A variation on the unique product property

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0. Introduction.

This paper is motivated by an attempt to understand the unique product property of groups from a more geometric point of view, in the hope of providing a context in which the results might be generalised. The main interest in the unique product property arises from the fact that it implies that the corresponding group rings have no zero-divisors. It is easy to see that right orderable groups have the former property. Recently, groups having some kind of treelike structure have received much attention. Here, we shall describe a variation on the unique product property that seems to fit more naturally into that framework. We give alternative arguments for the some of the known examples of unique product groups, and describe how further examples may be obtained.

There are many open questions relating to the structure of group rings. A standard reference is [Pas]. In particular, the zero-divisor conjecture of Kaplanski asserts that if R is a ring without zero-divisors, and Γ is a torsion-free group, then the group ring, $R\Gamma$, also has no zero-divisors (i.e. $xy = 0$ implies $x = 0$ or $y = 0$). We are not particularly concerned here about the ring R — the case where R is $\mathbf{Z}$ or $\mathbf{R}$, for example, is already interesting.

The notion of a "unique product" or "u.p." group arose in this context. Thus, Γ is said to have the *u.p.* property if given any nonempty finite sets, $A, B \subseteq \Gamma$, then at least one element, say c, of $AB = \{ab \mid a \in A, b \in B\}$ can be written as a product, $c = ab$, in only one way, i.e. for a unique $a \in A$ and $b \in B$. It is easily seen that this implies the zero-divisor conjecture for Γ for any ring R. It is, however, more restrictive. Examples of torsion-free groups without the u.p. property were found by Rips and Segev [RS]. A simpler example was given by Promislow [Pr]. The latter example can be shown, by other means, to satisfy the zero-divisor conjecture, at least in the case where R is an intergral domain [Pas], or more generally, an Ore domain [KLM]. The question as to whether this particular group is u.p. seems to have originated in [BuH].

Another, apparently more restrictive, condition is the "two unique products" or "t.u.p." condition. This asserts that if $A, B \subseteq \Gamma$ satisfy $2 \leq \operatorname{card} A < \infty$ and $2 \leq \operatorname{card} B < \infty$, then there are at least two elements of AB each of which has a unique expression as a product. In fact, it has been shown that t.u.p. is equivalent to u.p. [S].

The best known examples of groups satisfying u.p. (or t.u.p.) are right orderable groups. This property has been much studied, and is related to Higman's notion of local indicability. Every locally indicable group is right orderable [BuH], and a converse for soluble groups is given in [CK] (where some further references on the subject are given). It is easily seen that free groups and surface groups are locally indicable, and it is also known that one-relator groups have this property [Br,H]. However, the only proofs I know that local indicability implies right orderability are non-constructive, and depend on the Axiom of Choice even for explicit examples. In another direction, it has been shown directly by Dehornoy that braid groups are right orderable [Deh].

In the light of the recent surge of interest in treelike structures of various sorts, in particular **R**-trees, it seems appropriate to reset these results in this context. Thus, it is well known, and not hard to see, that most surface groups act freely isometrically on **R**-trees. It is in turn, not hard to see that groups admitting such actions are u.p. The theory of group actions on **R**-trees was developed by Morgan and Shalen, and since then by Rips and many other authors. See [Pau] and [Be] for recent surveys.

To explore this in more detail, we shall introduce the notion of what we shall call a "diffuse" torsion-free group. Such groups are u.p., and the class of such groups is closed under extensions and free products. We shall restrict attention to torsion-free groups, since it is not clear what the natural generalisation to arbitrary groups should be. One advantage of the notion of diffusion is that it is also applicable to group actions, and therefore allows us to set the arguments in a more geometric context.

We shall also introduce the notion of a "concentrated" subgroup of a torsion free group, by analogy with Lichtman's notion of "inertia" in the context of u.p. groups [Lic]. We shall see that amalgamated free products of diffuse groups over concentrated subgroups are diffuse. We apply this to surface groups in Section 3. Analogous results for u.p. groups subgroups are discussed in [Lic] and [F].

The main constructions of diffuse groups described here rest ultimately on some kind of treelike structure. It would be nice to find other settings in which these kind of arguments can be made to work. For example, cube complexes might be one obvious place to try. As was pointed out to me by Ralph Strebel, one can see quite easily that locally indicable groups are diffuse — thereby bypassing the usual non-constructive route passing via right orderability.

We remark that there are other, more sophisticated, approaches to the zero-divisor conjecture. A number of these are described in [Pas] and [Lin]. In particular, it is shown in [KLM] that if R is an Ore domain (i.e. embeds in a division ring) and Γ is soluble-by-finite, then $R\Gamma$ has no zero-divisors.

This paper is structured as follows. In Section 1, we define a "diffuse group", and state the main general results. In Section 2, we define a "diffuse set", and prove the main general results. In Section 3, we apply some of this to surface groups. In Section 4, we give a sufficient criterion for concentration of a cyclic subgroup of a free group. In Section 5, we show that groups acting on hyperbolic spaces with large translation lengths are hyperbolic. In Section 6, we briefly discuss the non-diffuse group featuring in Promislow's paper.

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1. Diffuse groups.

Let Γ be a torsion-free group with identity 1. Suppose $A \subseteq \Gamma$. Let $A^{-1} = \{a^{-1} \mid a \in A\}$. We say that A is *antisymmetric* if $A \cap A^{-1} \subseteq \{1\}$. Let $\Delta(A) \subseteq A$ be the set of $a \in A$ such that Aa^{-1} is antisymmetric. Put another way, $a \in A \setminus \Delta(A)$ if and only if there is some $\gamma \in \Gamma \setminus \{1\}$ such that $\gamma a, \gamma^{-1} a \in A$. We can thus think of $\Delta(A)$ as consisting, in some sense, of "extreme" elements of A.

Definition : We say that Γ is *diffuse* if, given any subset $A \subseteq \Gamma$ with $2 \leq \operatorname{card} A < \infty$, we have $\operatorname{card} \Delta(A) \geq 2$.

An obvious variation on this is:

Definition : We say that Γ is *weakly diffuse* if, given any nonempty finite set $A \subseteq \Gamma$, the set $\Delta(A)$ is nonempty.

Although the second definition is easier to state, the first appears more natural, and seems to have nicer (at least more easily verifiable) closure properties. Clearly both properties are inherited by subgroups. Also, any increasing union of (weakly) diffuse groups is (weakly) diffuse. The last observation means that we can often restrict attention to finitely generated groups in verifying diffusion.

Any right orderable group is diffuse: if A is a finite subset of a right orderable group, then $\{\min A, \max A\} \subseteq \Delta(A)$. We also note:

Proposition 1.1 : *A diffuse group satisfies the t.u.p. property.*

Proof : Suppose $A, B \subseteq \Gamma$ are finite sets each with at least 2 points. Let $C = AB$. Thus, $\operatorname{card} C \geq 2$ and so $\operatorname{card} \Delta(C) \geq 2$. We claim that if $c \in \Delta(C)$, then c has a unique expression as a product.

Suppose, to the contrary, that $c = a_0 b_0 = a_1 b_1$, where $a_0, a_1 \in A$, $b_0, b_1 \in B$ and $a_0 \neq a_1$. Let $\gamma = a_0 a_1^{-1} \in \Gamma \setminus \{1\}$. Then, $\gamma c = a_0 a_1^{-1} a_1 b_1 = a_0 b_1 \in C$ and $\gamma^{-1} c = a_1 a_0^{-1} a_0 b_0 = a_1 b_0 \in C$, contradicting $c \in \Delta(C)$. $\diamondsuit$

The same argument shows that a weakly diffuse group satisfies u.p.

It is interesting to speculate on the relationship of these four properties. In particular, in view of the result of [S] that u.p. implies t.u.p., it is natural to ask if weakly diffuse implies diffuse.

One reason for introducing this notion is that the following facts are fairly easily verified, as we shall see in Section 2.

Theorem 1.2 : *Let Γ, G and G' be torsion-free groups.*
(1) Suppose $N \triangleleft \Gamma$, and that N and Γ/N are both diffuse. Then Γ is diffuse.
(2) If G and G' are both diffuse, then the free product $G * G'$ is diffuse.
(3) If Γ acts freely isometrically on an $\mathbf{R}$-tree, then Γ is diffuse.

In fact, (2) and (3) follow by similar arguments. It would not be hard to give a unified statement that encompasses both, though we shall not bother to formulate one here. One can also generalise (2) to infinite free products, as we discuss shortly. The u.p. property is also known to be closed under extensions and free products [S,Lic]. Note that this result shows, for example, that pure braid groups and torsion-free locally nilpotent groups are diffuse.

It is well known that (compact) surface groups (except for the fundamental groups of a direct sum of one, two or three copies of the projective plane) act freely isometrically on **R**-trees. This is also true of free and finitely generated free-abelian groups. Such groups are therefore dealt with by part (3). Conversely the theorem of Rips (conjectured by Shalen) classifies all finitely generated groups admitting such actions. They are all free products of surface, free and free abelian groups.

One can generalise Theorem 1.2(2) to allow for amalgamated free products. For this, we shall need the notion of a "concentrated" subgroup (by analogy with Lichtman's notion of "inertia"). Suppose that Γ is a torsion-free group, and $H \leq \Gamma$ is any subgroup. A *left coset* of H in Γ is a set of the form aH, for $a \in \Gamma$.

Definition : We say that H is *concentrated* in Γ if the following holds. Suppose $A \subseteq \Gamma$ is a finite union of left cosets of H, comprising at least two such cosets. Then, there are elements, $a_1, a_2 \in A$, lying in distinct left cosets, each with the property that if γ is an element of Γ with $\gamma a_i, \gamma^{-1} a_i \in A$, then $\gamma \in a_i H a_i^{-1}$.

This is therefore effectively saying the Γ is "diffuse relative to H". We note that if $N \triangleleft \Gamma$, then N is concentrated in Γ if and only Γ/N is diffuse. In particular, Γ is diffuse if and only if the trivial group is concentrated in Γ. Clearly, every group is concentrated in itself.

Suppose $H \leq G \leq \Gamma$. If H is concentrated in Γ, then it's also concentrated in G. Also, a proper finite index subgroup can never be concentrated. It follows that any concentrated subgroup of a group must be maximal among commensurable subgroups.

We shall prove the following result, which generalises Theorem 1.2, parts (1) and (2).

Theorem 1.3 :

(1) Suppose Γ is a torsion-free group, and $H \leq G \leq \Gamma$ are subgroups. If H is concentrated in G, and G is concentrated in Γ, then H is concentrated in Γ.

(2) Suppose that a torsion-free group, Γ, can be expressed as the fundamental group of a (possibly infinite) graph of groups, with the property that every edge group is (intrinsically) diffuse, and concentrated in each incident vertex group. Then, Γ is diffuse. Moreover, any concentrated subgroup of a vertex group is also concentrated in Γ.

We make a few remarks about Theorem 1.3. Firstly, it is immediate from (1) that if $H \leq \Gamma$ is (intrinsically) diffuse and concentrated in Γ, then Γ is diffuse. It therefore generalises Theorem 1.2(1). In the hypotheses of part (2), it would be enough to assume diffusion for any one of the vertex or edge groups. The fact that all the other vertex and edge groups are diffuse then follows by repeated application of part (1), together with the fact that diffusion is inherited by subgroups. Note also that we may conclude immediately from part (2) that all the vertex and edge groups are concentrated in Γ. Finally, we observe that part (1) may, in fact, be viewed as a corollary to part (2). To see this, consider the trivial splitting of Γ with three vertex groups and two edge groups, namely $\Gamma \cong \Gamma *_G G *_H H$. The hypotheses of (2) are satisfied, so the vertex group, H, is concentrated in Γ.

For the same reason as given in [Lic] in the case of inertia, we see that any isolated subgroup, H, of a torsion-free locally nilpotent group, Γ, is concentrated. ("Isolated" means that $x^n \in H$ implies $x \in H$.)

2. Diffuse sets.

To prove Theorem 1.2, it seems more natural to define diffusion in the category of Γ-sets, i.e. sets admitting an action by the (torsion-free) group Γ. Here are some formal definitions.

A Γ-*set* is a set, X, together with an action by the group Γ. Given $x \in X$, we write $\Gamma(x) = \{\gamma \in \Gamma \mid \gamma x = x\}$. Note that if $N \triangleleft \Gamma$ acts trivially on X, then X also has the structure of a (Γ/N)-set.

A *morphism* between two Γ-sets, X and Y, is a map $f : X \longrightarrow Y$, satisfying $f(\gamma x) = \gamma f(x)$ for all $x \in X$ and $\gamma \in \Gamma$. Clearly, if $y \in Y$, then the preimage, $f^{-1}y$, has the structure of a $\Gamma(y)$-set.

Suppose that X is a Γ-set. Given any $A \subseteq X$, let $\Delta_\Gamma(A) \subseteq A$ be the set of $a \in A$ such that if $\gamma \in \Gamma$ satisfies $\gamma a, \gamma^{-1}a \in A$, then $\gamma a = a$.

Definition : We say that X is *diffuse* if given any $A \subseteq X$ with $2 \leq \operatorname{card} A < \infty$, then $\operatorname{card} \Delta_\Gamma(A) \geq 2$.

Clearly, any Γ-invariant subset of a diffuse Γ-set is also diffuse. Also, any disjoint union or increasing union of diffuse Γ-sets is diffuse.

Note that any group, Γ, has the structure of a Γ-set, where Γ acts by left multiplication. In this case, the above definitions agree with those of Section 1. In particular, Γ is diffuse as a group if and only if it is diffuse as a Γ-set.

More generally, if $H \leq \Gamma$ is any subgroup, then Γ acts by left multiplication on the set, Γ/H, of left cosets of H. The Γ-set Γ/H is diffuse if and only if H is concentrated in Γ. Note that if X is a Γ-set with just one orbit, then X is isomorphic, as a Γ-set, to Γ/H, where H is the stabiliser of a point of X.

Note that if X is a Γ-set and $N \triangleleft \Gamma$ acts trivially on X, then X is diffuse as a Γ-set if and only if it is diffuse as a (Γ/N)-set.

Lemma 2.1 : *Suppose that X and Y are Γ-sets, and $f : X \longrightarrow Y$ is a morphism. Suppose that Y is diffuse, and that $f^{-1}y$ is diffuse as a $\Gamma(y)$-set for all $y \in Y$. Then X is diffuse.*

Proof : Suppose $A \subseteq X$ with $2 \leq \operatorname{card} A < \infty$. Let $B = f(A) \subseteq Y$.

Suppose, first, that $\operatorname{card} B \geq 2$, so that $\operatorname{card} \Delta_\Gamma(B) \geq 2$. Let $b \in \Delta_\Gamma(B)$. Suppose that $\operatorname{card}(A \cap f^{-1}b) \geq 2$. Then $\Delta_{\Gamma(b)}(A \cap f^{-1}b) \neq \emptyset$. Let $a \in \Delta_{\Gamma(b)}(A \cap f^{-1}b)$. We claim that $a \in \Delta_\Gamma(A)$. For, if $\gamma \in \Gamma$ and $\gamma a, \gamma^{-1}a \in A$, then $\gamma b, \gamma^{-1}b \in B$ and so $\gamma b = b$ (since $b \in \Delta_\Gamma(B)$). Thus $\gamma \in \Gamma(b)$, and so $\gamma a, \gamma^{-1}a \in A \cap f^{-1}b$, and so $\gamma a = a$ (since $a \in \Delta_{\Gamma(b)}(A \cap f^{-1}b)$). This proves the claim. By a similar argument, we see that if $f^{-1}b = \{a\}$, for some $a \in A$, then $a \in \Delta_\Gamma(A)$. In either case, we obtain $\Delta_\Gamma(A) \cap f^{-1}b \neq \emptyset$. Since $\operatorname{card} \Delta_\Gamma(B) \geq 2$, we deduce that $\operatorname{card} \Delta_\Gamma(A) \geq 2$.

Finally, if $B = \{b\}$ for some $b \in B$, then $A = f^{-1}b$, and we see that $\Delta_\Gamma(A) = \Delta_{\Gamma(b)}A$, so again we have $\operatorname{card} \Delta_\Gamma(A) \geq 2$. $\diamondsuit$

This immediately proves Theorem 1.2(1). To see this, suppose Γ, N are as in the hypotheses, and let $f : \Gamma \longrightarrow \Gamma/N$ be the quotient map. Thus, f is a morphism of Γ-sets. Now, Γ/N is diffuse as a (Γ/N)-set, and hence as a Γ-set. Also, if $y \in \Gamma/N$, then $\Gamma(y)$ is conjugate to N in Γ, and $f^{-1}y$ is isomorphic to N viewed as an N-set. Thus $f^{-1}y$ is diffuse. It follows that Γ is diffuse as a Γ-set.

We approach Theorem 1.2(2) from the viewpoint of actions on simplicial trees. First, we recall some definitions.

Let T be a simplicial tree, and denote the vertex and edge sets by $V(T)$ and $E(T)$ respectively. Given $x \in V(T)$, let $E(x) \subseteq E(T)$ be the set of incident edges.

Given $x, y \in V(T)$, let $[x, y] \subseteq T$ be the unique arc connecting them. If $x \neq y$, we write $e(x, y) \in E(x)$ for the edge of $[x, y]$ incident on x. We write ρ for the combinatorial metric on $V(T)$, i.e. $\rho(x, y) = \operatorname{card}([x, y] \cap V(T)) - 1$. Given $x, y, z \in V(T)$, there is a unique point in $[x, y] \cap [y, z] \cap [z, x]$ called the *median* of x, y, z and denoted by $\operatorname{med}(x, y, z)$.

Suppose $x, y, z \in V(T)$ satisfy $\rho(m, x) < \min\{\rho(m, y), \rho(m, z)\}$, where $m = \operatorname{med}(x, y, z)$. Then it's easily seen that for any $a \in V(T)$, we have $\rho(a, x) < \max\{\rho(a, y), \rho(a, z)\}$.

Now, suppose Γ acts simplicially without edge-inversions on T. (The following discussion will apply also to isometric actions on $\mathbf{R}$-trees, which we will discuss later.)

Suppose $\gamma \in \Gamma$. Then the fixed point set, $\operatorname{fix} \gamma$, of γ is a (possibly empty) subtree of T. If it is nonempty, then γ is called *elliptic*. In this case, if $x \in V(T)$ is any vertex, then $m = \operatorname{med}(\gamma^{-1}x, x, \gamma x)$ is fixed by γ^2. (To see this, let y be the nearest point of $\operatorname{fix} \gamma$ to x. Now, $m = \operatorname{med}(\gamma^{-1}x, y, \gamma x)$, and so γ^2 fixes the segement $[y, m]$.) If it happens that $\operatorname{fix} \gamma^2 = \operatorname{fix} \gamma$, then m is fixed by γ, and it in this case, it follows that m is equidistant from x, γx and $\gamma^{-1}x$. On the other hand, if the fixed point set is empty, we say that γ is *hyperbolic*. In this case, there is a unique γ-invariant axis, which is translated by γ. If $x \in V(T)$ is any vertex, then $m = \operatorname{med}(\gamma^{-1}x, x, \gamma x)$ lies on this axis. It follows that $\rho(m, x) < \rho(m, \gamma x) = \rho(m, \gamma^{-1}x)$. Thus, from the inequality described in the last paragraph, we see that if $a \in V(T)$ is any other vertex, then $\rho(a, x) < \max\{\rho(a, \gamma x), \rho(a, \gamma^{-1}x)\}$.

We are now ready for:

Proposition 2.2 : *Suppose that Γ acts simplicially and without edge-inversions on a simplicial tree, T. If $E(x)$ is diffuse as a $\Gamma(x)$-set for all $x \in V(T)$, then $V(T)$ and $E(T)$ are both diffuse as Γ-sets.*

Proof : We first note that if $\gamma \in \Gamma$ is elliptic, then $\operatorname{fix}\gamma^2 = \operatorname{fix}\gamma$. Otherwise, there is an edge, e, fixed by γ^2 but not γ, and with one endpoint, x, fixed by γ. Now, the set $\{e, \gamma e\}$ is γ-invariant, an so violates the condition that $E(x)$ is diffuse.

Now, suppose $A \subseteq V(T)$ with $2 \leq \operatorname{card} A < \infty$. Fix any $a \in V(T)$, and let $M = \max\{\rho(a,x) \mid x \in A\}$. Let $B = \{x \in A \mid \rho(a,x) = M\}$ — in other words, the set of points of A a maximal distance from a. Let $\Sigma = \bigcup_{b \in B}[a,b]$. Thus, Σ is the minimal subtree of T containing $B \cup \{a\}$.

Given $x \in V(\Sigma)$, let $E_\Sigma(x)$ be the set of edges of Σ incident on x. If x is terminal in Σ, i.e. $\operatorname{card} E_\Sigma(x) = 1$, then we set $D(x) = E_\Sigma(x)$. Otherwise, we set $D(x) = \Delta_{\Gamma(x)} E_\Sigma(x)$. Since $E_\Sigma(x)$ is finite, and $E(x)$ is diffuse, we see that in the latter case, $\operatorname{card} D(x) \geq 2$.

Now, by starting at a and following an edge of $D(x)$ whenever we reach a branch point of Σ, we eventually arrive at a point $b \in B$ with the property that if x is any vertex in $[a,b] \setminus \{b\}$, then $e(x,b) \in D(x)$. (At this point, we are using the fact $E(x)$ is diffuse, rather than just weakly diffuse.) We claim that $b \in \Delta_\Gamma(A)$.

To see this, suppose for contradiction, that there is some $\gamma \in \Gamma$ with $\gamma b, \gamma^{-1} b \in A$, but with $\gamma b \neq b$. Now, if γ were hyperbolic, we would have $\max\{\rho(a,\gamma b), \rho(a, \gamma^{-1} b)\} > \rho(a,b) = M$, contradicting the maximality of $\rho(a,b)$. Thus, γ is elliptic, and as observed above, $\operatorname{fix}\gamma^2 = \operatorname{fix}\gamma$. In particular, $m = \operatorname{med}(\gamma^{-1}b, b, \gamma b)$ is fixed by γ, and so $\rho(m, \gamma b) = \rho(m, \gamma^{-1} b) = \rho(m,b)$. Since $\gamma b \neq b$, $m \neq b$. Now, m must lie in $[a,b]$, otherwise we would again get $\max\{\rho(a,\gamma b), \rho(a,\gamma^{-1} b)\} > \rho(a,b)$, contradicting the maximality of $\rho(a,b)$. Thus, from the choice of b, we have $e = e(m,b) \in D(m) \subseteq E_\Sigma(m)$. Clearly, $\gamma e = e(m, \gamma b)$ and $\gamma^{-1} e = e(m, \gamma^{-1} b)$. We claim that $\gamma e, \gamma^{-1} e \in E_\Sigma(m)$. To see that γe lies in Σ, note that either $m \in [a, \gamma b]$, in which case $\rho(a,\gamma b) = \rho(a,b) = M$, so $\gamma b \in B$, so $\gamma e = e(m,\gamma b)$ lies in Σ as stated, or else $m \notin [a,\gamma b]$, in which case $\gamma e = e(m,\gamma b) = e(m,a)$ which again lies in Σ. The same argument works for $\gamma^{-1} e$, proving the claim. Now since $e \in D(m) = \Delta_{\Gamma(m)} E_\Sigma(m)$, it follows that $\gamma e = e$, contradicting the fact that $m = \operatorname{med}(\gamma b, b, \gamma^{-1} b)$. We therefore conclude that $b \in \Delta_\Gamma(A)$ as claimed.

In summary, we have shown that for any $a \in V(T)$, the set $\Delta_\Gamma(A) \setminus \{a\} \neq \emptyset$. Thus, $\operatorname{card} \Delta_\Gamma(A) \geq 2$, as required. This shows that $V(T)$ is diffuse.

To see that $E(T)$ is diffuse, we apply the result just proven to the binary subdivision, T', of T. This is obtained by introducing a new vertex at the middle of each edge of T. We can thus identify $V(T')$ with $V(T) \sqcup E(T)$ as Γ-sets. Now if $x \in V(T')$ is a newly introduced vertex, then $\Gamma(x)$ acts trivially on the unordered pair, $E(x)$. Thus $E(x)$ is diffuse as a $\Gamma(x)$-set. The hypotheses of the first part of the theorem are now satisfied, so we see that $V(T')$ and hence $E(T)$ are diffuse, as required. $\diamondsuit$

We can now prove Theorem 1.2(2). Let G, G' be (torsion-free) groups, and let $\Gamma = G * G'$. Let T be the associated Bass-Serre tree. Thus, Γ acts simplicially on T, without edge-inversions, with trivial edge stabilisers, and with one orbit of edges. Moreover, we can write $V(T)$ as a Γ-invariant disjoint union $V(T) = W \sqcup W'$, with each edge incident on both W and W', and with $\Gamma(x)$ isomorphic to either G or G' depending on whether $x \in W$ or $x \in W'$. Note that for any $x \in V(T)$, there is precisely one orbit in $E(x)$. Thus, as a $\Gamma(x)$-set, $E(x)$ is isomorphic to $\Gamma(x)$ itself under left multiplication. It follows that if both G and G' are diffuse, then $E(x)$ is diffuse as a $\Gamma(x)$-set for all $x \in V(T)$. By Proposition 2.2, we see that $E(T)$ is diffuse as a Γ-set. But $E(T)$ is isomorphic to Γ itself. Thus Γ is diffuse as required.

We next need to prove Theorem 2.1(3). This follows by a similar but simpler argument. Recall that an **R**-*tree* can be defined as a metric space, (T, ρ), such that any pair of points, $x, y \in T$, are connected by a unique arc, $[x, y] \subseteq T$, and this arc is isometric to a real closed interval. Most of the earlier discussion of simplicial trees applies equally well to **R**-trees. In particular, we have a notion of median. Moreover, if Γ acts isometrically on T, we can classify the elements of Γ as elliptic or hyperbolic. If Γ acts freely, then each element $\gamma \in \Gamma \setminus \{1\}$ is hyperbolic. In particular, if $a, b \in T$, then $\rho(a, b) < \max\{\rho(a, \gamma b), \rho(a, \gamma^{-1} b)\}$.

The proof of Theorem 2.1(3) is now straightforward. Suppose Γ acts freely isometrically on T. Let X be the Γ-orbit of any point in T. Thus X is isomorphic as a Γ-set to Γ itself. Suppose that $A \subseteq X$ with $2 \leq \operatorname{card} A < \infty$. Choose any $a \in T$, and then choose any $b \in A$ so as to maximise $\rho(a, b)$. It follows that $b \in \Delta_\Gamma(A)$, for if $\gamma \in \Gamma \setminus \{1\}$ with $\gamma b, \gamma^{-1} b \in A$, then we would contradict maximality of $\rho(a, b)$ by the inequality of the previous paragraph. This shows that for all $a \in T$, we have $\Delta_\Gamma(A) \setminus \{a\} \neq \emptyset$. Thus $\operatorname{card} \Delta_\Gamma(A) \geq 2$. It follows that X is diffuse as required.

We finally need to prove Theorem 1.3. Both parts are simple variations on the corresponding parts of Theorem 1.2.

To prove Theorem 1.3(1), let f be the Γ-set morphism from Γ/H to Γ/G which sends the coset aH to aG. The stabiliser of a point of Γ/G is conjugate, and hence isomorphic, to G. Viewed as a G-set, its preimage in Γ/H is isomorphic to G/H. Since H is concentrated in G, this preimage is diffuse. Since G is concentrated in Γ, the Γ-set Γ/G is also diffuse. Thus, Lemma 2.1 tells us that Γ/H is diffuse and so H is concentrated in Γ as required. (We have already noted that Theorem 1.3(1) is also a corollary of Theorem 1.3(2), though this is perhaps not such a natural way of viewing it.)

The proof of Theorem 1.3(2) is again just a variation on Theorem 1.2(2). Let T be the Bass-Serre tree corresponding to the splitting. If $x \in V(T)$, then $E(x)$ is a disjoint union of $\Gamma(x)$-sets, each isomorphic to $\Gamma(x)/\Gamma(e)$ for some $e \in E(x)$. Thus $E(x)$ is diffuse as a Γ-set, and so Proposition 2.2 tells us that $V(T)$ and $E(T)$ are both diffuse. If $x \in V(T)$, then the Γ-orbit of x in $V(T)$ is isomorphic to $\Gamma/\Gamma(x)$ as a Γ-set. Thus, $\Gamma/\Gamma(x)$ is diffuse, so $\Gamma(x)$ is concentrated in Γ. Now, suppose $G \leq \Gamma$ is concentrated in $\Gamma(x)$. By trivially refining our splitting if necessary (using $\Gamma(x) \cong \Gamma(x) *_G G$), we can assume that G is, itself, a vertex of our splitting, and hence concentrated in Γ, by what we have already proven. Finally, to see that Γ is itself diffuse, take G to be the trivial group. Since $\Gamma(x)$ is diffuse, $\{1\}$ is concentrated in $\Gamma(x)$ and hence in Γ. Thus, Γ is diffuse. This completes the proof of Theorem 1.3.

3. Surface groups.

We noted in the indroduction that surface groups (with the exception of the projective plane) are locally indicable, hence right orderable and hence diffuse. However proofs of right-orderability tend to be non-constructive. Alternatively, it is well known that most (compact) surface groups act freely on **R**-trees, and hence diffuse by Theorem 1.2(3). This is a constructive, but still "infinite", procedure. In this section, we give a more explicit proof of diffusion using Theorem 1.3(2). In the course of the argument, we will describe when simple closed curves support concentrated subgroups.

Let P^2 denote the projective plane, and nP^2 denote the connected sum of n copies thereof. The only surface groups that do not act freely on any $\mathbf{R}$-tree are $\pi_1(P^2)$, $\pi_1(2P^2)$ and $\pi_1(3P^2)$. Now, $\pi_1(P^2) = \mathbf{Z}_2$ is finite, and so certainly not diffuse. We shall see below that the remaining examples are diffuse.

Suppose that C is an essential (i.e. not homotopically trivial) simple closed curve on a surface, S. We shall identify the infinite cyclic group $\pi_1(C)$ with its image in $\pi_1(S)$. As a subgroup if $\pi_1(S)$, this is well-defined up to conjugacy. It is convenient to allow S to have boundary. If we like, we can assume C to lie in the interior of S, and in this case, it can be either one-sided or two-sided. If it is two-sided, it can be either separating or non-separating. If C bounds a Möbius band in S, then $\pi_1(C)$ lies in the strictly larger cyclic subgroup of $\pi_1(S)$ supported by the band. It follows that, in this case, $\pi_1(C)$ cannot be concentrated in $\pi_1(S)$. We shall see that there is only one other exceptional case.

In the statement of the theorem we do not assume that S is compact. However, the extra generality is spurious, since $\pi_1(S)$ can always be expressed as an increasing union of compact surface groups. We therefore give the proof under the assumption that S is compact.

Theorem 3.1 : *Let S be a surface other than the projective plane. Then $\pi_1(S)$ is diffuse. Suppose C is an essential simple closed curve on S which does not bound a Möbius band. Then, $\pi_1(C)$ is concentrated in $\pi_1(S)$, unless S is the Klein bottle and C is one-sided.*

Proof : A "pair of pants" is the compact surface consisting of a sphere with three open discs removed. By a "twisted pants" we mean a projective plane with two discs removed. In either case, the fundamental group is free of rank 2. Moreover, if C is a boundary component, then the free generators can be chosen so as to include the generator of $\pi_1(C)$. It follows (by Theorem 1.3(2), or by a simple direct argument) that the theorem holds in this case.

Now, suppose that S is any compact surface of negative Euler characteristic. Then, we can find a system of two-sided simple closed curves on S which cut S into pairs of pants and twisted pants. This gives a representation of $\pi_1(S)$ as a graph of groups where the edge groups are supported on these curves. Thus, by Theorem 1.3(2), $\pi_1(S)$ is diffuse. Moreover, if C is a boundary curve of S, then $\pi_1(C)$ is a concentrated subgroup of one of the vertex groups, and is hence concentrated in $\pi_1(S)$.

Now, assume that S is again of negative Euler characteristic, and that C is an essential non-peripheral two-sided curve in the interior of S, which does not bound a Möbius band. Then, cutting S along C, we obtain one or two surfaces, each again of negative Euler characteristic. Thus, applying the result of the previous paragraph together with Theorem 1.3(2), we see that $\pi_1(C)$ is diffuse in $\pi_1(S)$.

Now, suppose that C is one-sided. Then C lies in a twisted pants in S, so Theorem 1.3(2) reduces us to this case. It is easy to see that we can choose free generators for the fundamental group of a twisted pants so as to include the generator of the subgroup supported on any given one-sided curve. This deals with all cases where the Euler characteristic is negative.

It remains to deal with the cases where the Euler characteristic of S is nonnegative. The only one that calls for comment is the Klein bottle, $2P^2$. We can write $\Gamma = \pi_1(2P^2)$ as an HNN extension $\langle a, t \mid tat^{-1} = a^{-1} \rangle$. The elements a, t and t^2 correspond, respectively, to separating, two-sided non-separating, and one-sided simple closed curves. The HNN presentation tells us (by Theorem 1.3(2)) that Γ is diffuse, and that $\langle a \rangle$ is concentrated in Γ. This completes the proof. $\diamondsuit$

We can also represent $\Gamma = \pi_1(2P^2)$ as an amalgamated free product, $\langle t, u \mid t^2 = u^2 \rangle$, by putting $u = at$, in the notation of the last paragraph of the proof. Since, $\langle t^2 \rangle \cup u\langle t^2 \rangle$ is $\langle u \rangle$-invariant, we see that $\langle t^2 \rangle$ is not diffuse in Γ. Thus, the case of a one-sided curve on a Klein bottle is genuinely exceptional.

4. Cyclic subgroups of free groups.

In this section, we show that a "generic" element of a free group generates a concentrated subgroup. There is no doubt scope for improving this result. Indeed it might be true for any element which is not a proper power. We note however, that it is sufficient to give an alternative way of dealing with most surface groups from that described in the last section. It also gives rise to many more examples of diffuse groups as amalgamated free products, etc. — though how many of these are of genuine interest is less clear.

Let Γ be a free group. A conjugacy class in Γ can be canonically represented as a cyclically reduced cyclic word in the free generators of Γ. We say that such a word has a *square factor* if (up to cyclic reordering) it has the form $x^2 y$, where $x, y \in \Gamma$, and $x \neq 1$. We say it is *square-free* if it has no square factor. Of course, this will depend on the free generating set. We can allow for a more general definition if we reinterpret this in terms of graphs.

Let Z be a connected graph. We can allow Z to have loops and multiple edges. Its fundamental group, $\Gamma = \pi_1(Z)$, is free. A path in Z can be thought of as a sequence of directed edges of Z, with the condition that the terminal vertex of each edge equals the initial vertex of the following edge. We denote by $\pi\xi$ the concatenation of the paths π and ξ, whenever this is possible. A *cycle* is a cyclic path. We say that a cycle has *no backtracking* if no edge is immediately followed by the same edge with the opposite orientation. Thus, a conjugacy class in Γ is canonically represented by a cycle with no backtracking. We say that a cycle, σ, is *square-free* if it cannot be written in the form $\pi^2\xi$, where π and ξ are subpaths with all endpoints equal, and where π is non-trivial. ¿From this, it can be deduced that if π^2 is a subpath of σ^n for any n, then $\pi = \sigma^m$ for some $m \geq 0$. Note that our original definition in terms of free generators corresponds to taking Z to be a wedge of circles.

Let T be the universal cover of Z. Thus, T is a simplicial tree on which Γ acts freely. Suppose σ is a cycle in Z. Then, σ generates an infinite cyclic subgroup of Γ, well defined up to conjugacy. We assume this to be non-trivial. Let Π be the set of lifts of σ to T. We can assume that σ has no backtracking, and so each element of Π is a bi-infinite arc. Distinct elements of Π intersect, if at all, in a finite interval or point. We shall suppose that

σ is not a proper power. In this case, the "square-free" hypothesis can be reinterpreted as saying that if $\gamma \in \Gamma$ with $\gamma p \neq p$ then $p \cap \gamma^2 p = \emptyset$.

Proposition 4.1 : *Let Z be a connected graph, and suppose $g \in \pi_1(Z)$ is represented by a square-free cycle (with no backtracking) in Z. Then, the cyclic subgroup, $\langle g \rangle$, is concentrated in $\pi_1(Z)$.*

Proof : Let $\Gamma = \pi_1(Z)$, and let T be the universal cover of Z. Let Π be the set of lifts of the cycle corresponding to g. Note that, if $p \in \Pi$, then its stabiliser $\Gamma(p)$, is conjugate to $\langle g \rangle$. Thus, Π is a Γ-set isomorphic to $\Gamma / \langle g \rangle$. We therefore want to show that Π is diffuse.

Suppose, for contradiction, that $A \subseteq \Pi$ satisfies $2 \leq \operatorname{card} A < \infty$ and $\operatorname{card} \Delta_\Gamma(A) \leq 1$. We first observe that there exist $p, p' \in A$ with $p \cap p' = \emptyset$. To see this, choose any $r \in A \setminus \Delta_\Gamma(A)$. There is some $\gamma \in \Gamma$ with $\gamma r, \gamma^{-1} r \in A$ and with $\gamma r \neq r$. Now set $p = \gamma r$ and $p' = \gamma^{-1} r$. Thus, $p \cap p' = \gamma r \cap \gamma^{-2}(\gamma r) = \emptyset$.

We now choose $q \in A$ so as to maximise the combinatorial distance to p (i.e. the number of edges in the arc connecting p to q). It is possible that $h^m q \in A$ for a (finite) number of $m \in \mathbf{Z}$, where h is a generator of $\Gamma(p)$. However, we can clearly choose q such that all such m are nonnegative. We now claim that $q \in \Delta_\Gamma(A)$. Suppose, to the contrary, that there is some $\gamma \in \Gamma$ with $\gamma q, \gamma^{-1} q \in A$ and with $\gamma q \neq q$. We see immediately that $\gamma \notin \Gamma(p)$. The "square-free" hypothesis tells us that $\gamma q \cap \gamma^{-1} q = \emptyset$ and $\gamma p \cap \gamma^{-1} p = \emptyset$. Let s be the axis of γ in T. Given any $r \in A$, let $I(r)$ be the projection of r to s (i.e. $r \cap s$ if $r \cap s \neq \emptyset$, otherwise the nearest point on s to r). Thus, $I(r)$ is a point or finite subarc of s, unless it happens that $r = s$, in which case $I(r) = s$. Since $p \cap q = \emptyset$, $\gamma p \cap \gamma^{-1} p = \emptyset$ and $\gamma q \cap \gamma^{-1} q = \emptyset$, we see easily that $I(p)$ cannot meet both $I(\gamma q)$ and $I(\gamma^{-1} q)$. Now, $I(\gamma q)$, $I(q)$ and $I(\gamma^{-1} q)$ are just translated copies of some finite interval in s, so we can assume, without loss of generality, that $I(\gamma q)$ is strictly further than $I(q)$ from $I(p)$. Thus, γq is strictly further than q from p, contradicting the choice of q. This shows that $q \in \Delta_\Gamma(A)$ as claimed. We have shown that $\Delta_\Gamma(A) \setminus \{p\} \neq \emptyset$.

Now, replacing p by p' we see similarly that $\Delta_\Gamma(A) \setminus \{p'\} \neq \emptyset$. Thus $\operatorname{card} \Delta_\Gamma(A) \geq 2$. This shows that Π is diffuse, as required. $\diamondsuit$

We may use this result to show that if S is a surface other than the Möbius band, and C is a boundary curve of S, then $\pi_1(C)$ is concentrated in $\pi_1(S)$, thereby bypassing part of the proof of Theorem 3.1. Except in the trivial cases of the disc and annulus, we can take Z to be a spine for the surface, with no loops, and with every vertex trivalent. We see that the boundary curve, C, gives a square-free cycle in Z, and so the result follows.

5. Hyperbolic spaces.

In this section, we explain how some of the arguments of Section 3 can be applied to actions on hyperbolic spaces which have large translation lengths (cf. [Del]). This is based on the observation that the proof of Theorem 1.2(3) is founded on a simple metric criterion. We thus have:

Lemma 5.1 : *Suppose that a group Γ acts isometrically on a metric space, (X, ρ). Suppose that for any $\gamma \in \Gamma$ and $a, x \in X$, the inequality $\rho(a, x) \geq \max\{\rho(a, \gamma x), \rho(a, \gamma^{-1}x)\}$ implies that $\gamma x = x$. Then X is diffuse as a Γ-set.*

Proof : Suppose $A \subseteq X$ is finite and non-empty. Choose $x, y \in A$ so as to maximise $\rho(x, y)$. As in the proof of Theorem 1.2(3), we see that $\{x, y\} \subseteq \Delta_\Gamma(A)$. ♢

In particular, if some point of X has trivial stabiliser, then Γ is itself diffuse.

Note that the hypothesis of 5.1 can be rephrased as follows. Given distinct points $a, b \in X$, let $H(a, b)$ the closed "half-space" of points closer to b, i.e. $H(a, b) = \{x \in X \mid \rho(x, b) \leq \rho(x, a)\}$. The criterion states that if $\gamma \in \Gamma$ and $x \in X$ is not fixed by γ, then $H(x, \gamma x) \cap H(x, \gamma^{-1}x) = \emptyset$.

We can apply this immediately to hyperbolic path metric spaces in the sense of Gromov [G]. Suppose that X is Gromov hyperbolic, and Γ acts isometrically on X. Given $\gamma \in \Gamma$, set $l(\gamma) = \inf\{\rho(x, \gamma x) \mid x \in X\}$. We have:

Corollary 5.2 : *Given any $k \geq 0$, there is a constant $K(k) \geq 0$, with the property that if Γ acts freely on a Gromov hyperbolic space of hyperbolicity constant k such that for all $\gamma \in \Gamma \setminus \{1\}$ we have $l(\gamma) \geq K(k)$, then Γ is diffuse.*

Proof : Given the treelike nature of Gromov hyperbolic spaces, it is a simple exercise to adapt the argument of Section 3 to verify the hypotheses of Lemma 5.1. ♢

In fact, the constant $K(k)$ can be taken to be a fixed multiple of k. In [Del], it is shown directly that a group satisfying the hypothesis of Corollory 5.2 has the t.u.p. property, and an explicit bound on $K(k)$ is given.

In the case of actions on constant curvature hyperbolic space, $\mathbf{H}^n$, we can be more specific. Note that, in this case, any isometry, γ, with $l(\gamma) > 0$, is loxodromic, and translates its axis a distance $l(\gamma)$.

Theorem 5.3 : *Suppose that M is a complete hyperbolic (i.e. constant curvature -1) manifold of injectivity radius greater than $\log(1 + \sqrt{2})$. Then, $\pi_1(M)$ is diffuse.*

Proof : $\Gamma = \pi_1(M)$ acts freely on $\mathbf{H}^n$. If $\gamma \in \Gamma \setminus \{1\}$, then $l(\gamma) > 2\log(1 + \sqrt{2})$. The proof therefore follows from the following geometric lemma. ♢

Lemma 5.4 : *Suppose that γ is a loxodromic isometry of $\mathbf{H}^n$, and that there exists $x \in \mathbf{H}^n$ with $H(x, \gamma x) \cap H(x, \gamma^{-1}x) \neq \emptyset$. Then, $l(\gamma) \leq 2\log(1 + \sqrt{2})$.*

Proof : Let β be the axis of γ. Clearly $x \notin \beta$. Let y be the nearest point on β to x. Let α^- be the geodesic ray based at y and containing the point x. Let α^+ be the ray based at y in the opposite direction. Let σ be the involution on $\mathbf{H}^n$ which reflects everything through the geodesic $\alpha = \alpha^- \cup \alpha^+$. Thus, $\sigma x = x$ and $\sigma\gamma\sigma = \gamma^{-1}$, and so $\sigma H(x, \gamma x) = H(x, \gamma^{-1}x)$. Now, if b lies in $H(x, \gamma x) \cap H(x, \gamma^{-1}x)$, then so does σb, and hence, by the convexity of these half-spaces, so also does the midpoint, a, of the geodesic segment $[b, \sigma b]$. By the definition of σ, the point a lies on α. Clearly $a \notin \alpha^-$, so we have shown that $\alpha^+ \cap H(x, \gamma x) \cap H(x, \gamma^{-1} x) \neq \emptyset$.

Now, let δ be the loxodromic isometry with axis β, whose restriction to β agrees with γ, and which sends each normal vector to β to the antipode of its parallel translate along β. Now, γx and δx both lie in a $(n-2)$-dimensional sphere centred on $\gamma y = \delta y$ and contained in a hyperplane orthogonal to β. Also δx lies in the 2-plane, π, containing $\beta \cup \alpha$, and is on the same side of β as the point a. ¿From this it follows that $\rho(a, \delta x) \leq \rho(a, \gamma x)$. Note also that $\rho(a, \delta^{-1} x) = \rho(a, \delta x)$ and $\rho(a, \gamma^{-1} x) = \rho(a, \gamma x)$. Thus, replacing γ by δ and $\mathbf{H}^n$ by $\pi \equiv \mathbf{H}^2$, we are reduced to the 2-dimensional situation, where γ is a glide reflection with axis β.

Let $m \in \beta$ be the midpoint of the segment $[y, \gamma y]$, which is also the midpoint of $[x, \gamma x]$. Now, the boundary of $H(x, \gamma x)$ is a geodesic through m meeting the geodesic segment $[x, m]$ orthogonally. Since $\alpha^+ \cap H(x, \gamma x) \neq \emptyset$, this geodesic must meet α^+ in some point c. In summary, we have a triangle, xmc, with a right angle at m, and with y as the orthogonal projection of m to the side $[x, c]$. We claim that $\rho(y, m) \leq \log(1 + \sqrt{2})$. This is simple trigonometry, observing that one of the angles xmy or cmy is at least $\pi/4$. Suppose, for example that it is xmy. The worst case would be if x were an ideal point, and $xmy = \pi/4$. Since $xym = \pi/2$, we would have $\cosh \rho(y, m) \sin(\pi/4) = 1$, and so $\rho(y, m) = \log(1 + \sqrt{2})$, proving the claim.

Finally, note that $l(\gamma) = \rho(y, \gamma y) = 2\rho(y, m) \leq 2 \log(1 + \sqrt{2})$ as required. ◊

It seems likely that one should be able to adapt the arguments of Theorem 5.3 to prove a similar result in the more general context of CAT(-1) spaces, though we shall not pursue that here.

6. A non-diffuse group.

We briefly comment on the torsion-free group, Γ, which Promislow showed to be non-u.p. by explicitly exhibiting a 14-element subset, S, such that SS has no unique products [Pr]. I don't know how Promislow found his subset, and I've not been able to improve on it by hand calculation. It is, however, easy to see that Γ is not (weakly) diffuse. One can find a 6-element subset, A, such that $\Delta(A) = \emptyset$. (This is the smallest such set one can hope for in a torsion-free group.) It's perhaps most illuminating to describe this set geometrically.

One can define Γ by the presentation $\langle a, b \mid ba^2b^{-1} = a^{-2}, \quad ab^2a^{-1} = b^{-2} \rangle$. The subgroup generated by $\{a^2, b^2, (ab)^2\}$ is normal and isomorphic to $\mathbf{Z}^3$. The quotient is the Klein-four group.

In fact, Γ is well-known as one of the six 3-dimensional torsion-free orientation preserving crystallographic groups. (The other five are all extensions of $\mathbf{Z}^2$ by $\mathbf{Z}$.) To express it in this way, we use coordinates (x_1, x_2, x_3) for $\mathbf{R}^3$, and interpret subscripts mod 3. Let T_i be the affine isometry defined by the coordinate transformations $[x_i \mapsto 1 + x_i]$, $[x_{i+1} \mapsto 1 - x_{i+1}]$, $[x_{i+2} \mapsto -x_{i+2}]$. Thus, T_i is a glide translation with axis given by $x_{i+1} = \frac{1}{2}$, $x_{i+2} = 0$, and T_i^2 is a translation. Now, $\{T_1^2, T_2^2, T_3^2\}$ generates our free-abelian subgroup. Every element of Γ is either a translation or glide translation, and the axes of the glides form three non-intersecting systems of lines parallel to the coordinate axes. To get the above presentation, set $a = T_1$ and $b = T_2$.

The lattice $\Lambda = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \equiv 1 \bmod 2\}$ can be viewed as a Γ-set isomorphic to Γ itself. Let $A = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. It is easy to see that $\Delta_\Gamma(A) = \emptyset$. For example, $T_1^{\pm 1}(0, 1, 0) = (\pm 1, 0, 0)$, and so $(0, 1, 0)$ lies "between" $(-1, 0, 0)$ and $(1, 0, 0)$. Combinatorially, each of a pair of opposite vertices of A will lie between a second pair of opposite vertices. The elements of this second pair, in turn, both lie between the third pair. We finally cycle back to the first pair of points. This is the simplest possible combinatorial arrangement for a set violating the weak diffusion assumption in any torsion-free group. (A fairly simple argument shows that any other arrangement must involve more than 6 elements.)

Note that Γ has three index-2 subgroups, respectively generated by the subsets $\{T_i, T_{i+1}^2, T_{i+2}^2\}$. Such a group can be expressed as an extension of $\mathbf{Z}^2$ (generated by $\{T_{i+1}^2, T_{i+2}^2\}$) by $\mathbf{Z}$ (generated by T_i). It is therefore diffuse. Thus, the property of diffusion is not passed to order-2 extensions. In particular, it's not a commensurability invariant of torsion-free groups.

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