M$. It is *locally finite* if every bounded subset is finite.) Moreover, M is quasi-isometric to a bounded geometry graph if and only if for every sufficiently separated W, the cardinality of every bounded subset is bounded above by some function of its diameter. In this case, we say that M itself has *bounded geometry.*

Definition. We say that a geodesic space, M, has *fast growth* if every point of M is a bounded distance from the image coarsely embedded 3-regular tree.

(Here the parameters of the coarse embeddings need not be uniform.)

Proposition 3.1. *Let M be a geodesic space of fast growth, let N be a bounded geometry hyperbolic space, and let $f : M \longrightarrow N$ be a coarse embedding. Then the limit set of $f(M)$ in ∂N is perfect.*

Proof. Since compositions of coarse embeddings are coarse embeddings, and since the closure of a union of perfect sets is perfect, we can reduce to the case where $M = G(M)$ is the 3-regular tree. Up to quasi-isometry, we can replace f by the map $G(f) : G(M) \longrightarrow G(N)$. Since this is a coarse embedding, it is easily seen that this if bounded-to-one. So we can now apply the observation at the end of Section 2. □

Examples of spaces of spaces with fast growth are "bushy" hyperbolic spaces:

Definition. We say that a hyperbolic space, M, is *bushy* if there is some $k \geq 0$ such that for all $p \in X$, there exist $x, y, z \in \partial M$ such that $\langle x, y \rangle_p \leq k$, $\langle y, z \rangle_p \leq k$ and $\langle z, x \rangle_p \leq k$.

This can be paraphrased by saying that every point is (a bounded distance) from the centre of an ideal (quasi)geodesic triangle.

It's not hard to see that bushiness is equivalent to asserting that (for some k) for all $p \in M$ and all $r \geq 0$, we can always find $x, y, z \in S(p, r) \subseteq M$ with $\langle x, y \rangle_p \leq k$,

$\langle y,z\rangle_p \leq k$ and $\langle z,x\rangle_p \leq k$. Using hyperbolicity, this is equivalent to asserting that for some $k \geq 0$, for all $w,p \in M$ and all $r \geq 0$, there exist $x,y \in S(p,r)$ with $\langle x,y\rangle_p \leq k$, $\langle w,x\rangle_p \leq k$ and $\langle w,y\rangle_p \leq k$.

Lemma 3.2. *If M is a bushy hyperbolic space, then every point of M lies in a uniformly quasi-isometrically embedded tree.*

Proof. Choose r sufficiently large in relation to the hyperbolicity constant, as described below. Let $p \in M$. Let $x,y,z \in S(p,r) \subseteq M$ be as given above. Now, choose $x_0, x_1 \in S(x,r)$ so that $\langle x_0,x_1\rangle_x$, $\langle p,x_0\rangle_x$, $\langle p,x_1\rangle_x$ are all bounded. Similarly choose, y_0,y_1,z_0,z_1. Now choose $x_{01}, x_{01} \in S(x_0,r)$ with $\langle x_{00},x_{01}\rangle_{x_0}$, $\langle x,x_{00}\rangle_{x_0}$, $\langle x,x_{01}\rangle_{x_0}$. Similarly we get x_{10}, x_{11} etc. Continuing outwards, this gives us a map of the vertex set $V(T)$ of the 3-regular tree, T, into X. We extend to T by mapping each edge to a geodesic (of length r). To see that this is a quasi-isometric embedding, we just need to note that arcs in T are sent to quasigeodesics. This is a consquence of the fact that this is a local property, in the following sense. Suppose that $(w_i)_i$ is a sequence of points in M with $d_M(w_i,w_{i+1}) \geq r$ and $\langle w_{i-1},w_{i+1}\rangle_{w_i} \leq k$ for all i, then $(w_i)_i$ is a quasigeodesic, provided r is chosen large enough in relation to k and the hyperbolicity constant. (See for example, [GhH].) By construction this holds for the vertices in any path in T. □

In particular this shows that a bushy hyperbolic space has fast growth. As as an immediate consequence of Lemma 3.2 and Proposition 3.1, we get:

Proposition 3.3. *Let M be a bushy hyperbolic space, let N be a bounded geometry hyperbolic space, and let $f : M \longrightarrow N$ be a coarse embedding. Then the limit set of $f(M)$ in ∂N is perfect.*

As observed in Section 1, this applies if M and N are non-elementary hyperbolic groups, or pinched Hadamard manifolds of dimension at least 2.

The fact that M has bounded geometry is essential. For example, one can easily construct Hadamard manifolds with curvature bounded away from 0, containing uniformly embedded horospheres which also have negative curvature bounded away from 0.

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