Archimedean actions on median pretrees

By Brian H. Bowditch

Faculty of Mathematical Studies, University of Southampton,
Southampton, SO17 1BJ, UK
e-mail: bhb@maths.soton.ac.uk

and John Crisp

Laboratoire de Topologie, Université de Bourgogne, UMR 5584 du CNRS,
B.P. 47 870, 21078 Dijon, France

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Abstract

In this paper we consider group actions on generalised treelike structures (termed "pretrees") defined simply in terms of betweenness relations. Using a result of Levitt, we show that if a countable group admits an archimedean action on a median pretree, then it admits an action by isometries on an ℝ-tree. Thus the theory of isometric actions on ℝ-trees may be extended to a more general setting where it merges naturally with the theory of right-orderable groups. This approach has application also to the study of convergence group actions on continua.

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1 Introduction

An ℝ-tree can be defined as a metric space in which every pair of points are connected by a unique arc, and where each such arc is isometric to a real interval. They were introduced by Morgan and Shalen, and the theory of isometric actions on ℝ-trees, as developed by Rips and others, has become a powerful tool in geometric group theory. (See for example [**P**] and [**Be**] for recent surveys.)

For some applications (eg. [**Bo2**]), it has become important to generalise these results to groups acting by homeomorphisms. The strongest result in this direction [**Le**] shows how a topological action (that is, an action by homeomorphisms) on an ℝ-tree which satisfies the non-nesting condition gives rise to an action by isometries on a related ℝ-tree, so that the Rips theory applies. (An action is "non-nesting" if no element maps any interval into a proper subinterval of itself.)

More generally still, we show in this paper (using Levitt's result), how the same theory applies to "archimedean actions" on "median pretrees". This would also seem to be the natural context in which to phrase various results concerning the application of actions on treelike structures to studying the structure of certain groups. A median pretree is a set with a ternary betweenness relation satisfying a set of axioms which describe its treelike nature. This notion has appeared many times in the literature under a variety of different names, one of the earliest references being [**Sh**]. (See, for example, [**Bo1**] for other references.) An action on such a tree is "archimedean" if, given any point of the tree and any group element, then any interval in the tree contains only finitely many images of this point under iterates of the given element. For topological actions on $\mathbb{R}$-trees this is equivalent to the non-nesting hypothesis, but for general median pretree automorphisms it is slightly more restrictive.

Formulated in this general way, the Rips machinery can be applied directly to median pretees, without the necessity of explicitly reproducing the constructions of topological actions on $\mathbb{R}$-trees. It can thus be used to streamline some of the arguments of [**Bo2**] etc. This is particulary relevant to the analysis of actions of groups on certain continua, such as boundaries of hyperbolic and relatively hyperbolic groups, where cutpoints play an important role. In this context, pretrees arise quite naturally (cf. [**W**]).

By an *edge* of a median pretree T we mean any subset which consists of a pair of distinct points and all of the points which lie between them. (The edges of an $\mathbb{R}$-tree are precisely the subsets homeomorphic to $[0,1] \subset \mathbb{R}$.) Given an action of a group Γ on T, we write $\Gamma(x)$ for the stabilizer of a point $x \in T$, and let $\mathcal{E}_T$ denote the set $\{\Gamma(x) \cap \Gamma(y) : x \neq y \in T\}$ of edge stabilizers. The action is said to be *nontrivial* if no point is fixed by the whole group, and *2-nontrivial* if every Γ-orbit contains strictly more than 2 points. These notions are equivalent for actions on $\mathbb{R}$-trees. Our main result can be stated as follows:

Theorem 1.1. *If a countable group Γ admits a 2-nontrivial archimedean action by pretree automorphisms on a median pretree T then it admits a nontrivial archimedean (equivalently, nontrivial non-nesting) action by homeomorphisms on an $\mathbb{R}$-tree Ψ. If $\mathcal{E}_T$ does not contain any infinite ascending chain, then we may arrange that $\mathcal{E}_\Psi \subseteq \mathcal{E}_T$.*

The construction is performed by taking a countable Γ-invariant median subpretree Q of T and by showing that Q embeds Γ-equivariantly in an $\mathbb{R}$-tree Ψ. The condition that $\mathcal{E}_T$ contains no infinite ascending chain implies that the stability condition of Bestvina and Feighn holds for the action of Γ on T (see [**BeF**], Proposition 3.2). Note that if $\mathcal{E}_T$ does not contain any infinite ascending chain, then neither does $\mathcal{E}_\Psi$ and so the action on Ψ will also be stable in the sense of [**BeF**].

In view of the Theorem of Levitt, [**Le**], which constructs an isometric action on some $\mathbb{R}$-tree Σ from the homeomorphic action on Ψ, we now have:

Theorem 1.2. *If a finitely presented group Γ admits a 2-nontrivial archimedean action by pretree automorphisms on a median pretree T then it admits a nontrivial action by isometries on an $\mathbb{R}$-tree Σ. If $\mathcal{E}_T$ does not contain any infinite ascending chain, then any subgroup of Γ fixing a pair of distinct points (i.e : an edge) in Σ also fixes a pair of distinct points in T.*

We observe that if $G_1, \dots, G_n$ is a finite set of finitely generated subgroups of Γ, each fixing some point of T, then the constructions used in the following sections allow us to choose the $\mathbb{R}$-tree Σ of Theorem 1.2 so that each group G_i fixes a point of Σ (c.f: [**Le**], Corollary 6). Combining Theorem 1.2 with the work of Bestvina and Feighn on stable isometric actions, [**BeF**] Theorem 9.5, we have the following.

Theorem 1.3. *Suppose that Γ is a finitely presented group with a 2-nontrivial archimedean action on a median pretree T such that $\mathcal{E}_T$ does not contain any infinite ascending chain and every element of $\mathcal{E}_T$ is slender. Then, either Γ preserves setwise an arc in T and is an extension of the (pointwise) stabilizer of this arc by an abelian group, or else it splits over a G-by-cyclic group where G fixes an edge of T. In the latter case, if $F \subseteq T$ is any finite set of points with finitely generated stabilizers then one can take the splitting to be relative to $\{\Gamma(x) \mid x \in F\}$.*

A group is said to be *slender* if all its subgroups are finitely generated. The assumption that each element of $\mathcal{E}_T$ is slender is then equivalent to saying that every group fixing an edge of T is finitely generated. Note that all polycyclic groups are slender.

Theorem 1.3 illustrates how certain results about isometric actions on $\mathbb{R}$-trees giving rise to group splittings (see [**P**, **Be**]) should generalise directly to actions on median pretrees. One complication is that these results typically require the (isometric) action to be stable, and it is not known whether stability of the action is a property which is preserved by Levitt's construction. This is why the hypothesis that edge stabilizers be slender is needed for Theorem 1.3. However, for many applications (e.g. those of [**Bo2**]), the issue of whether the action is stable does not even arise, since all relevant actions on $\mathbb{R}$-trees are necessarily stable. In these cases, the class of subgroups that arise as edge stabilisers contains no infinite ascending chain.

The study of group actions on median pretrees also merges naturally with the theory of right orderable groups. For example, there is a classical result which states that a right ordered group is archimedean if and only if it is a subgroup of the additive reals (proven with varying degrees of generality by Hölder, Frege and Huntington – see [**ADN**] for a discussion). As a result of Theorem 1.3 we now have the following (compare with [**Le**]):

Corollary 1.4. *If a finitely presented group of order greater than 2 admits a free archimedean action on a median pretree T then either it is free abelian of finite rank or it splits over a cyclic group (which is either trivial, order 2, or infinite cyclic). The two cases correspond to whether or not the group leaves invariant a totally ordered subtree of T while respecting the order.*

We remark that there are examples of groups with free archimedean actions on $\mathbb{R}$-trees, but with no free isometric actions [**Li**].

It seems likely that one could replace the assumption of finite presentability with one of finite generation in Theorems 1.2 and 1.3 and Corollary 1.4. However, Levitt uses finite presentability in his proof, and we have not attempted to generalise his argument.

2 Median pretrees

In this section, we recall some basic facts about (median) pretrees. Median pretrees, under the name of "trees" were described by Sholander [**Sh**] in terms of a betweenness relation (slightly different from the formulation given below). The term "pretree" is used in [**Bo1**] to describe the more general structure defined by Ward [**W**] and Adeleke and Neumann [**AN**]. Further elaboration on some of the statements made here can be found in [**Bo1**].

A *pretree* is a set, T, together with a ternary relation, denoted xyz for $x, y, z \in T$, satisfying the following axioms:

(T0) xyz implies $x \neq z$,

(T1) xyz holds if and only if zyx holds,

(T2) xyz and xzy cannot hold simultaneously, and

(T3) if xyz holds and $w \neq y$, then either xyw or wyz holds.

A *(pretree) automorphism* of T is a bijection $T \to T$ which respects the ternary relation.

The intuitive interpretation of the statement xyz is that y lies strictly "between" x and z. The axioms express the treelike nature of this betweenness relation. Specifically, we have the following useful lemma ([**AN**] or [**Bo1**]):

Lemma 2.1. *If $F \subseteq T$ is finite, then we can embed F in a finite simplicial tree, Σ, such that if $x, y, z \in F$, then xyz holds in T if and only if y separates x from z in Σ.*

(There are variations on this result for arbitrary subsets of T — see [**C**].)

Given $x, y \in T$, we write $(x, y) = \{z \in T \mid xzy\}$, $[x, y) = (y, x] = \{x\} \cup (x, y)$ and $[x, y] = \{x, y\} \cup (x, y)$, and refer to such subsets as *intervals*. The points x and y are called *limit points* of the interval. An interval which may be written $[x, y]$ is said to be *closed* with *endpoints* x and y. Note that $[x, y] = [u, v]$ implies that $\{x, y\} = \{u, v\}$. On the other hand, we remark that a given interval (even a closed one) may be defined in terms of more than one pair of limit points. This will become an issue, for example, in the proof of Proposition 3.10.

A *median* of $x, y, z \in T$ is an element of $[x, y] \cap [y, z] \cap [z, x]$. If such a point exists, then it is unique (for example, using Lemma 2.1). In such a case, we write it as $\mathrm{med}(x, y, z)$.

Definition 2.2. A *median pretree* is a pretree in which every triple of points has a median.

Given a pretree, T, and points $x, y, z \in T$, we say that x, y and z are *collinear* if at least two of them are equal or if xyz or yzx or zxy holds. A subset, $A \subseteq T$ is *linear* if every three points of A are collinear. A *direction* on a linear set is a total order, $<$, such that $x < y < z$ implies xyz. A *directed linear set* is a linear set together with a direction. Each nontrivial linear set has precisely two directions. An *endpoint* of a linear set is a maximal or minimal element under such an order. Given $x, y, z, w \in T$, we write $xyzw$ to mean xyz and yzw (which implies also xyw and xzw). We extend this convention to larger linear sets.

A subset A of T is *full* if $[x, y] \subseteq A$ for all $x, y \in A$. An *arc* is a nonempty full linear subset. Every interval is an arc. (In the context of $\mathbb{R}$-trees the word "arc" is sometimes used only to refer to a nontrivial closed interval, however we will avoid this terminology by referring to a nontrivial closed interval as an *edge*.) An arc is *maximal* if it is not contained in any larger arc, and *bounded* if it is contained in an interval.

A pretree is *complete* if every arc is an interval. More commonly, it is *interval complete* if every closed interval is complete (equivalently, if every bounded arc is an interval). Note that for arcs, this weaker notion of completeness coincides with the standard one for total orders.

A subset Q of a pretree T is *dense* in T if, given any distinct $x, y \in T$, $(x, y) \cap Q \neq \emptyset$. We say that a pretree is *dense* if it is dense in itself. Note that if Q is dense in T then any subset containing Q is also dense in T. In particular, T is dense. However, Q being dense (in itself) does not necessarily imply that Q is dense in T. A dense interval complete pretree is necessarily median (by the same argument given in [**Bo1**], Lemma 2.11, for dense complete pretrees). A pretree T is *separable* if it has a countable subset which is dense in T. In particular, a separable pretree is necessarily dense.

Definition 2.3. A *real pretree* is one where every interval is order isomorphic to a subset of the real line. Such a pretree is dense and interval complete, and hence median (cf. [**Bo1**], Lemma 2.11). Also, any interval complete separable pretree is a real pretree (cf. [**Bo1**], Lemma 2.13). We remark that in fact, any interval complete separable pretree can be given the structure of an $\mathbb{R}$-tree [**MO**, **Bo1**], and in such a way that every pretree automorphism is a homeomorphism (but NOT an isometry).

There is a canonical procedure for embedding any pretree in a complete median pretree (described in [**Bo1**] using "flows"). In Section 4 we build on this to obtain a canonical procedure for embedding a countable median pretree in a real pretree (c.f: Theorem 4.5).

Suppose that T is a median pretree and that $A \subseteq T$ is full. If $x \in T$, we say that $b \in A$ is a *projection* of x to A if $[x, b] \cap A = \{b\}$. If such a projection exists, then it is unique. Moreover, it is characterised by the fact that $b \in [x, c]$ for all $c \in A$. Also, if $a, b, c \in A$ such that abc then b is a projection of x to A if and only if $b = \mathrm{med}(a, x, c)$.

Definition 2.4. Let α be a directed linear subset of the median pretree T, and let $x \in T$. We write $\{x\} \prec \alpha$ if xbc for all $b < c$ in α, and $\alpha \prec \{x\}$ if bcx for all $b < c$ in T. (In fact, this extends to define a relation amongst directed linear sets by putting $\alpha \prec \beta$ if both $\{a\} \prec \beta$ for all $a \in \alpha$ and $\alpha \prec \{b\}$ for all $b \in \beta$. However we shall not need this generality in what follows).

Define subtrees $T^+(\alpha) = \{x \in T : \alpha \prec \{x\}\}$, and $T^-(\alpha) = \{x \in T : \{x\} \prec \alpha\}$, and let $T^0(\alpha)$ denote the subtree of T consisting of those elements whch have a projection to α.

Lemma 2.5. *Let α be a directed arc of the median pretree T. Then, for every $x \in T$, either $\{x\} \prec \alpha$ or $\alpha \prec \{x\}$ or x has a projection to α. That is, T is the union of subtrees $T^-(\alpha)$, $T^+(\alpha)$ and $T^0(\alpha)$.*

Proof. Suppose throughout that x has no projection to α. Then $x \notin \alpha$ and α is nontrivial (for if $\alpha = \{a\}$ then a is a projection for every $x \in T$). Also, given elements $b < c$ of α we have either xbc or bcx, for otherwise $\mathrm{med}(b, x, c)$ lies in (b, c) and so is a projection of x to α.

Now suppose that neither $\{x\} \prec \alpha$ nor $\alpha \prec \{x\}$. The first condition implies that there exists $b < c$ such that xbc fails, and therefore bcx holds. The second implies, similarly, that there exist $d < e$ such that xde. Let $m = \mathrm{med}(c, x, d)$. Then m lies in α with $b < c \le m \le d < e$ and must be a projection of x to α, a contradiction. □

It's easily seen that $T^0(\alpha)$ is a full subset of T. If α has a maximal endpoint m then $T^0(\alpha)$ will also contain $T^+(\alpha)$, all of whose elements project to m, but one cannot expect $T^+(\alpha)$ to be a full subtree in this case. A similar statement holds for $T^-(\alpha)$ in the case of a minimal endpoint of α. However, one does have the following.

Lemma 2.6. *If α is a directed arc with no endpoints, in a median pretree T, then $T^-(\alpha)$, $T^+(\alpha)$ and $T^0(\alpha)$ are mutually disjoint full subsets of T (whose union is T). Moreover, the subsets $T^-(\alpha)$ and $T^+(\alpha)$ have full complements in T.*

Proof. If x has a projection $b \in \alpha$ then since b is not an endpoint one can find $a, c \in \alpha$ such that $a < b < c$, from which it follows that neither $\{x\} \prec \alpha$ nor $\alpha \prec \{x\}$. Thus $T^0(\alpha)$ is disjoint from $T^-(\alpha)$ and $T^+(\alpha)$. It is clear that $T^-(\alpha)$ is disjoint from $T^+(\alpha)$. We now show that these sets are full.

Suppose $x, y \in T^+(\alpha)$ and take $z \in [x, y]$. For any $b, c \in \alpha$ with $b < c$, we have bcx and bcy and, applying Lemma 2.1 to the set $\{b, c, x, y, z\}$, we see that either bcz or $z = c$. If the latter case ever arises, it follows that $c = \max \alpha$ is an endpoint of α, a contradiction. Thus $z \in T^+(\alpha)$ and $T^+(\alpha)$ is full. The argument for $T^-(\alpha)$ is identical.

Finally, observe that if x^-, x^0 and x^+ are elements of $T^-(\alpha)$, $T^0(\alpha)$ and $T^+(\alpha)$ respectively, then $m = \mathrm{med}(x^-, x^0, x^+)$ is the projection of x^0 to α, and lies strictly between x^- and x^+. Thus neither $x^0x^-x^+$ nor $x^-x^+x^0$ hold. Consequently, the subsets $T^0(\alpha) \cup T^+(\alpha)$ and $T^0(\alpha) \cup T^-(\alpha)$ are full. □

Definition 2.7. Given an arc α in a median pretree T, write $\mathrm{proj}_\alpha : T^0(\alpha) \to \alpha$ for the map which takes each element to its projection on α. Observe that $\mathrm{proj}_\alpha([x, y]) = [\mathrm{proj}_\alpha(x), \mathrm{proj}_\alpha(y)]$ for $x, y \in T^0(\alpha)$. Also, the projection is canonical. Thus, if g is an automorphism of T and α a $\langle g\rangle$-invariant arc, then $g(\mathrm{proj}_\alpha(x)) = \mathrm{proj}_\alpha(gx)$ for $x \in T^0(\alpha)$. Note that an arc, α, is maximal if and only if both $T^+(\alpha)$ and $T^-(\alpha)$ are empty. In this case, we get a projection, $\mathrm{proj}_\alpha : T \to \alpha$, of the whole tree.

We finish this section with some observations about subpretrees. Note that every subset of a pretree has itself the structure of a pretree, simply by restricting the ternary relation. In general, a subpretree of a median pretree need not be median. However, we have the following:

Lemma 2.8. *Suppose that T is a median pretree and let S be any non-empty subset of T. Then the set* $\mathrm{med}(S) = \{\mathrm{med}(x,y,z) | x,y,z \in S\}$ *is a median subpretree of T.*

Proof. Given three elements of $\mathrm{med}(S)$, written $a_i = \mathrm{med}(a_i^1, a_i^2, a_i^3)$ for $i = 1,2,3$, we need to show that $\mathrm{med}(a_1,a_2,a_3) \in \mathrm{med}(S)$. Note that we do not assume that the elements a_i^j are necessarily distinct. Let A be a finite set containing all of the a_i^j for $i,j \in \{1,2,3\}$, and consider $\mathrm{med}(A) = \{\mathrm{med}(x,y,z)|x,y,z \in A\}$, which is a finite subpretree of $\mathrm{med}(S)$ containing each a_i together with the set A. (Note that each $x \in A$ appears in $\mathrm{med}(A)$ as the element $\mathrm{med}(x,x,x)$.) By Lemma 2.1 we may embed $\mathrm{med}(A)$ in a finite simplicial tree Σ. Now there exists a unique element $c = \mathrm{med}_\Sigma(a_1,a_2,a_3)$ in Σ, and it remains to show that $c \in \mathrm{med}(A)$. It is easy to see that, in Σ, one interval $[a_1^{j_1}, c]$, for $j_1 \in \{1,2,3\}$, contains a_1. Similarly choose $a_2^{j_2}$ and $a_3^{j_3}$. It now follows that $c = \mathrm{med}_\Sigma(a_1^{j_1}, a_2^{j_2}, a_3^{j_3})$ and so must be the element $\mathrm{med}(a_1^{j_1}, a_2^{j_2}, a_3^{j_3})$ of $\mathrm{med}(A)$, completing the proof. □

Corollary 2.9. *If the median pretree T admits an action by a countable group Γ then it contains a countable Γ-invariant median subpretree.*

Proof. Let S be a single orbit of the action of Γ on T, and hence a countable subset of T. Then the median subpretree $\mathrm{med}(S)$ of T given by Lemma 2.8 is clearly also countable and Γ-invariant. □

3 Automorphisms of a median pretree

The purpose of this section is to give a useful characterisation of an archimedean automorphism (see Definition 3.5) of a median pretree. In so doing, we find that archimedean automorphisms have essentially the same dynamical properties as non-nesting homeomorphisms, or for that matter isometries, of an $\mathbb{R}$-tree (as in, for example, [**Le**] Theorem 3).

We begin with some general definitions. Throughout this section we shall let T denote a median pretree and g an automorphism of T. We write $\langle g \rangle$ for the cyclic group of automorphisms generated by g, and $\langle g \rangle.x$ for the $\langle g \rangle$-orbit of a point $x \in T$. Write $\mathrm{Fix}(g)$ for the set of points fixed by g. We say that g *translates* a point $x \in T$ if $gx \in (x, g^2x)$. In this case, a straightforward induction shows that $\langle g \rangle.x$ is a linear set which may be directed so that

$$\cdots < g^{-1}x < x < gx < g^2x < g^3x < \cdots . \quad (1)$$

Note that if g translates x then so does g^n for any $n \in \mathbb{Z} \setminus \{0\}$.

Definition 3.1. Given an automorphism, g, of T, a subset, $\alpha \subseteq T$, is said to be an *axis* of g, or g-*axis*, if it is a minimal $\langle g \rangle$-invariant arc, and contains an element which is translated by g.

Since a nonempty intersection of two $\langle g \rangle$-invariant arcs is also a $\langle g \rangle$-invariant arc, it follows that the minimal $\langle g \rangle$-arcs and hence the axes of g must be mutually disjoint. Note that an axis has no endpoints (since removing them would give a smaller invariant arc). Note also that a g-axis is also a γ-axis for all nontrivial $\gamma \in \langle g \rangle$. The following lemma gives a concrete description of the axes of an automorphism $g : T \to T$.

Lemma 3.2. *If g translates the element $x \in T$ then the set*

$$\alpha_x = \bigcup_{n \in \mathbb{Z}} g^n([x, gx])$$

is a g-axis, in fact the unique g-axis containing x. Moreover, every g-axis is of this form, and every element of a g-axis is translated by g. In particular, the union of all axes of g is precisely the set of elements of T translated by g.

Remark: We shall generally assume that each g-axis α is directed so that $x < gx$ for each $x \in \alpha$.

Proof. Since the orbit of x under g is linearly ordered as in (1), we may write $\alpha_x = \bigcup_{k \in \mathbb{N}} I_k$ where $I_k = [g^{-k}x, g^k x]$ is an increasing sequence of nested intervals (that is $I_k \subseteq I_{k+1}$ for all k). It follows that α_x is an arc, which is clearly also $\langle g \rangle$-invariant. Note, also, that any $\langle g \rangle$-invariant arc containing x must contain all of α_x. Hence, to show that α_x is a g-axis it suffices to show that any $\langle g \rangle$-invariant subarc, α' say, contains x. Take an element y of α'. Then $g^k y \in [x, gx]$ for some $k \in \mathbb{Z}$, and indeed x lies in $g^k([g^{-1}y, y])$ which is a subinterval of α' by $\langle g \rangle$-invariance and fullness of the arc. Therefore α_x is a g-axis.

Now, any axis containing x must equal α_x, and clearly every element of α_x is translated by g. □

Suppose α is an axis for an automorphism g of the median pretree T. Applying Lemma 2.6, we see that T is a disjoint union of full $\langle g \rangle$-invariant subtrees $T^0(\alpha)$, $T^+(\alpha)$ and $T^-(\alpha)$. It follows that if β is another g-axis then β lies wholly in one of these three subtrees. In fact:

Lemma 3.3. *If α and β are distinct g-axes then either $\beta \subseteq T^-(\alpha)$ or $\beta \subseteq T^+(\alpha)$. In particular, $\alpha \cup \beta$ is a linear subset of T.*

Proof. Suppose the lemma is false. Then $\beta \subseteq T^0(\alpha)$ and α, β are disjoint. Take $b \in \beta$ and let $a = \mathrm{proj}_\alpha(b)$. Since the projection is canonical one has $g(a) = \mathrm{proj}_\alpha(g(b))$. But then both $bag(a)$ and $g(b)g(a)a$, from which it follows that $a \in [b, g(b)] \subseteq \beta$, a contradiction. Since the same argument shows that either $\alpha \subseteq T^-(\beta)$ or $\alpha \subseteq T^+(\beta)$, it follows that $\alpha \cup \beta$ is linear. □

Lemma 3.4. *Let g be an automorphism of a median pretree T. For each $x \in T$, the element $\mathrm{med}(g^{-1}x, x, gx)$ is either fixed or translated by g^2.*

Proof. Put $m = \mathrm{med}(g^{-1}x, x, gx)$. Since $m \in [g^{-1}x, x]$ one has $gm \in [x, gx]$. Thus both m and gm lie in $[x, gx]$, and either m is fixed by g or, for a suitable choice of direction $<$ on $[x, gx]$, one has either (i) $x \leq m < gm \leq gx$, or (ii) $x \leq gm < m \leq gx$.

Applying g^{-1} to case (i) gives $g^{-1}x \leq g^{-1}m < m < gm \leq gx$ for a suitable direction $<$ on $[g^{-1}x, gx]$. Thus m is translated by g.

In case (ii), applying g^{-1} shows that both gm and $g^{-1}m$ lie in $[x, m)$. If $gm = g^{-1}m$ then m is fixed by g^2. Otherwise we may suppose, without loss of generality, that $g^{-1}(m)g(m)m$. From this we may deduce that $g^{-1}(m)g(m)g^2(m)m$ and then $g^{-1}(m)g(m)g^3(m)g^4(m)g^2(m)m$, and hence that g^2 translates m. □

Definition 3.5. Let g be an automorphism of a pretree T.

(i) We say that g is *archimedean* if, for all $x, y, z \in T$, $\langle g \rangle.z \cap [x, y]$ is a finite set.

(ii) We say that g (or rather $\langle g\rangle$) is *non-nesting* if no element of $\langle g\rangle$ maps any closed interval $[x, y]$ of T properly into itself.

Observe that g is archimedean if some nontrivial element of $\langle g\rangle$ is archimedean, and only if every element of $\langle g\rangle$ is archimedean. (Any $\langle g\rangle$-orbit is the disjoint union of finitely many $\langle g^n\rangle$-orbits for any $n \neq 0$.) If g maps a closed interval properly into itself then so does g^n for any $n > 1$. It follows that g is non-nesting if some nontrivial element of $\langle g\rangle$ is non-nesting, and only if every element of $\langle g\rangle$ is non-nesting.

If $\gamma \in \langle g\rangle$ and γ maps the interval $[x, y]$ properly into itself, then the $\langle\gamma\rangle$-orbit of one of x or y will intersect $[x, y]$ in an infinite set. Thus, if g is archimedean then it is non-nesting. The converse to this statement is not necessarily true in general. However, if T is interval complete then the two notions are equivalent (see Proposition 3.10, to follow).

In what follows we show that one may classify archimedean automorphisms of a median pretree into the following two types.

Definition 3.6. Let g be an automorphism of the median pretree T.

(i) We say that g is *strongly elliptic* if no element of $\langle g\rangle$ translates any element of T.

(ii) We say that g is *loxodromic* if T possesses a maximal arc which is an axis for g. If such an axis exists then, by Lemma 3.3, it is the only g-axis in T and shall be referred to as *the loxodromic axis* of g.

Note that if g is strongly elliptic then, by Lemma 3.4, g^2 fixes $\mathrm{med}(g^{-1}x, x, gx)$ for every $x \in T$. We might say that that g is 'weakly elliptic' if g^2 fixes a point. In the case of a homeomorphism of an $\mathbb{R}$-tree this weaker notion is equivalent to the notion of ellipticity used by Levitt [**Le**], namely that g has a fixed point. In this context our strongly elliptic is equivalent to non-nesting with a fixed point (see Lemma 3.7 and Proposition 3.10, to follow).

By considering the projection onto the loxodromic axis, one sees that a loxodromic automorphism can have no periodic points, that is no finite orbits. On the other hand, any automorphism of finite order is strongly elliptic, since, by (1), the orbit of a translated point is infinite.

Lemma 3.7. *Let g be a strongly elliptic automorphism of a median pretree T. Then g is archimedean. Moreover, if $\gamma \in \langle g\rangle$ has a fixed point then Fix(γ) is a full subtree of T, each element of T has a projection to Fix(γ), and, for each $x \in T$, the interval $[x, \gamma x]$ intersects Fix(γ) in precisely one point (namely the common projection of x and γx).*

Proof. First, note that g is non-nesting, for if $\gamma \in \langle g\rangle$ maps the interval $[x, y]$ properly into itself then γ^2 must translate either x or y.

Suppose that $\gamma \in \langle g\rangle$ has a fixed point. This is true at least for $\gamma = g^2$. By the non-nesting property, we see that $\mathrm{Fix}(\gamma)$ is a full subset of T. Suppose $a \in T$. Given any $b \in \mathrm{Fix}(\gamma)$, let $c = \mathrm{med}(b, a, \gamma a)$. Now, again by non-nesting, it is clear that c must be fixed by γ. Thus, $[a, c] \cap \mathrm{Fix}(\gamma) = \{c\}$. In other words, c is a projection of a to $\mathrm{Fix}(\gamma)$. Moreover, $[a, \gamma a] \cap \mathrm{Fix}(\gamma) = \{c\}$. Finally we show that γ, and in particular g^2 and hence g, is archimedean. Suppose $x, y, z \in T$, and choose any $b \in \mathrm{Fix}(\gamma)$. By non-nesting, $[b, x] \cap \langle\gamma\rangle.z$ can have at most one element, and so also can $[b, y] \cap \langle\gamma\rangle.z$. Since $[x, y] \subseteq [b, x] \cup [b, y]$, the result follows. □

Lemma 3.8. *Let g be an automorphism of a median pretree T.*

(i) If g is loxodromic then it is archimedean.

(ii) Moreover, g is loxodromic if some $\gamma \in \langle g\rangle$ is loxodromic, and only if every nontrivial $\gamma \in \langle g\rangle$ is loxodromic.

Proof. (i) Let g be loxodromic with axis ℓ and suppose that g is not archimedean. Then some interval $[y, z]$ in T contains infinitely many iterates of an element x under g. Projecting onto ℓ one finds infinitely many iterates (under g) of $\mathrm{proj}_\ell(x)$ inside $[\mathrm{proj}_\ell(y), \mathrm{proj}_\ell(z)]$. But it follows from Lemma 3.2 that any interval inside a g-axis contains only finitely many elements of any g-orbit, giving a contradiction. Therefore g is archimedean.

(ii) It is clear that if g is loxodromic then so is every nontrivial element of $\langle g \rangle$. We show that if $\gamma = g^n$ is loxodromic, for some $n \in \mathbb{Z}$, then so is g. Let ℓ denote the loxodromic axis of γ. Then $g(\ell)$ is also a maximal arc and a γ-axis (since g and γ commute). Thus $g(\ell) = \ell$ by uniqueness of the loxodromic axis. It follows that the points x, gx and g^2x are collinear, and distinct (since if g^2 fixes a point then so does γ^2, a contradiction). Since γ and hence g is non-nesting, it is clear that g must translate x and so ℓ is a loxodromic axis for g. □

Theorem 3.9. *An automorphism of a median pretree is archimedean if and only if it is either strongly elliptic or loxodromic.*

Proof. By Lemmas 3.7 and 3.8, elliptic and loxodromic elements are archimedean. For the converse, suppose that g is an archimedean automorphism of a median pretree T. We may suppose also that g is not strongly elliptic, in which case there must exist a γ-axis α, for some $\gamma \in \langle g \rangle$. If α is not a maximal arc, then there exists $x \in T$ such that either $\{x\} \prec \alpha$ or $\alpha \prec \{x\}$. But this contradicts the fact that γ (equivalently g) is archimedean, by considering $\langle \gamma \rangle.z \cap [z, x]$ for any $z \in \alpha$. Therefore α is a maximal arc and γ is loxodromic. But then so is g, by Lemma 3.8(ii). □

Proposition 3.10. *An automorphism of an interval complete median pretree is archimedean if and only if it is non-nesting.*

Proof. The necessity is generally true; we need only prove the sufficiency. Suppose that g is a non-nesting automorphism of T, an interval complete median pretree. Either g is strongly elliptic, and hence archimedean, or some $\gamma \in \langle g \rangle$ translates a point x with axis α. Either α is maximal and γ loxodromic, in which case γ and hence g is archimedean as required, or one of $T^+(\alpha)$ or $T^-(\alpha)$ is nonempty. Suppose, without loss of generality, that $T^+(\alpha)$ is nonempty. Then, by interval completeness, the arc $I_x = \{u \in \alpha : x \leq u\}$ is an interval (since it is contained in $[x, a]$ for any choice of $a \in T^+(\alpha)$). Since α has no maximal element, I_x is not a closed interval, but may be written $I_x = [x, b)$ for some b in T. Note that, since $x < \gamma x$, I_x is mapped properly into itself by γ, and in fact $I_x = [x, \gamma x] \cup \gamma(I_x) = [x, \gamma b)$. Let $m = \mathrm{med}(x, b, \gamma b)$. If $m \in I_x$ then $I_x = [x, m]$ which contradicts the fact that I_x is not closed. On the other hand, if $m \notin I_x$, we must have $b = m = \gamma b$. But then γ maps $[x, b]$ properly into itself, contradicting non-nesting. □

In general, of course, non-nesting actions (on median pretrees) need not be archimedean. Suppose that A is a directed $\langle g \rangle$-invariant arc on which $x < gx < g^2x$ for all $x \in A$. Thus A is just a disjoint union of directed g-axes arranged 'head to tail' in a linear order. We shall call such an arc a *g-axial arc.* The action of g on A is non-nesting, but fails to be archimedean unless A consists of a single g-axis. In fact one can show without too much difficulty (the proof is left to the reader) that an automorphism g of a median pretree T is non-nesting if and only if either it is strongly elliptic or T contains a g-axial maximal arc. Such an arc must necessarily contain every g-axis in T.

4 From median pretrees to real pretrees

In this section, we describe a canonical procedure for embedding a countable median pretree in a real pretree. Another embedding is described in [**C**], though it is unclear how to make the latter construction canonical.

In [**Bo1**] it is shown how one may embed an arbitrary pretree, T, in a complete median pretree, $P = P(T)$, with the property (amongst others) that every closed interval of P contains a point of T. The pretree P is described as the set of flows on T with a natural pretree relation. Note that a complete median pretree P can have no loxodromic automorphisms, for any maximal arc must have endpoints and hence cannot be an axis. However, this problem may be overcome by removing "terminal" points from P. The resulting pretree will still be interval complete.

An element p of a pretree T is said to be *terminal* if there do not exist $x, y \in T$ with xpy. Fix some q in T different from p. Then p is terminal if (and only if) there does not exist $x \in T$ with xpq. (If xpy for some $x, y \in T$ then, by (T3), either xpq or qpy).

Lemma 4.1. *Given an arbitrary pretree T there exists an interval complete median pretree $\Theta = \Theta(T)$ with the following properties:*

(C0) T is embedded as a subpretree of Θ,

(C1) the terminal points of Θ all lie in T,

(C2) every nontrivial closed interval of Θ contains an element of T.

Moreover, the construction of Θ is canonical in the sense that any automorphism of T extends uniquely to an automorphism of Θ.

Proof. We take Θ to be $P \setminus \{x \in P \mid x \notin T \text{ and } x \text{ is terminal}\}$ where $P = P(T)$ is the complete median pretree described by Theorem 3.19 of [**Bo1**]. It follows from the construction in [**Bo1**] that any automorphism of T extends to an automorphism of P and hence Θ. Uniqueness may also be deduced from there, or may be seen from properties (C1)-(C2) as follows. Without loss of generality, suppose that the automorphism $h : \Theta \longrightarrow \Theta$ restricts to the identity on T. Take $p \in \Theta \setminus T$. If $h(p) \neq p$ then, by (C2), $pah(p)$ for some $a \in T$. Since, by (C1), p cannot be terminal we have bpc for some $b, c \in \Theta$ and so, by axiom (T3), we may assume that bpa. Thus $bpah(p)h(b)$. In particular $[b, p]$ is disjoint from $h([b, p])$ and so cannot intersect T contradicting (C2). Therefore h is the identity on Θ. □

It is inevitable that pairs of adjacent points will appear in the above "interval completion" Θ of T, that is pairs $p, q \in \Theta$ such that $p \neq q$ and $(p, q) = \emptyset$. In order to obtain a dense pretree it is necessary to fill all the gaps between adjacent points. To obtain a separable pretree, we will also need the following:

Lemma 4.2. *If T is a countable pretree then there are at most countably many distinct pairs of adjacent points in $\Theta(T)$.*

Proof. Although this fact is true in $P(T)$ and follows from the account in [**Bo1**], we give here an elementary proof using the properties (C1) and (C2). If p, q are adjacent in Θ then, by (C2), at least one of them, say p, is an element of T. If q is not already in T then, by (C1), q is not terminal and it follows that pqx for some $x \in \Theta$. But then, by (C2), pqy for some $y \in T$. In any case $q \in [p, y]$ for some $y \in T$. Moreover, if q' is adjacent to p and $q' \neq q$ then $q' \notin [p, y]$. The lemma is now proved by observing that there are at most countably many pairs $p, y \in T$. $\square$

Note that if T is a median pretree, then the median of any three non-collinear points of Θ must lie in T. (In fact, in this case, it is possible to modify the construction of Θ so that both elements of any adjacent pair must lie in T. However, we shall not need to bother with this.) Note also that if $p, q \in \Theta$ are adjacent, and $x \in \Theta \setminus \{p, q\}$, then either pqx or xpq.

We now give a construction which "fills the gaps" in our pretree. A similar construction has been used by Swenson [**Sw**]. Given a median pretree Θ, define

$$\widehat{\Theta} = \Theta \cup \bigcup_{\{p,q\} \text{ adjacent in } \Theta} R(p, q)$$

where each $R(p, q)$ is an isomorphic copy of the real line $\mathbb{R}$, thought of as a directed arc. We shall take $R(q, p)$ to denote the arc $R(p, q)$ with the opposite direction. For each $x, z \in \widehat{\Theta}$ we define a set $(\!(x, z)\!) = (\!(z, x)\!) \subseteq \widehat{\Theta}$, which will serve as an "open interval" in $\widehat{\Theta}$, as follows. Firstly, $(\!(x, x)\!) = \emptyset$. Otherwise, for $x \neq z$:

(J0) If $x < z \in R(p, q)$, then $(\!(x, z)\!) = \{y \in R(p, q) \mid x < y < z\}$.

(J1) If $x, z \in \Theta$, then $y \in (\!(x, z)\!)$ if and only if, either $y \in \Theta$ with xyz, or $y \in R(p, q)$ where p and q both lie in the interval $[x, z]$ of Θ.

(J2) If $x \in R(p, q)$, $z \in \Theta$, and without loss of generality pqz, then

$$(\!(x, z)\!) = \{y \in R(p, q) \mid x < y\} \cup \{q\} \cup (\!(q, z)\!)\,,$$

where $(\!(q, z)\!)$ is as defined in (J1).

(J3) If $x \in R(p, q)$, $z \in R(r, s)$, and without loss of generality $pqrs$, then

$$(\!(x, z)\!) = \{y \in R(p, q) \mid x < y\} \cup \{q\} \cup (\!(q, r)\!) \cup \{r\} \cup \{y \in R(r, s) \mid y < z\}\,,$$

where $(\!(q, r)\!)$ is as defined in (J1).

The ternary relation, $\langle\rangle$, is now defined, for $x, y, z \in \widehat{\Theta}$, such that $\langle xyz\rangle$ if and only if $y \in (\!(x, z)\!)$.

Lemma 4.3. *The relation, $\langle\rangle$, defines a median pretree structure on $\widehat{\Theta}$ which contains Θ as a subpretree.*

Proof. Axioms (T0) and (T1) are obvious from the definition. Observe that, for $x, z \in \widehat{\Theta}$, we have that $x, z \notin (\!(x, z)\!)$ and

$$\text{if } y \in (\!(x, z)\!) \text{ then } (\!(x, z)\!) \text{ is a disjoint union of } (\!(x, y)\!),\ \{y\}, \text{ and } (\!(y, z)\!). \tag{2}$$

(these facts follow by inspection of (J0)–(J3) above). Axiom (T2) now follows, namely $y \in (\!(x, z)\!) \implies z \notin (\!(x, y)\!)$.

For $x, z \in \widehat{\Theta}$, write $[\![x, z]\!] = \{x\} \cup (\!(x, z)\!) \cup \{z\}$. We now show that, for any three points $x, z, w \in \widehat{\Theta}$, one has

$$[\![x, z]\!] \cap [\![x, w]\!] \cap [\![w, z]\!] = \{c\} \quad \text{for some } c \in \widehat{\Theta}\,. \tag{3}$$

Clearly, by (2), this is true if the three points lie in a common interval, say $w \in [\![x, z]\!]$. We suppose otherwise. In particular the three points are distinct, and no two of them can lie in the same $R(p, q)$, for then, by (J2)–(J3), one lies between the other two. Suppose that $w \in R(p, q)$ for p, q adjacent in Θ. Then, without loss of generality, both $[\![x, w]\!]$ and $[\![z, w]\!]$ contain p and not q, for otherwise w lies in $[\![x, z]\!]$. Put $w' = p$ and $\delta(w) = (\!(p, w)\!) \cup \{w\}$. If, on the other hand, $w \in \Theta$ put $w' = w$ and $\delta(w) = \emptyset$. Define x', $\delta(x)$, and z', $\delta(z)$ similarly. Now $\delta(x)$, $\delta(z)$ and $\delta(w)$ are mutually disjoint, and $[\![x, z]\!] = \delta(x) \cup [\![x', z']\!] \cup \delta(z)$ etc. Therefore, $[\![x, z]\!] \cap [\![x, w]\!] \cap [\![w, z]\!] = [\![x', z']\!] \cap [\![x', w']\!] \cap [\![w', z']\!]$ which contains an element $c = \mathrm{med}_\Theta(x', z', w')$. By (2), this element is unique.

Now, (T3) is equivalent to the statement that $(\!(x, z)\!) \subseteq (\!(x, w)\!) \cup \{w\} \cup (\!(w, z)\!)$ for any $x, z, w \in \widehat{\Theta}$, which follows from (2) and (3). Finally, $\widehat{\Theta}$ is median by (3), and it follows from (J1) that Θ is a subpretree of $\widehat{\Theta}$. □

We now revert to the usual notation for expressing intervals and the betweenness relation in $\widehat{\Theta}$.

Lemma 4.4. *If Θ is interval complete then so is $\widehat{\Theta}$.*

Proof. Note firstly that, for p, q adjacent in Θ, the interval $[p, q]$ is order isomorphic to a closed interval of $\mathbb{R}$. Thus any arc of $\widehat{\Theta}$ which is disjoint from Θ is automatically an interval.

Any closed interval of $\widehat{\Theta}$ is contained in one whose endpoints lie in Θ (by simply attaching the interval $[p, q]$ if an endpoint happens to lie between points p, q adjacent in Θ). Thus any bounded arc of $\widehat{\Theta}$ lies in a closed interval with endpoints in Θ. Now take any bounded arc A which, by the opening remarks, we may suppose intersects Θ nontrivially, and let $A \subseteq [x, y]$ for some $x, y \in \Theta$. Since Θ is interval complete $A_0 = A \cap \Theta$ is an interval of Θ. It follows that $A_1 = A_0 \cup \{z \mid z \in R(p, q) \text{ for } p, q \in A_0 \text{ adjacent}\}$ is an interval of $\widehat{\Theta}$, having limit points in common with A_0. Then A is a disjoint union of the interval A_1 and at most two subarcs which are disjoint from Θ and hence also intervals. Since an arc which is the union of finitely many intervals is itself an interval, it now follows that A is an interval as required. □

Theorem 4.5. *Let T be a median pretree. Then T is embedded as a subpretree of an interval complete median pretree Ψ, namely $\Psi = \widehat{\Theta}$ where Θ is the pretree $\Theta(T)$ of Lemma 4.1. Moreover:*

(i) If T is countable then Ψ is a real pretree.

(ii) The terminal points of Ψ all lie in T. In particular, every closed interval of Ψ is contained in a closed interval with endpoints in T.

(iii) The embedding of T in Ψ is canonical up to a choice of a family of order isomorphisms

$$\{h_{p,q} : \mathbb{R} \longrightarrow R(p, q) \mid p, q \text{ adjacent in } \Theta\}$$

such that $h_{q,p}(x) = h_{p,q}(-x)$ for all $x \in \mathbb{R}$ and p, q adjacent in Θ. That is, any automorphism g of T extends uniquely to an automorphism g_Ψ of Ψ which respects the subtree Θ and such that $g_\Psi \circ h_{p,q} = h_{g_\Psi(p), g_\Psi(q)}$ for all p, q adjacent in Θ.

Proof. That T embeds canonically in Ψ follows from Lemma 4.1 and the construction of $\widehat{\Theta}$ from Θ. We consider statements (ii), then (i).

(ii) By (C1), the terminal points of Θ all lie in T, and it is clear that the construction of $\widehat{\Theta}$ from Θ does not introduce any new terminal points. As in the proof of Lemma 4.4 any closed interval of Ψ lies in a closed interval $[u,v]$ with $u,v \in \Theta$. Applying both (C1) and (C2) one can easily show that $[u,v]$ lies in a closed interval with endpoints in T. (Choose any $p \in T$ and find an $r \in T$ such that $u \in [p,r]$ and, similarly, an $s \in T$ such that $v \in [p,s]$. Then $[u,v]$ is contained in one of the intervals with endpoints p, r or s.)

(i) Since, by Lemma 4.4, we know that Ψ is interval complete, it suffices, by Lemma 2.13 of [**Bo1**], to show that Ψ is separable. Now, each $R(p,q) \cong \mathbb{R}$ contains a countable subset $Q(p,q) \cong \mathbb{Q}$ which is dense in $R(p,q)$. Moreover, since T is countable and since, by Lemma 4.2, there are countably many pairs of adjacent elements in $\Theta = \Theta(T)$, the set

$$\widehat{T} = T \cup \bigcup_{\{p,q\} \text{ adjacent in } \Theta} Q(p,q)$$

is a countable subset of Ψ. We claim that $\widehat{T}$ is dense in Ψ, and hence that Ψ is separable. To see this, take any distinct pair $a,b \in \Psi$. If $b \in R(p,q)$ say, then (a,b) contains an open interval of $R(p,q)$ and hence an element of $\widehat{T}$, by the density of $Q(p,q)$ in $R(p,q)$. Thus we may suppose that both $a,b \in \Theta$. In this case either a and b are adjacent and so $(a,b) = R(a,b)$ contains elements of $\widehat{T}$, or one has acb for some $c \in \Theta$. Again, either (c,b) contains an element of $\widehat{T}$ or $acdb$ for some $d \in \Theta$. But then, by (C2), $x \in [c,d]$ for some $x \in T$. Therefore $(a,b) \cap \widehat{T}$ is nonempty and so Ψ is separable, completing the proof. □

Given an archimedean automorphism $g : T \to T$ of a median pretree, we wish to establish that the canonically induced automorphism $g_\Psi : \Psi \to \Psi$ of Theorem 4.5 is also archimedean. Note that g_Ψ permutes the intervals $R(p,q)$, and if $g_\Psi(R(p,q)) = R(p,q)$ then it acts on $R(p,q)$ as -1 or the identity.

Lemma 4.6. *With the notation introduced above, let α_Ψ denote a g_Ψ-axis in Ψ. Then*

(i) α_Ψ contains an element of T, and

(ii) the subtree $\Psi^+(\alpha_\Psi)$ (resp. $\Psi^-(\alpha_\Psi)$) is either empty or contains an element of T.

Proof. (i) The axis α_Ψ cannot be wholly contained inside some $R(p,q)$ for then, by Lemma 4.5 (iii), g_Ψ^2 fixes $R(p,q)$ pointwise, a contradiction. Thus α_Ψ contains some $x \in \Theta$ and hence some element of T in the interval $[x, g_\Psi x]$, by (C2).

(ii) Suppose now that $\Psi^+(\alpha_\Psi)$ is nonempty. Since $\Psi \setminus \Psi^+(\alpha_\Psi)$ is full, by Lemma 2.6, if an interval $R(p,q)$ intersects $\Psi^+(\alpha_\Psi)$ then one of p or q must lie in $\Psi^+(\alpha_\Psi)$. Thus $\Psi^+(\alpha_\Psi)$ contains at least one element $x \in \Theta$. Either x is terminal in Θ and hence an element of T by (C1), or $\Psi^+(\alpha_\Psi)$ contains a second element $y \in \Theta$ (using the fullness of $\Psi \setminus \Psi^+(\alpha_\Psi)$ in the same manner as just above). Using (C2) again, we now have an element of T inside the interval $[x,y]$ which lies wholly in $\Psi^+(\alpha_\Psi)$ by fullness. The argument for $\Psi^-(\alpha_\Psi)$ is identical. □

Theorem 4.7. *Let T be a median pretree and Ψ as in Theorem 4.5. Suppose that $g : T \to T$ is an archimedean automorphism. Then*

(i) the canonically induced $g_\Psi : \Psi \to \Psi$ is archimedean, and

(ii) if g_Ψ fixes a distinct pair of points x, y in Ψ, then g fixes a distinct pair of points a, b in T such that $[x,y] \subseteq [a,b]$. Moreover, the pair a, b may be chosen to lie in any given interval which contains $[x,y]$ and has endpoints in T.

Proof. (i) Suppose that some element g_Ψ^n of $\langle g_\Psi \rangle$ translates some point, along an axis α_Ψ say. By Lemma 4.6(i), $\alpha_\Psi \cap T$ is nonempty and consists of points translated by g^n. Thus, if g is strongly elliptic, it follows that g_Ψ is too. On the other hand, if g is loxodromic then there exists a maximal arc α in T which is a g-axis. By Lemma 3.2, it is clear that α is equal to $\alpha_\Psi \cap T$ for some g_Ψ-axis α_Ψ. It now follows, by Lemma 4.6(ii), that α_Ψ is a maximal arc in Ψ, and hence that g_Ψ is loxodromic. The claim now follows by Theorem 3.9.

(ii) Suppose that g_Ψ fixes distinct points $x, y \in \Psi$. By Theorem 4.5(ii), we may also suppose that $[x, y]$ is contained in an interval $[u, v]$ with $u, v \in T$. Let $a = \mathrm{med}(u, g_\Psi u, v)$ and $b = \mathrm{med}(v, g_\Psi v, u)$. Then a and b are elements of T which are fixed by g_Ψ (or g) and $[x, y] \subseteq [a, b]$. (Apply Lemma 2.1 to the set $\{u, v, g_\Psi u, g_\Psi v, a, b, x, y\}$.) □

5 Application to group actions

We finally consider implications of these constructions for archimedean actions. Suppose the group Γ acts by automorphism on the median pretree T.

Definition 5.1. We say that the action of Γ is *archimedean* if every element of Γ is an archimedean automorphism.

The following should be compared with Theorem 3 (part (3)) of [**Le**] in the case of homeomorphic actions on $\mathbb{R}$-trees, and with the well-known corresponding result for isometric actions.

Lemma 5.2. *Suppose that we have an archimedean action of a finitely generated group Γ on a median pretree. Then the action is 2-nontrivial if and only if Γ has loxodromic elements.*

Proof. The sufficiency is clear, since every orbit of a loxodromic element is infinite (it projects to an infinite orbit in the loxodromic axis). We now prove that the action fails to be 2-nontrivial unless there is a loxodromic element.

We suppose that Γ has no loxodromic elements. Since the action is archimedean this means that, by Theorem 3.9, every element of Γ is strongly elliptic. Take a finite generating set $\{g_1, .., g_n, h_1, .., h_m\}$ for Γ, where each $\mathrm{Fix}(g_i) \neq \emptyset$ and each $\mathrm{Fix}(h_j) = \emptyset$. By Lemma 3.4, each $\mathrm{Fix}(h_j^2) \neq \emptyset$. Let

$$A = (\bigcap_{i=1}^{n} \mathrm{Fix}(g_i)) \cap (\bigcap_{j=1}^{m} \mathrm{Fix}(h_j^2)) .$$

We first show that A is nonempty. Suppose otherwise. Then we have elements $\gamma, \delta \in \Gamma$ with $\mathrm{Fix}(\gamma)$ and $\mathrm{Fix}(\delta)$ nonempty and disjoint. Let x be the projection of a point in one of these fixed sets to the other fixed set. Then, using Lemma 3.7, it is not hard to show that the element $\gamma\delta$ translates x, contradicting the fact that it is strongy elliptic. Thus A must be nonempty.

Note that A is full, as it is the intersection of full sets, and clearly invariant under each h_i (so Γ-invariant). We show by induction that there exists a nontrivial closed interval $[x, y]$ in A such that $h_i(x) = y$ for all $i = 1, .., m$. It follows that the action of Γ fails to be 2-nontrivial since the set $\{x, y\}$ is Γ-invariant. The induction step is proved as follows. Suppose that for some $r < m$ we have $[x_r, y_r] \subseteq A$ with $h_1(x_r) = .. = h_r(x_r) = y_r$. By non-nesting each of $h_1, .., h_r$ acts in the same fashion on the whole interval $[x_r, y_r]$. Let $z = \mathrm{med}(x_r, y_r, h_{r+1}(x_r))$. Then, since $h_{r+1}^2(x_r) = x_r$, we have $h_{r+1}(z) \in [x_r, h_{r+1}(x_r)]$. Similarly, $h_1(z) = ... = h_r(z)$ lies in $[x_r, y_r]$. Thus z, $h_1(z)$ and $h_{r+1}(z)$ are collinear. If $h_1(z) = h_{r+1}(z)$ then the induction step is completed by putting $x_{r+1} = z$. On the other hand, if the three points are distinct, it is not too hard to show that $h_1 h_{r+1}$ translates the point z, contradicting the fact that it is strongly elliptic. $\square$

Proof of Theorem 1.1: Suppose that Γ is a countable group which admits a 2-nontrivial archimedean action on a median pretree T. Note that the action of Γ is still archimedean and 2-nontrivial when restricted to any Γ-invariant median subpretree. Thus we obtain, in view of Corollary 2.9, a 2-nontrivial archimedean action of Γ on a countable median subpretree Q of T. Moreover, it is clear that $\mathcal{E}_Q \subseteq \mathcal{E}_T$. We shall assume henceforth that T is already a countable median pretree.

Now, the construction of Section 4 gives us a canonical embedding of T in a real pretree Ψ, and the induced action of Γ on Ψ is archimedean (hence non-nesting) by Theorem 4.7(i). Moreover, by Lemma 5.2, Γ has elements which are loxodromic on T and hence (as in the proof of Theorem 4.7(i)) loxodromic on Ψ. Thus the action on Ψ is nontrivial. As described in [**Bo1**], since Ψ is separable, one can canonically embed it in a dendrite. This gives rise to a topology on Ψ which is induced from an $\mathbb{R}$-tree structure. Of course, this $\mathbb{R}$-tree metric will not be Γ-invariant, so we still need Levitt's result to prove Theorem 1.2. However Γ does act by homeomorphisms on the $\mathbb{R}$-tree Ψ.

Now suppose that $\mathcal{E}_T$ does not contain any infinite ascending chain. We show that $\mathcal{E}_\Psi \subseteq \mathcal{E}_T$. Take any pair of distinct points $x, y \in \Psi$, and let H denote the edge stabilizer $\Gamma(x) \cap \Gamma(y)$. Since H is countable, we may label its elements by natural numbers. Thus $H = \{h_1, h_2, h_3, ...\}$. By Theorem 4.7(ii), we may choose a sequence of closed intervals $I_i = [a_i, b_i]$ with $a_i, b_i \in T$ such that h_i fixes I_i (pointwise) and $[x, y] \subseteq I_{i+1} \subseteq I_i$, for each $i \geq 1$. Let H_i denote the edge stabilizer $\Gamma(a_i) \cap \Gamma(b_i)$, an element of $\mathcal{E}_T$. Then, for each $i \geq 1$, we have $h_i \in H_i$ and $H_i < H_{i+1} < H$. Since $\mathcal{E}_T$ does not contain any infinite ascending chain, it follows that $H = H_n$ for some n. But then $H \in \mathcal{E}_T$. $\square$

Proof of Theorem 1.2: Now suppose Γ is finitely presentable. Starting from the homeomorphic action on the $\mathbb{R}$-tree Ψ of Theorem 1.1, the construction of Levitt [**Le**] gives an isometric action of Γ on a different $\mathbb{R}$-tree Σ. By Levitt's theorem, any subgroup G of Γ fixing an edge of Σ also fixes an edge of Ψ. That is $G \subseteq H$ where $H \in \mathcal{E}_\Psi$. If $\mathcal{E}_T$ does not contain any infinite ascending chain then by Theorem 1.1 we have $H \in \mathcal{E}_T$, so that G also fixes an edge of T. $\square$

Proof of Theorem 1.3: The conclusion of Theorem 1.3 follows directly from Theorem 9.5 of [**BeF**] and Theorem 1.2 above, once we can show that the isometric action of Γ on the $\mathbb{R}$-tree Σ given by Theorem 1.2 is stable. To this end, it suffices (by Proposition 3.2 (1) of [**BeF**]) to show that for any decreasing sequence of nested edges $[x_1, y_1] \supseteq [x_2, y_2] \supseteq [x_3, y_3] \supseteq \cdots$ there is an integer N such that the edge stabilizers $\Gamma(x_i) \cap \Gamma(y_i)$ are equal to $\Gamma(x_N) \cap \Gamma(y_N)$ for all $i > N$.

We will need to use the following properties of Levitt's construction − we refer to Section 2 of [**Le**]. There is a certain finite simplicial subtree (i.e: a connected union of finitely many closed intervals) K in Ψ such that Ψ is the union of translates of K under the group action, and a corresponding finite simplicial subtree K_0 in Σ such that Σ is the union of translates of K_0. Moreover, these subtrees are such that every closed interval of Σ (resp. Ψ) is contained in the union if finitely many translates of K_0 (resp. K). There is defined a "collapsing map" $\pi : K \to K_0$ which is monotonous (certain full subsets are simply collapsed to a point). To each edge α in K_0 is associated the edge $\widehat{\alpha}$ in K which is the smallest edge that collapses to α under π. By monotonicity, $\widehat{\alpha} \subseteq \widehat{\beta}$ if $\alpha \subseteq \beta$. Finally, Levitt shows that any subgroup of Γ which fixes an edge α in K_0 also fixes $\widehat{\alpha}$.

We suppose now that $\mathcal{E}_T$ does not contain any ascending chain and consists of slender subgroups. By Theorem 1.1, the same is true for $\mathcal{E}_\Psi$ since then $\mathcal{E}_\Psi \subseteq \mathcal{E}_T$. Take any decreasing sequence of nested edges $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ in Σ and write G_i for the stabilizer of I_i. Since I_1 is contained in the union of finitely many translates of K_0, at least one of these translates, gK_0 say, intersects every I_i in at least 2 points. We write α_i for the edge $I_i \cap gK_0$. Now $gK_0 \supseteq \alpha_1 \supseteq \alpha_2 \supseteq \alpha_3 \supseteq \cdots$ and each α_i is fixed by G_i. Taking the corresponding edges $\widehat{\alpha}$ in the subtree gK of Ψ, we have $\widehat{\alpha}_1 \supseteq \widehat{\alpha}_2 \supseteq \widehat{\alpha}_3 \supseteq \cdots$ and each $\widehat{\alpha}_i$ fixed by G_i. For each $i \geq 1$, let H_i denote the stabilizer of $\widehat{\alpha}_i$. So $G_i < H_i$. By the ascending chain condition in Ψ there is some N such that $H_i = H_N$ for all $i \geq N$. Then we have

$$G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots \subseteq G \subseteq H_N$$

where G denotes the subgroup generated by the union of the G_i. Since H_N is slender G is finitely generated and it now follows that the sequence of groups $\{G_i\}$ stabilizes. This completes the proof that the action of Γ on Σ is stable. □

Proof of Corollary 1.4: Finally we observe that Corollary 1.4 follows directly from Theorem 1.3. Note that if Γ admits a free archimedean action on a median pretree then the order of any nontrivial cyclic subgroup is either 2 or infinite – a loxodromic element has infinite order, while if g is elliptic then either g or g^2 fixes a point and so must be trivial by freeness of the action.

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