Singular Euclidean Structures on Surfaces

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1. Introduction.

By a *singular euclidean structure* on a compact surface, we mean a metric structure locally modelled on the euclidean plane, except at a finite number of points, where we may have "cone singularities" (Figure 1). Such singularities may be thought of as local concentrations of curvature. The cone angle is allowed to take any positive value. A non-singular point can be regarded as a cone point of angle 2π.

A typical singular euclidean surface is obtained by gluing together euclidean triangles along sides of the same length. Of course, different triangulations might give rise to the same structure.

Such structures exist on a surface S of genus g provided the Euler characteristic $\chi(S \setminus P)$ is negative, where P is a finite set of cone points. We shall restrict attention to the cases when $P \neq \emptyset$, i.e. we are disallowing euclidean tori with no cone points. We write $p = |P|$. We regard two such metrics as equivalent if one is a scalar multiple of the other. That is to say we are working with euclidean similarity structures which have an underlying euclidean metric. We may always normalise so that, for example, the area of each surface is equal to 1. We denote by $\mathcal{S}_g^p$ the space of marked singular euclidean structures quotiented out by scaling. *Marked* means that two structures on $S \setminus P$ are regarded as equal if they are related by a similarity which is isotopic to the identity rel P.

The aim of this paper is to give a cellulation of the space $\mathcal{S}_g^p$ by convex euclidean polyhedra.

If (S, P) is a singular euclidean surface, then the sum of the cone angles over each point y of P must be equal to $-2\pi\chi$, where $\chi = \chi(S \setminus P) = 2 - 2g - p$. If $\mathrm{int}\Delta^{p-1}$ is the simplex of maps $\{x : P \to (0,1) \mid \sum_{y \in P} x(y) = 1\}$, then we may define a projection $\lambda : \mathcal{S}_g^p \longrightarrow \mathrm{int}\Delta^{p-1}$ by

$$\lambda(S,P)(y) = \frac{1}{(-2\pi\chi)} \times \text{cone angle of } S \text{ at } y \in P.$$

Our cellulation has the property that for each $x \in \mathrm{int}\Delta^{p-1}$, $\lambda^{-1}x$ intersects each polyhedral cell in an affine subspace.

In [EP], a construction is given to associate, to a non-compact, finite-volume hyperbolic manifold, a singular euclidean structure, which, in the case of surfaces, coincides with our notion. The ideas are geometric, and involve a convex hull construction in Minkowski space. It is not difficult to see that their construction can be used to establish a homeomorphism between $\mathcal{S}_g^p$ and $\mathrm{int}\mathcal{T}_g^p = T_g^p \times \mathrm{int}\Delta^{p-1}$, where T_g^p is the Teichmüller space of finite-area complete hyperbolic structures on $S \setminus P$.

An analytic approach to this is to define a map $h : \mathcal{S}_g^p \longrightarrow T_g^p \times \mathrm{int}\Delta^{p-1}$, by taking λ as the second coordinate, and the conformal structure induced from the euclidean metric as the first. Troyanov [Tr] shows that h is bijective.

In Section 6, we show that we may obtain, geometrically, a cell decomposition of $\mathrm{int}\mathcal{T}_g^p$ which is identical to that of $\mathcal{S}_g^p$. This establishes a third homeomorphism between these two spaces. This homeomorphism identifies the map $\lambda : \mathcal{S}_g^p \longrightarrow \mathrm{int}\Delta^{p-1}$ with projection onto the second coordinate of $\mathrm{int}\mathcal{T}_g^p$.

All the above constructions are natural in the sense that they commute with the action of the mapping class group of $S \setminus P$.

The cellulation of $\mathcal{S}_g^p$ can be thought of as a PL subset of the complex of arc-systems described by Harer [H] and later in [BE]. That is to say, our complex is obtained by removing pieces from each simplex in Harer's complex, so as to make a complex of polyhedral cells.

The ideas of this paper are related to those of [BE]. We shall use similar notation and terminology. Where arguments are only outlined here, more details can be found in that paper.

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2. The Spine and Delaunay Triangulation.

Let (S, P) be a singular euclidean surface.

Given $u \in S \setminus P$, let $w(u)$ be the number of distinct shortest paths to P. Clearly, each of these shortest paths must be a euclidean line segment whose interior lies in $S \setminus P$. Let $\Sigma = \{u \in S \setminus P \mid w(u) \geq 2\}$ and $V = \{u \in S \setminus P \mid w(u) \geq 3\}$.

Lemma. *V is a finite set of points. $\Sigma \setminus V$ is a finite collection $E(\Sigma)$ of open euclidean line segments. Each point of V is the endpoint of at least three directed line segments of $E(\Sigma)$. (The two endpoints of an undirected line segment are regarded as the heads of two distinct directed line segments.) There is a natural deformation retraction of $S \setminus P$ onto Σ.*

Proof : This is a simple exercise, c.f. [BE, Lemma 2.2.1]. $\diamondsuit$

We call Σ the *spine* of S. It has the structure of a graph, with edges $E(\Sigma)$, and vertices $V(\Sigma) = V$. We may extend this graph naturally to a triangulation of S by adding *ribs*, i.e. for each vertex v of $V(\Sigma)$, we add in the shortest line segments joining v to P.

This *spinal triangulation* is characterised by the following properties.

(i) It is a triangulation of S by euclidean triangles.

(ii) If T is a closed triangle of the triangulation, then $T \cap P$ consists of a single vertex of T.

(iii) Each vertex in $S \setminus P$ has at least six incident edges.

(iv) If e is an edge which does not terminate in P, then the two triangles on opposite sides of e are congruent via reflection in e.

(c.f. [BE, Theorem 5.1].)

Associated to each edge e of $E(\Sigma)$, there is a topological arc joining two points of P, and meeting Σ only in e at a single transverse intersection. Its homotopy class in $S \operatorname{rel} P$ is well defined, and it has a unique realisation, e^*, as a straight line segment with interior in $S \setminus P$.

The lines $\{e^* \mid e \in E(\Sigma)\}$ constitute the 1-skeleton for a cell decomposition of S, with 0-skeleton equal to P. The cells are in bijective correspondence to the the vertices of Σ. Each cell is a cyclic polygon, since its corners are all equidistant from the corresponding vertex of Σ. Generically, when each vertex of Σ has degree 3, the cells are all triangles. In this case, we get the *Delaunay triangulation* of S. (The idea originates in [D].) It has the property that each triangle T is inscribed in a closed euclidean disc D, with $D \cap P = \partial D \cap T$. (The interior of D is imbedded in the universal cover of $S \setminus P$, though it may just be immersed in $S \setminus P$.)

3. Coordinates.

First, we introduce the following notation, to be used throughout the rest of this paper.

Given any finite set X, and any $Y \subseteq \mathrm{R}$, we denote by $\operatorname{Map}(X,Y)$, the set of maps from X into Y. Given $k \in \mathrm{R}$, we write

$$\operatorname{Map}(X,Y;k) = \{f \in \operatorname{Map}(X,Y) \mid \sum_{x \in X} f(x) = k\}.$$

Thus, $\operatorname{int}\Delta^{p-1} = \operatorname{Map}\big(P,(0,1);1\big)$.

Now, to each singular euclidean surface, (S,P), with spine Σ, we will associate a map

$$\gamma \in \operatorname{Map}\big(E(\Sigma),(0,\infty)\big)$$

as follows.

For each edge $e \in E(\Sigma)$, there are two triangles, T_1 and T_2, of the spinal triangulation, abutting along e. The triangles are congruent with respect to reflection in e. Thus, we may define $\gamma(e)$ to be the angle opposite e in either one of the triangles T_i.

If Σ is generic, we may define γ in terms of the Delaunay triangulation. Let Q_1 and Q_2 be the two triangles meeting along e^*. Let α_i be the angle opposite e^* in Q_i. Then $\gamma = \pi - (\alpha_1 + \alpha_2)$ (Figure 2).

If Σ is not generic, Q_1 and Q_2 may be cyclic polygons. In this case, α_i is the angle subtended by e^* at any corner of Q_i, other than the two endpoints of e^*. Then, $2\alpha_i$ is the angle subtended at the circumcentre of Q_i. Again $\gamma = \pi - (\alpha_1 + \alpha_2)$.

Note that $\gamma(e) \in (0,\pi)$ for all $e \in E$.

Now, the spinal triangulation has precisely $2|E|$ triangles, so their angles together sum to $2\pi|E|$. Of this sum, $2\pi|V|$ is contributed by the angles around the points of V, and $2\sum_{e \in E} \gamma(e)$ is contributed by the angles around points of P. We thus have

$$2\sum_{e \in E} \gamma(e) + 2\pi|V| = 2\pi|E|.$$

So,

$$\sum_{e \in E} \gamma(e) = \pi(|E| - |V|) = -\pi\chi.$$

Let $W \subseteq V$ be any non-empty proper subset. Let $E_W \subseteq E$ be the set of edges with at least one endpoint in W. By summing all the angles around the vertices of W, we see that

$$\sum_{e \in E_W} 2\big(\pi - \gamma(e)\big) > 2\pi|W|,$$

or

$$\sum_{e \in E_W} \gamma(e) < \pi(|E_W| - |W|).$$

Thus, we see that the set of possible values of γ lies inside a certain convex subset of $\mathrm{Map}\big(E(\Sigma), (0,\infty), -\pi\chi\big)$.

4. The Cell Complex.

Let S_T be a fixed topological surface of genus g. Let $P_T \subseteq S_T$ be a non-empty finite set of points. A *marked* singular euclidean structure is a homeomorphism $(S_T, P_T) \longrightarrow (S, P)$ defined up to isotopy rel P_T, where (S, P) is a singular euclidean surface. Let $\mathcal{S}_g^p$ be the space of such structures defined up to scale.

We shall describe the cellulation of $\mathcal{S}_g^p$ as a PL subset of the simplicial complex Harer calls A (see [H]). It was subsequently described in [BE]. Briefly, a vertex of A corresponds to a non-trivial homotopy class of arcs in S, with endpoints in P. A k-simplex σ of A is then a disjoint union of such arcs in different homotopy classes.

If σ cuts the surface into discs, then σ is dual to a *spine* Σ on S_T, i.e. $S_T \setminus P_T$ deformation retracts onto Σ. We have a natural correspondence $E(\Sigma) \equiv E(\sigma)$ (the set of arcs of σ) where corresponding arcs intersect once transversely. Suppose σ and τ cut the surface into discs, and that τ is a face of σ (i.e. $\tau \subseteq \sigma$). Let Σ and T be the dual spines to σ and τ respectively. There is a homotopy equivalence $j : \Sigma \longrightarrow T$, with $j^{-1}(v)$ being a tree $F_v \subseteq \Sigma$, for each $v \in V(T)$. The inverse of an edge is always a single edge. We may identify $E(\Sigma) \equiv E(T) \cup E(F)$, where $F = \bigcup_{v \in V(T)} F_v$. The resulting inclusion $E(T) \hookrightarrow E(\Sigma)$ is the same as the inclusion $E(\tau) \subseteq E(\sigma)$.

Realising each $\sigma \in A$ as $|\sigma| = \mathrm{Map}\big(E(\sigma), (0,\infty); 1\big)$ gives us a simplicial complex $|A|$. For each $\sigma \in A$, we define $C(\sigma)$ as follows.

If σ is not dual to a spine (some component of $S \setminus \bigcup \sigma$ is not contractible) then $C(\sigma) = \emptyset$.

If σ is dual to Σ, let $C(\sigma)$ be the set of $\theta \in \mathrm{Map}\big(E(\Sigma), (0,\infty); 1\big)$ satisfying

(i) $\theta(e) < 1/(|E| - |V|)$ for all $e \in E$, and

(ii) $\sum_{e \in E_W} \theta(e) < (|E_W| - |W|)/(|E| - |V|)$, whenever $\emptyset \neq W \neq V$.

(Note $|E| - |V| = -\chi$.)

We see that $C(\sigma)$ is an open subset of $\mathrm{int}|\sigma|$. Let $C(A) = \bigcup_{\sigma \in A} C(\sigma)$.

Proposition. *$C(A)$ is an open subset of $|A|$.*

Proof : Let $\theta \in C(\tau)$. Since $C(\tau) \neq \emptyset$, τ is dual to some spine T. If τ is a face of some $\sigma \in A$, then σ is obtained by adding finitely many additional arcs. There are a finite number of possibilities for these arcs. Hence, τ is a face of only finitely many simplices of A. So it is enough to show that for any simplex σ with $\tau \subseteq \sigma$, all points of $\mathrm{int}|\sigma|$ sufficiently near θ lie in $C(\sigma)$.

We now fix σ with $\tau \subseteq \sigma$. Let T and Σ be the respective dual spines. We have $T = j(\Sigma)$, and $E(\Sigma) \equiv E(T) \sqcup E(F)$, where j collapses Σ along the "forest" F. We may write $V(\Sigma) = V(F) = \bigcup_{v \in V(T)} V(F_v)$.

Given $W \subseteq V(\Sigma)$, let $W' = \{v \in V(T) \,|\, V(F_v) \subseteq W\}$, i.e. $v \in W'$ means that all the vertices of the corresponding tree lie in W. We have $E_W \setminus E(F) \equiv j(E_W) \supseteq E_{W'}$. We claim that

$$|E_W \setminus E(F)| - |W'| \leq |E_W| - |W|,$$

or equivalently,

$$|E_W \cap E(F)| \geq |W| - |W'|.$$

We have $|E_W \cap E(F)| = \sum_{v \in V(T)} |E_W \cap E(F_v)|$, and $|W| - |W'| = \sum_{v \in V(T)} (|W \cap V(F_v)| - \delta(v))$, where $\delta(v) = 0$ if $v \notin W'$, and $\delta(v) = 1$ if $v \in W'$. Now, if $v \in W'$, then $V(F_v) \subseteq W$, and so $E(F_v) \subseteq E_W$. In this case, $|E_W \cap E(F_v)| = |E(F_v)| = |V(F_v)| - 1 = |W \cap V(F_v)| - \delta(v)$. If, on the other hand, $v \notin W'$, then $W \cap V(F_v)$ is a proper subset of $V(F_v)$. In this case it is easy to see (by induction on the size of $V(F_v)$, for example) that $|E_W \cap E(F_v)| \geq |W \cap V(F_v)|$. Summing over all $v \in V(T)$ gives the required inequality.

We have $\theta \in C(\tau)$. So, we may find $\epsilon > 0$ so that for all $e \in E(T)$,

$$\theta(e) < \frac{1}{|E| - |V|} - 2\epsilon$$

and, for all non-empty proper subsets U of V,

$$\sum_{e \in E_U} \theta(e) < \frac{|E_U| - |U|}{|E| - |V|} - 2\epsilon.$$

Under the identification of τ as a subset of $|\sigma|$, we have $\theta(e) = 0$ for all $e \in E(\sigma) \setminus E(\tau) \equiv E(F)$.

Let θ' be a point of $\mathrm{int}|\sigma|$ such that $|\theta'(e) - \theta(e)| < \epsilon/2|E(\Sigma)|$. Then, $\theta'(e) < 1/(|E| - |V|)$ for all $e \in E(\Sigma)$, and given a non-empty proper subset W of $V(\Sigma)$, we have

$$\begin{aligned}
\sum_{e \in E_W} \theta'(e) &\leq \sum_{e \in E_W \setminus E(F)} \theta(e) + \epsilon \\
&\leq \frac{1}{|E| - |V|}(|E_W \setminus E(F)| - |E_{W'}|) + \sum_{e \in E_{W'}} \theta(e) + \epsilon \\
&< \frac{1}{|E| - |V|}(|E_W \setminus E(F)| - |E_{W'}|) + \frac{1}{|E| - |V|}(|E_{W'}| - |W'|) \\
&= \frac{1}{|E| - |V|}(|E_W \setminus E(F)| - |W'|) \\
&\leq \frac{1}{|E| - |V|}(|E_W| - |W|).
\end{aligned}$$

This shows that $\theta' \in C(\sigma)$. (Recall $|E(\Sigma)| - |V(\Sigma)| = |E(T)| - |V(T)| = -\chi$.) $\diamondsuit$

If $g = p = 1$, then $\mathcal{S}_g^p$ is the space of euclidean tori which we may identify with the hyperbolic plane, H^2. In this case, it is easily seen that $C(A) = |A|$, and our cellulation of $\mathcal{S}_g^p$ gives us a tesselation of H^2 by ideal triangles. For more complicated surfaces, however, $C(A)$ will be a proper subset of $|A|$. Figure 3 gives an example, with g=2 and p=1, of a spine Σ and coordinates θ, in the corresponding simplex $|\sigma|$, so that $\theta \notin C(\sigma)$. Note that the inequality (ii) fails if we take W to consist of the three points as shown. An immersion of $S_T \setminus P_T$ in the plane is obtained by thickening up Σ as it is drawn.

There is a natural map $\mu : |A| \longrightarrow \Delta^{p-1} = \mathrm{Map}(P, [0,1]; 1)$ defined as follows. Given $\theta \in |\sigma|$ and $y \in P$, let

$$\mu(\theta)(y) = \frac{1}{2} \sum \{\theta(e) \,|\, e \text{ is incident on } y\}.$$

In this sum, we count twice any arc with both endpoints at y. Clearly, μ is well defined and continuous, and $\mu(C(A)) \subseteq \mathrm{int}\Delta^{p-1}$. It turns out that $\mu(C(A)) = \mathrm{int}\Delta^{p-1}$, since it is always possible to find a singular euclidean structure with any given set of cone angles.

5. The Cellulation of $\mathcal{S}_g^p$.

We can define a map $g : \mathcal{S}_g^p \longrightarrow C(A)$ by associating to a singular euclidean structure $(S_T, P_T) \longrightarrow (S, P)$, a point $g(S, P)$ in the simplex $\mathrm{int}|\sigma|$, where σ is dual to the spine Σ for (S, P). The coordinates θ for $g(S, P)$ are given by $\theta(e) = \gamma(e)/(-\pi\chi)$, where $\gamma(e)$ is the angle opposite e, as defined in Section 3. We see that $\theta \in C(\sigma)$, and that $\mu \circ g = \lambda$, where λ and μ are the maps to $\mathrm{int}\Delta^{p-1}$ defined above (Sections 1 and 4 respectively).

To show that g is surjective, we need to reconstruct a singular euclidean surface from the coordinates $\theta \in C(\sigma)$ and the combinatorial information of the spine Σ.

Theorem. *Suppose that Σ_T is a topological spine for (S_T, P_T), and suppose we have $\gamma \in \mathrm{Map}\big(E(\Sigma_T), (0, \pi); -\pi\chi\big)$ with $\sum_{e \in E_W} \gamma(e) < \pi(|E_W| - |W|)$ for all non-empty proper subsets W of $V(\Sigma_T)$. Then, there is a singular euclidean structure $(S_T, P_T) \longrightarrow (S, P)$, unique up to scale, such that the associated spine Σ is a geometric realisation of Σ_T, and for which each coordinate $\gamma(e)$, with $e \in E(\Sigma) \equiv E(\Sigma_T)$ gives the angle opposite the edge e in the spinal triangulation.*

The idea of the proof is as follows.

We choose a function $r \in \mathrm{Map}\big(V(\Sigma_T), (0, \infty); 1\big)$. For each edge e of Σ_T, we construct two congruent triangles, each with vertex angle equal to $\gamma(e)$ and adjacent sides of length $r(v)$ and $r(w)$, where v and w are the endpoints of e in Σ_T. We can then piece together these triangles, as dictated by the combinatorics of Σ_T, to give a singular euclidean surface. In general, we will have cone singularities at the vertices of Σ_T. We will need to show that there is a unique choice of r for which each of the cone angles is equal to 2π.

We define

$$f : \mathrm{Map}\big(V(\Sigma_T), (0, \infty); 1\big) \longrightarrow \mathrm{Map}\big(V(\Sigma_T), (0, \infty)\big)$$

by taking $f(r)(v)$ to be the cone angle at v in the above construction. By summing over all triangles, we see that $\sum_{v \in V(\Sigma)} f(r)(v) = 2\pi|V|$, so that $f(r)$ lies in the simplex $\mathrm{Map}\big(V, (0,\infty); 2\pi|V|\big)$. In fact, we shall show that for any fixed γ, f is injective, and that its image is the open convex polyhedron Q defined by

$$Q = \{s \in \mathrm{Map}\big(V, (0,\infty); 2\pi|V|\big) \mid \\ \sum_{v \in W} s(v) < 2\pi|E_W| - 2\sum_{e \in E_W} \gamma(e) \text{ whenever } \emptyset \neq W \subseteq V\}.$$

We shall need to express $f(r)(v)$ as a sum of angles coming from the component triangles in our construction.

Let $\phi(r, s, \gamma)$ be defined as in Figure 4. Then, $f(r)(v) = 2\sum_e \phi\big(r(v), s(w), \gamma(e)\big)$, where the sum is taken over all edges e incident on v. The point w is the other endpoint of e. Throughout, we use the convention that we count twice any edge that has both its ends at v.

Note that ϕ satisfies the following.

(1) $\partial\phi/\partial s > 0$

(2) $\partial\phi/\partial r < 0$

(3) $\phi(r, s, \gamma) < \pi - \gamma$

(4) $\phi(r, s, \gamma) + \phi(s, r, \gamma) = \pi - \gamma$

(5) If $r \to 0$, s bounded away from 0, then $\phi(r, s, \gamma) \to \pi - \gamma$

(6) If $s \to 0$, r bounded away from 0, then $\phi(r, s, \gamma) \to 0$

(7) For all $k > 0$, $\phi(kr, ks, \gamma) = \phi(r, s, \gamma)$.

Property (7) shows that our construction is independent of how we choose to normalise.

We can now check the relevant facts about f.

(i) $f(r) \in Q$.

Let W be a non-empty proper subset of V. We write $E_W = E_W^1 \sqcup E_W^2$, where E_W^1 and E_W^2 are the sets of edges of Σ with, respectively, one or both endpoints in W. Using (3) and (4), we see that

$$\begin{aligned}\sum_{w \in W} f(r)(w) &= 2\sum_{e \in E_W^2} \big(\pi - \gamma(e)\big) \\ &\qquad + 2\sum_{e \in E_W^1} \{\phi\big(r(w), r(v), \gamma(e)\big) \mid e \in E_W^1,\ w \in W,\ w < e,\ v < e\} \\ &< 2\sum_{e \in E_W} \big(\pi - \gamma(e)\big) \\ &= 2\pi|E_W| - 2\sum_{e \in E_W} \gamma(e).\end{aligned}$$

Here, $v < e$ means that v is a vertex incident on e.

(ii) f *is injective.*

Suppose $r \neq r'$. Let $W = \{v \,|\, r(v) < r'(v)\}$, so that W is a non-empty proper subset of V. From (1), (2) and (4), we check that

$$\sum_{w \in W} f(r)(w) > \sum_{w \in W} f(r')(w),$$

so that $f(r) \neq f(r')$.

(iii) f *is proper, as a map from* $\mathrm{Map}\big(V, (0,\infty); 1\big)$ *into* Q.

Given a sequence r_n in $\mathrm{Map}\big(V, (0,\infty); 1\big)$ tending to the boundary of the simplex, we need to find a subsequence r'_n, with $f(r'_n) \to \partial Q$. We may assume there exists a non-empty proper subset W of V so that $r_n(v) \to 0$ for all $v \in W$, and with $r_n(v)$ bounded away from 0 for all $v \notin W$. Using (4) and (5), we get

$$\begin{aligned}\sum_{w \in W} f(r_n)(w) = 2 &\sum_{e \in E^2_W} \big(\pi - \gamma(e)\big) \\ &+ 2 \sum \{\phi\big(r(w), r(v), \gamma(e)\big) \mid e \in E^1_W,\ w \in W,\ w < e,\ v < e\} \\ \to 2 &\sum_{e \in E_W} \big(\pi - \gamma(e)\big),\end{aligned}$$

and so $f(r_n) \to \partial Q$.

Taking the one-point compactifications of $\mathrm{Map}\big(V(\Sigma_T), (0,\infty); 1\big)$ and Q respectively, we get a continuous injective map between two spheres of the same dimension. A Brouwer-degree argument now shows that f is surjective.

To prove the theorem, we need only check that $(2\pi, \ldots, 2\pi) \in Q$. But this follows since $\sum_{e \in E_W} \gamma(e) < \pi(|E_W| - |W|)$, and so $\sum_{w \in W}(2\pi) = 2\pi|W| < 2\pi|E_W| - 2\sum_{e \in E_W} \gamma(e)$, for any non-empty proper subset W of V.

6. The cellulation of $\mathrm{int}\mathcal{T}^p_g$

We stated in the introduction that our complex $C(A)$ may be used to triangulate the space of finite-area hyperbolic structures with a certain weight assigned to each cusp. The proof will be identical to that for singular euclidean structures, and is based on the observation that the function ϕ', that is appropriate to this situation, and which corresponds to ϕ in the euclidean case, has the same qualitative properties as ϕ.

Let $\mathrm{int}\mathcal{T}^p_g = T^p_g \times \mathrm{int}\Delta^{p-1}$, where T^p_g is the Teichmüller space of a surface of genus g with p cusps. A point of $\mathrm{int}\mathcal{T}^p_g$ can be thought of as a marked, finite-area, complete hyperbolic surface, S, with a fixed horoball chosen about each cusp. These horoballs are chosen by the second coordinate $x \in \mathrm{int}\Delta^{p-1}$. In [BE], we took the horoball $B(x,y)$ about $y \in P$ so that the length of $\partial B(x,y)$ was equal to $c_0 x(y)$, where c_0 is a universal constant small enough so that these horoballs are always disjoint. There is an alternative interpretation of the second coordinate which is more appropriate to the present situation and will be discussed below. For the moment, however, we will stick with these "standard horoballs". We will write $B(x) = \bigcup_{y \in P} B(x,y)$.

Given such a surface, S, with standard horoballs $B(x)$, we may associate a spine $\Sigma \subseteq S$ in an exactly analogous manner to that for singular euclidean structures given in Section 2, where distance from a cone point is replaced by distance from a horoball. We may construct a spinal triangulation by adding ribs between the vertices of Σ and the points of P. In [BE], we then chose coordinates for S which gave us an identification of $\mathrm{int}\mathcal{T}_g^p$ with a certain subset of the complex $|A|$. Here, we shall give different coordinates, which will identify $\mathrm{int}\mathcal{T}_g^p$ with precisely the subset $C(A)$, obtained for singular euclidean structures.

Given S, x, B, Σ as above, and $e \in E(\Sigma)$, let $\gamma(e)$ be the area of one of the triangles of the spinal triangulation which has e as an edge. (The two such triangles are isometric to each other reflection in e.) Equivalently, $\gamma(e)$ is equal to π minus the sum of the two angles of this triangle adjacent to e. Now, by the Gauss-Bonnet theorem, we have that

$$\sum_{e \in E(\Sigma)} \gamma(e) = \frac{1}{2}\mathrm{areaS} = -\pi\chi.$$

Moreover, if $W \subseteq V(\Sigma)$ is a proper non-empty subset, then by summing the angles around the vertices of W, we once more obtain

$$\sum_{e \in E_W} \gamma(e) < \pi(|E_W| - |W|).$$

Thus, $(\sigma, -\frac{1}{\pi\chi}\gamma) \in C(\sigma)$, where σ is dual to Σ. We get a map $\hat{g} : \mathrm{int}\mathcal{T}_g^p \longrightarrow C(A)$ in exactly the same way as we defined g in section 5.

The proof that $\hat{g}$ is bijective proceeds as in the singular euclidean case. Suppose we are given a topological spine Σ_T, and coordinates $\gamma \in \mathrm{Map}\big(E(\Sigma_T), (0,\pi); -\pi\chi\big)$, satisfying the same conditions as in the statement of the main theorem of Section 5. We piece together hyperbolic triangles, each with area determined by γ, but with variable shape. This gives a hyperbolic surface with cone singularities. The argument of Section 5 will show that there is a unique way to make each of these angles equal to 2π.

To see that this works, we define $\phi' : (0,\infty)^2 \times (0,\pi) \longrightarrow (0,\pi)$, where $\phi'(r,s,\gamma)$ is the angle in the hyperbolic triangle $T'(r,s,\gamma)$, of area γ, defined by Figure 5. Here, $\log r$ and $\log s$ are the signed distances of the vertices from the horocycle. Each one is allowed a negative value, so that ϕ' is defined for r and s in $(0,\infty)$. We take logrithms for convenience, so that ϕ' satisfies precisely the same properties, (1)–(7) of Section 5, as ϕ.

The only property that is not immediate is that $\partial\phi'/\partial r > 0$. (Since $\phi'(r,s,\gamma) + \phi'(s,r,\gamma) = \pi - \gamma$ it follows then that $\partial\phi'/\partial s < 0$.) We need also check that the triangle $T'(r,s,\gamma)$ is uniquely determined by the parameters r, s and γ. However, both of these statements may be easily deduced from the following observation.

Lemma. *Let C be a horocircle about the ideal point b of H^2. Let $c \in \mathrm{H}^2$ be fixed, and let a vary on C. Then, the (positive) area of the triangle abc is a strictly increasing function of the distance between a and c.*

Proof : We may assume that c lies outside the open horodisc bounded by C, otherwise we could reverse the roles of a and c. We imagine a as moving away from c, along C. While the angle at a remains greater than $\pi/2$, each triangle abc is strictly contained in any later one, and so the area is increasing. After the angle at a has become less than $\pi/2$, the angles at a and c are both decreasing with time, and so the area of abc continues to increase. $\diamondsuit$

We have shown that $\hat{g}$ is bijective.

Now, let $\mu : C(A) \longrightarrow \text{int}\Delta^{p-1}$ be the map defined at the end of Section 4. Let $(S, B(x)) \in \text{int}\mathcal{T}_g^p$ with spine Σ. It follows from our definitions that $\mu \circ \hat{g}(S, B(x))(y)$ is equal to $-\frac{1}{2\pi\chi}$ times the hyperbolic area of the component of $S \setminus \Sigma$ containing the cusp y. It would be more natural to reinterpret the coordinate x in terms of these areas.

Let $(S, P) \in T_g^p$ be a finite-area hyperbolic surface, and suppose $x \in \text{int}\Delta^{p-1}$. Let $B(x, y)$ be the standard cusps defined at the start of this Section. Let Σ be the corresponding spine. If $y \in P$, let $j_S(x)(y)$ be the area of the component of $S \setminus \Sigma$ containing y. By Gauss-Bonnet, we have

$$\sum_{y \in P} j_S(x)(y) = \text{areaS} = -2\pi\chi.$$

Thus, $j_S(x) \in \text{Map}(P, (0, \infty); -2\pi\chi)$.

The Proposition below establishes that $j_S : \text{int}\Delta^{p-1} \longrightarrow \text{Map}(P, (0, \infty); -2\pi\chi)$ is a homeomorphism for any surface S. Given this, we can set

$$g' = \hat{g} \circ (\pi_1, \psi)^{-1} : T_g^p \times \text{int}\Delta^{p-1} \longrightarrow C(A),$$

where π_1 is projection to the first coordinate, and $\psi(S, x) = -\frac{1}{2\pi\chi} j_S(x)$. Now, g' is a homeomorphism, and $\mu \circ g' : T_g^p \times \text{int}\Delta^{p-1} \longrightarrow \text{int}\Delta^{p-1}$ is projection to the second coordinate. This ties in more naturally with the case of singular euclidean structures.

Proposition. *Let $(S, P) \in T_g^p$ and $j_S : \text{int}\Delta^{p-1} \longrightarrow \text{Map}(P, (0, \infty); -2\pi\chi)$ be as above. Then, j_S is a homeomorphism.*

Proof : It is clear that j_S is continuous. We show that it is injective and proper, and thus a homeomorphism (by Brouwer degree).

Suppose that $x \in \text{int}\Delta^{p-1}$ and that $P = P_1 \sqcup P_2$ is a partition of P. For $i = 1, 2$, let $B(x, P_i) = \bigcup_{y \in P_i} B(x, y)$. Let

$$D(x, P_1) = \{z \in S \mid d(z, B(x, P_1)) \leq d(z, B(x, P_2))\},$$

where d is the hyperbolic metric on S. Thus, $D(x, P_1)$ is the closure of the union of all those components of $S \setminus \Sigma$ which contain some point of P_1. Thus,

$$\text{areaD(x, P}_1) = \sum_{\text{y} \in \text{P}_1} \text{j}_\text{S}(\text{x})(\text{y}).$$

Now, suppose that $x \neq x' \in \text{int}\Delta^{p-1}$. Let $P^+ = \{y \in P \mid x(y) < x'(y)\}$, and let $P^- = P \setminus P^+$. Then, $B(x, P^+)$ is strictly contained in $B(x', P^+)$, and $B(x', P^-)$ is strictly contained in $B(x, P^-)$. One checks readily that $D(x, P^+)$ is strictly contained in $D(x', P^+)$, and so

$$\begin{aligned}\sum_{y \in P^+} j_S(x)(y) &= \mathrm{areaD(x, P^+)} \\ &< \mathrm{areaD(x', P^+)} \\ &= \sum_{y \in P^+} j_S(x')(y).\end{aligned}$$

Thus, $j_S(x) \neq j_S(x')$, so we have shown that j_S is injective.

The proof that j_S is proper is similar. Suppose that $x_n \longrightarrow x \in \partial\Delta^{p-1}$. Let $P_0 = \{y \in P \mid x(y) = 0\}$, and $P_+ = P \setminus P_0$. Then $\bigcap_n B(x_n, P_0) = P_0$, whereas $\bigcap_n B(x_n, P_+)$ contains a neighbourhood of P_+. Again, it is readily checked that $\limsup_n D(x_n, P_0) = P_0$. Thus,

$$\sum_{y \in P_0} j_S(x)(y) = \mathrm{areaD(x_n, P_0)} \longrightarrow 0,$$

and so $j_S(x_n) \longrightarrow \partial\mathrm{Map}(P, (0, \infty); -2\pi\chi)$. We have shown that j_S is proper. $\diamondsuit$

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