A short proof that a subquadratic isoperimetric inequality implies a linear one.

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In this paper, we give a short proof of the result that path-metric space satisfying a subquadratic isoperimetric inequality must, in fact, satisfy a linear isoperimetric inequality, and is therefore hyperbolic in the sense of Gromov. This result was announced by Gromov in his original paper [G] on hyperbolic groups. A detailed proof was supplied by Ol'shanskii [O], in the context of combinatorial group theory. Somewhat shorter arguments have been given by Papasoglu (see [P1,P2]). This result is important for some applications—see for example [BeF].

The argument of this paper works for any path-metric space on which has been defined some notion of area satisfying two modest axioms, (A1) and (A2) described below. Note, in particular, that there is no need to assume any "quasihomogeneity", for example that the space be quasiisometric to the Cayley graph of a group. It is easy to construct complete riemannian metrics on the plane for which the isoperimetric function is asymptotic to $[x \mapsto x^t]$ for any real $t \geq 2$. On the other hand, all the most obvious groups satisfying an isoperimetric inequality which is $o(x^{n+1})$, for some natural number n, in fact satisfy one which is $O(x^n)$. However Rips and Sapir recently claimed an example of a finitely presented group for which the isoperimetric function is asymptotic to $x^2 \log x$. Since then, Bridson has produced examples where the isoperimetric function is asymptotic to x^r for certain rational non-integer values of $r > 2$ [Br]. An interesting question remaining is to describe the class of functions that can arise in this way.

For convenience, we shall assume that (X, d) is a length-space, i.e. any two points $x, y \in X$ are joined by a geodesic α, so that length $\alpha = d(x, y)$. We write $[x \to y]$ for some choice of geodesic from x to y. We write $[x, y] = \text{image}[x \to y] \subseteq X$. The arguments given here will work in a general path-metric space, with slight modification.

By a *loop* in X, we mean a rectifiable map $\gamma : S^1 \longrightarrow X$. We write $L(\gamma)$ for the length of γ. Suppose $x, y \in X$, and $\alpha_1, \alpha_2, \alpha_3$ are three rectifiable paths from x to y. We may form three loops $\gamma_1, \gamma_2, \gamma_3$ by $\gamma_i = \alpha_{i+1} \cup -\alpha_{i+2}$ (taking subscripts mod 3). Whenever three loops arise in this way, we say that they form a *theta-curve*.

As a particular instance, consider a loop $\gamma : S^1 \longrightarrow X$, and two points $t, u \in S^1$ which cut γ into two paths α_1 and α_2. Setting $\alpha_3 = [\gamma(t) \to \gamma(u)]$, we obtain a theta-curve $(\gamma_1, \gamma_2, \gamma_3)$, with $\gamma = \gamma_3$ and $L(\gamma_i) \leq L(\gamma)$ for $i = 1, 2$. We have thus cut γ into two smaller pieces.

Now, let Ω be a set of loops in X, closed under the above operations of cutting in two by geodesic segments. Let us suppose we have a map $A : \Omega \longrightarrow [0, \infty)$, satisfying the following two axioms:

(A1) (Triangle inequality for theta curves): If $\gamma_1, \gamma_2, \gamma_3 \in \Omega$ form a theta-curve, then $A(\gamma_3) \leq A(\gamma_1) + A(\gamma_2)$.

(A2) (Rectangle inequality): Suppose $\gamma \in \Omega$ is split into four subpaths, $\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$. Then, $A(\gamma) \geq d_1 d_2$, where $d_1 = d(\text{image}\,\alpha_1, \text{image}\,\alpha_3)$ and $d_2 = d(\text{image}\,\alpha_2, \text{image}\,\alpha_4)$.

Thus $A(\gamma)$ is thought of as the minimal area of a disc spanning γ. For example, if X is a riemannian manifold, we simply take riemannian area. Axiom (A2) then follows from Almgren's Coarea Formula (see [S]). For groups, we may interpret area as the number conjugates of relators needed to reduce a word representing the identity to the trivial word. (In this case, we may need to normalise by a multiplicative constant so that Axiom (A2) is satisfied.) Various notions of area for a general path-metric space are discussed in [Bo].

Given $x \in [0, \infty)$, let

$$f(x) = \sup\{A(\gamma) \mid \gamma \in \Omega, L(\gamma) \leq x\}.$$

Thus $f : [0, \infty) \longrightarrow [0, \infty)$ is the "isoperimetric function". Note that $f(x)$ is non-decreasing in x. Moreover:

Lemma 1 : *For all $x \in [0, \infty)$, there exist $p, q \in [0, \infty)$ such that the following hold:*

$$\begin{aligned} f(x) &\leq f(p) + f(q) \\ p, q &\leq \frac{3}{4}x + 3\sqrt{f(x)} \\ p + q &\leq x + 6\sqrt{f(x)}. \end{aligned}$$

Proof : Let $x \in [0, \infty)$. Suppose, for simplicity, that the supremum is attained, so that there is some $\gamma \in \Omega$, with $L(\gamma) = L \leq x$, and $A(\gamma) = f(x)$. (The general case, where the supremum is not attained, only introduces slight technical complication, which can be dealt with, for example, by considering appropriate sequences tending to the supremum.)

Let $\Delta \subseteq S^1 \times S^1$ be the set of pairs (t, u) such that $\gamma(t)$ and $\gamma(u)$ cut γ into two paths each of length at most $\frac{3}{4}L$. Let $l = \min\{d(\gamma(t), \gamma(u)) \mid t, u \in \Delta\}$. Choose $a = \gamma(t_0)$ and $b = \gamma(u_0)$ so that $(t_0, u_0) \in \Delta$ and $d(a, b) = l$. Thus, a, b cut γ into two paths β_0 and β_1. Without loss of generality, $L(\beta_1) \leq L(\beta_0) \leq \frac{3}{4}L$.

Now, take $a', b' \in [a, b]$ so as to cut $[a \to b]$ into three equal segments, $\alpha_1 = [a \to a']$, $\delta = [a' \to b']$ and $\alpha_2 = [b' \to b]$, each of length $l/3$. (See Figure.)

FIGURE

Clearly, $d(\text{image}\,\alpha_1, \text{image}\,\alpha_2) = l/3$. We claim that $d(\text{image}\,\delta, \text{image}\,\beta_0) \geq l/3$.

To see this, suppose $z = \gamma(v) \in \text{image}\,\beta_0$ and $z' \in \text{image}\,\delta$. Thus, z cuts β_0 into two subpaths β_2, β_3, so that $a \in \text{image}\,\beta_2$ and $b \in \text{image}\,\beta_3$. Without loss of generality, $L(\beta_2) \leq L(\beta_3)$. Let $y_i = L(\beta_i)$ for $i = 1, 2, 3$. Thus, $y_1 + y_2 + y_3 = L$, $y_1 \leq y_2 + y_3$ and $y_2 \leq y_3$. It follows that $y_1 \leq \frac{1}{2}L$ and $y_2 \leq \frac{1}{2}(L - y_1)$ and so $y_1 + y_2 \leq \frac{3}{4}L$. Since also $y_3 \leq \frac{3}{4}L$, we see that $(u_0, v) \in \Delta$. Thus $d(z, b) \geq l = d(a, b)$, and so $d(z, z') \geq d(a, z') \geq d(a, a') = l/3$. This proves the claim.

Now, the geodesic $[a, b]$ cuts γ into two loops γ_1 and γ_2 so that $\gamma_1 = \beta_0 \cup \alpha_1 \cup \delta \cup \alpha_2$. Let $p = L(\gamma_1)$ and $q = L(\gamma_2)$, so $p, q \leq x$. Axiom (A2) tells us that $A(\gamma_1) \geq (l/3)^2$, and Axiom (A1) tells us that $f(x) = A(\gamma) \leq A(\gamma_1) + A(\gamma_2) \leq f(p) + f(q)$. Now, $l \leq 3\sqrt{A(\gamma_1)} \leq 3\sqrt{f(p)} \leq 3\sqrt{f(x)}$. Thus, $p, q \leq \frac{3}{4}L + l \leq \frac{3}{4}x + \sqrt{f(x)}$ and $p + q \leq L + 2l \leq x + 6\sqrt{f(x)}$. $\diamondsuit$

Lemma 2 : *Suppose $f : [0,\infty) \longrightarrow [0,\infty)$ is an increasing function. Suppose there are constants $K > 0$, and $0 < \lambda < 1$, so that for all $x \in [0,\infty)$ there exist $p, q \in [0,\infty)$ with*

$$\begin{aligned} f(x) &\leq f(p) + f(q) \\ p, q &\leq \lambda x + K\sqrt{f(x)} \\ p + q &\leq x + K\sqrt{f(x)}. \end{aligned}$$

If $f(x) = o(x^2)$, then $f(x) = O(x)$.

Proof : After multiplying f by a constant, we may as well assume that $K = 1$. Let $\mu = \frac{1+\lambda}{2}$. There is some $x_0 \in [0,\infty)$ such that if $x > x_0$, then $f(x) \leq (1-\mu)^2 x^2$. Thus, if $x > x_0$, we have $p, q \leq \lambda x + (1-\mu)x = \mu x$. Also, we can always assume that $p, q \geq 1$ (say), by taking $x_0 \geq \max\{2, 1/\lambda, 1/(1-\mu)\}$.

For $x \geq 1$, set $g(x) = \frac{f(x)}{x}$. Thus, if $x > x_0$, we have

$$g(x) \leq \left(\frac{p}{x}\right) g(p) + \left(\frac{q}{x}\right) g(q).$$

Without loss of generality, $g(q) \leq g(p)$, and so

$$g(x) \leq \left(\frac{p+q}{x}\right) g(p) \leq \left(\frac{x + \sqrt{f(x)}}{x}\right) g(p) = \left(1 + \sqrt{\frac{g(x)}{x}}\right) g(p).$$

In other words, if $x > x_0$, there exists p with $1 \leq p \leq \mu x$ and $g(x) \leq \left(1 + \sqrt{\frac{g(x)}{x}}\right) g(p)$. We want to show that if $g(x) = o(x)$, then g is bounded.

Given $\epsilon > 0$, we can find $x_1 \geq x_0$ so that $g(x) \leq \epsilon^2 x$ for all $x > x_1$. Let $B = \max\{g(x) \mid 1 \leq x \leq x_1\}$. Thus if $x > x_1$, we have $g(x) \leq (1+\epsilon)g(p)$ with $1 \leq p \leq \mu x$. After iterating at most $n \leq 1 + \frac{\log(x/x_1)}{\log(1/\mu)}$ times, we get

$$g(x) \leq B(1+\epsilon)^n \leq B(1+\epsilon)(x/x_1)^r,$$

where $r = \frac{\log(1+\epsilon)}{\log(1/\mu)}$. We can choose ϵ small enough so that $r < 1$.

Let $0 < s < \frac{1}{2}(1-r)$, so that $r < 1 - 2s$. We have $g(x) = o(x^{1-2s})$, and so we can assume that $g(x) \leq x^{1-2s}$ for all $x > x_2 \geq x_1$. Thus, if $x > x_2$, we get

$$g(x) \leq (1 + x^{-s})g(p).$$

After iterating some number k times, we arrive at

$$g(x) \leq C(1 + x^{-s})(1 + \mu^{-s}x^{-s}) \cdots (1 + \mu^{-ks}x^{-s})$$

where $C = \max\{g(y) \mid 1 \leq y \leq x_2\}$, and $\mu^k x > x_2$. Thus,

$$\begin{aligned} \log g(x) &\leq \log C + x^{-s}(1 + \mu^{-s} + \cdots + \mu^{-ks}) \\ &= \log C + \frac{(\mu^k x)^{-s} - \mu^s x^{-s}}{1 - \mu^s} \\ &\leq \log C + \frac{x_2^{-s}}{1 - \mu^s}, \end{aligned}$$

and so g is bounded. $\diamondsuit$

Lemmas 1 and 2 together now give the result. In fact, the argument effectively shows that a quadratic isoperimetric inequality with sufficiently small multiplicative constant implies a linear one — as was observed by Gromov.

Note that, provided Ω is sufficiently large, e.g. if it consists of all rectifiable loops, and if (X, d) has a linear isoperimetic inequality with some notion of area satisfying axiom (A2), then (X, d) is Gromov hyperbolic (see, for example, [Bo]).

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