Hausdorff dimension and dendritic limit sets.

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Abstract.

Let Γ be a singly degenerate closed surface group acting properly discontinuously on hyperbolic 3-space, $\mathbf{H}^3$, such that $\mathbf{H}^3/\Gamma$ has positive injectivity radius. It is known that the limit set is a dendrite of Hausdorff dimension 2. We show that the cut-point set of the limit set has Hausdorff dimension strictly less than 2.

0. Introduction.

Let Σ be a closed orientable surface with $\mathrm{genus}(\Sigma) \geq 2$, and let $\Gamma = \pi_1(\Sigma)$. Suppose that Γ acts properly discontinuously on hyperbolic 3-space $\mathbf{H}^3$. Since Γ is torsion-free, the quotient $M = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold, with $\pi_1(M) \equiv \Gamma$. Its limit set, $\Lambda\Gamma$, is a subcontinuum of the 2-sphere $S^2 \equiv \partial \mathbf{H}^3$. Of particular interest is the case where M is "singly degenerate" so that $\partial \mathbf{H}^3 \setminus \Lambda\Gamma$ is a topological disc. If, in addition, M has positive injectivity radius, $\mathrm{inj}(M) > 0$, then $\Lambda\Gamma$ is known to be a dendrite [Min1,Min2]. Moreover, it is shown in [S] that its Hausdorff dimension equals 2.

In contrast, we show:

Theorem 0 : *The cut-point set of the dendrite* $\Lambda\Gamma$ *has Hausdorff dimension at most* $2 - \epsilon$, *where* $\epsilon > 0$ *depends only on* $\mathrm{inj}(M)$ *and* $\mathrm{genus}(\Sigma)$.

It clearly follows that the set of extreme (i.e. non-cut) points of $\Lambda\Gamma$ has Hausdorff dimension 2. (Since every conical limit point is an extreme point, this observation also follows from [BiJ].) In principle, a lower bound on ϵ is computable in terms of $\mathrm{inj}(M)$ and $\mathrm{genus}(\Sigma)$, but we shall not make any explicit estimate here. Theorem 0 will follow easily from a stronger statement, namely Theorem 2.1.

This result was inspired by a recent result of Miyachi [Miy] which shows that any pair of points in $\Lambda\Gamma$ are connected by a quasi-arc in $\Lambda\Gamma$. A quasi-arc has Hausdorff dimension strictly less than 2, and $\Lambda\Gamma$ is a countable union of such arcs. The additional ingredient is thus a uniform bound on these Hausdorff dimensions.

As in [Miy], the essential point is to show that any arc in $\Lambda\Gamma$ lies in the boundary of a half-plane quasi-isometrically embedded in $\mathbf{H}^3$. Instead of using the full force of the result of [Min2], we proceed directly from the singly degenerate assumption, using ideas of Mitra [Mit], as laid out explicitly in [Bow2].

1. Quasidendrites.

First we recall some basic facts about dendrites, and introduce the notion of a "quasi-dendrite" — by analogy with the standard notions of "quasicircle" and "quasi-arc". (Note that the prefix "quasi-" is used in two different senses in this paper: here from the association with quasiconformal maps, and elsewhere from the association with quasi-isometries.)

There are numerous equivalent definitions of a dendrite. For example, a dendrite, D, is a locally connected metrisable continuum in which any pair of points, $x, y \in D$, are connected by a unique arc, $I[x, y] \subseteq D$. (Here an "arc" is a subset homeomorphic to a closed real interval.) We write $I(x, y) = I[x, y] \setminus \{x, y\}$. One shows that a point $z \in D$ is a cut point if and only there exist $x, y \in D$ such that $z \in I(x, y)$. We write $\mathrm{cut}(D) \subseteq D$ for the set of cut points. A point of $D \setminus \mathrm{cut}(D)$ is called an *extreme* point.

Since D is metrisable, it contains a countable dense subset $P \subseteq D$. One can easily show that $\mathrm{cut}(D) \subseteq \bigcup_{x,y \in P} I[x, y]$. In particular, $\mathrm{cut}(D)$ is a countable union of open subarcs.

Given four points, x, y, z, w in the Riemann sphere, $\mathbf{C} \cup \{\infty\}$, we define their *crossratio*, $[x, y : z, w]$, as $\frac{(w-z)(y-x)}{(w-y)(z-x)}$. Thus, $[x, y : z, \infty] = \frac{y-x}{z-x}$.

First recall the notion of a K-quasicircle (cf. [LV]), for $K \geq 1$. This can be defined as a simple closed curve, $\alpha \subseteq \mathbf{C} \cup \{\infty\}$, such that whenever $\{x, z\} \subseteq \alpha$ is linked with $\{y, w\} \subseteq \alpha$, we have $|[x, y : z, w]| \leq K$. Mapping w to ∞ by a Möbius transformation, this means that if $\beta \subseteq \alpha \setminus \{\infty\}$ is any compact subarc with endpoints $\partial\beta$, then $\mathrm{diam}(\beta) \leq K \,\mathrm{diam}(\partial\beta)$, where diam denotes euclidean diameter. From this, one can deduce that the Hausdorff dimension of $\alpha \setminus \{\infty\}$ is at most $2 - \epsilon$, where $\epsilon > 0$ depends only on K. Since Möbius transformations are smooth, the same applies to α with respect to either the spherical or euclidean metrics.

We can similarly define a *quasi-arc*, γ, where x, y, z, w are assumed to occur in this order along γ. The same discussion applies. More generally:

Definition : Let $D \subseteq \mathbf{C} \cup \{\infty\}$ be a dendrite embedded in the Riemann sphere. We say that D is a K-*quasidendrite* if, given any distinct $x, y, z, w \in D$ such that y separates x from z and z separates y from w, then $|[x, y : z, w]| \leq K$.

Note that the condition on x, y, z, w is equivalent to asserting that $y, z \in I(x, w)$ and x, y, z, w occur in this order along $I[x, w]$. In other words, we are saying that each subarc of D is a K-quasi-arc. Since the cut-point set, $\mathrm{cut}(D)$, lies on a countable union of such quasi-arcs, we conclude:

Lemma 1.1 : *Given $K \geq 1$, there is some $\epsilon > 0$ such that if $D \subseteq \mathbf{C} \cup \{\infty\}$ is a K-quasidendrite, then the Hausdorff dimension of* $\mathrm{cut}(D)$ *is at most $2 - \epsilon$.*

We note that to verify that a dendrite D is a quasidendrite, it is enough, by continuity, to consider $x, y, z, w \in \mathrm{cut}(D)$.

We also note the following geometric interpretation of the crossratio bound, which is what we will verify in practice. First, identify $\mathbf{C} \cup \{\infty\}$ with boundary, $\partial \mathbf{H}^3$, of hyperbolic

3-space, $\mathbf{H}^3$. Given $x, y \in \mathbf{H}^3 \cup \partial \mathbf{H}^3$, we denote by $[x, y]$ the hyperbolic geodesic from x to y. We write d for the hyperbolic metric on $\mathbf{H}^3$. If $x, y, z, w \in \partial \mathbf{H}^3$, then $d([x, z], [y, w]) = |\log|\mu||$, where $\mu \in \mathbf{C} \setminus \{0\}$ satisfies $4[x, y : z, w] = \mu + \mu^{-1} - 2$. In particular, we see that an upper bound on $|[x, y : z, w]|$ is equivalent to an upper bound on $d([x, z], [y, w])$.

2. Kleinian groups.

We recall some basic facts about kleinian groups and give a more precise formulation of our main theorem.

By a *kleinian group*, we shall mean a group Γ acting freely and properly discontinuously on hyperbolic 3-space, $\mathbf{H}^3$. We write $\Lambda\Gamma \subseteq \partial \mathbf{H}^3$ for its limit set, and $M = M(\Gamma) = \mathbf{H}^3/\Gamma$ for the quotient 3-manifold. The *injectivity radius*, $\mathrm{inj}(M)$, of M is twice the infimum of lengths of essential closed curves in M.

An important class of groups arises when $\Gamma = \pi_1(\Sigma)$ is the fundamental group of a closed orientable surface, Σ, with $\mathrm{genus}(\Sigma) \geq 2$. (The non-orientable case is essentially the same, or can be dealt with passing to a double cover.) In this case, Bonahon [Bon] shows that M is geometrically tame (in particular, homeomorphic to $\Sigma \times \mathbf{R}$). If there are no parabolics, then each end of M is either geometrically finite or simply degenerate. Depending on whether M has 2, 0 or 1 simply degenerate ends, the limit set, $\Lambda\Gamma$, will be respectively, all of $\partial \mathbf{H}^3$, a quasicircle, or a continuum, D, with $\partial \mathbf{H}^3 \setminus D$ a disc. It is the last, "singly degenerate", case that interests us here (Γ is sometimes called a "Bers group"). It is conjectured that D is always a dendrite, and this is known to be the case if $\mathrm{inj}(M) > 0$ [Min2]. Moreover, under the same hypotheses, Sullivan [S] showed that the Hausdorff dimension of D equals 2. (For a much more general statement, see [BiJ].) In contrast, Miyachi [Miy] showed that every subarc of D is a quasi-arc. However, his argument does not give a uniform constant. We use a slightly different approach to give our main result:

Theorem 2.1 : *Let Σ be a closed orientable surface with* $\mathrm{genus}(\Sigma) \geq 2$. *Let* $\Gamma = \pi_1(\Sigma)$ *act on* $\mathbf{H}^3$ *as a singly degenerate kleinian group with* $\mathrm{inj}(\mathbf{H}^3/\Gamma) > 0$. *Then the limit set,* $\Lambda\Gamma$, *is a K-quasidendrite, where K depends only on* $\mathrm{inj}(\mathbf{H}^3/\Gamma)$ *and* $\mathrm{genus}(\Sigma)$.

Again, the dependence of K on $\mathrm{inj}(M)$ and $\mathrm{genus}(\Sigma)$ is, in principle, computable.

Putting Theorem 2.1 together with Lemma 1.1, we immediately deduce Theorem 0.

3. Riemannian half-planes.

As in [Miy] we shall prove the main result by constructing quasi-isometric maps of half-planes into $\mathbf{H}^3$. We begin with a general discussion of half-planes.

Let H be a complete path-metric space homeomorphic to $\mathbf{R} \times [0, \infty)$. (We can assume H to be riemannian if we want.) We suppose H to be hyperbolic in the sense of Gromov [Gr,GhH]. We write ∂H for its Gromov boundary, so that $H \cup \partial H$ is compact. To avoid any confusion, we shall refer to $\mathbf{R} \times \{0\}$ as the *frontier* of H, and denote it by $\mathrm{fr}(H)$. We shall assume that the closure of $\mathrm{fr}(H)$ in $H \cup \partial H$ is homeomorphic to a closed real interval with endpoints $\mathrm{end}(H) = \{a, b\} \subseteq \partial H$. Since H is one-ended, we know that ∂H is connected (see, for example, [GhH]). In fact:

Lemma 3.1 : *∂H is homeomorphic to a closed interval with endpoints* $\mathrm{end}(H)$.

Proof : Since ∂H is metrisable, it's enough to show that each point, c, of $\partial H \setminus \mathrm{end}(H)$ separates a from b. To this end, let γ be a geodesic ray with basepoint in $\mathrm{fr}(H)$ and converging on c. By cutting γ off at its last intersection point with $\mathrm{fr}(H)$, we can assume that γ meets $\mathrm{fr}(H)$ in a single point. This point cuts $\mathrm{fr}(H)$ into two rays, α and β, converging on a and b respectively.

Now γ cuts H into two half-planes, H_a and H_b, so that $\mathrm{fr}(H_a) = \alpha \cup \gamma$ and $\mathrm{fr}(H_b) = \beta \cup \gamma$. Now H_a and H_b are convex and hence intrinsically Gromov hyperbolic. We have $\partial H_a \cap \partial H_b = \{c\}$, $\partial H_a \cup \partial H_b = \partial H$, and $a \in \partial H_a$ and $b \in \partial H_b$. Moreover, ∂H_a and ∂H_b are both connected. It follows that c separates a from b in ∂H as required.

Lemma 3.2 : *Suppose $p, q, r, s \in \partial H$ are distinct and that p, q, r, s occur in this order along the interval ∂H. Suppose that $\alpha, \beta \subseteq H$ are bi-infinite geodesics connecting p to r and q to s respectively. Then $\alpha \cap \beta \neq \emptyset$.*

Proof : Simple planar topology shows that α bounds a half-plane, $H_\alpha \subseteq H$. This is convex and hence intrinsically Gromov hyperbolic. We have $\partial H_\alpha \subseteq \partial H$ and $\mathrm{end}(H_\alpha) = \{p, r\}$. In other words, ∂H_α is the subinterval of ∂H with endpoints p and r. Similarly, β bounds a convex half-space, H_β, with $\partial H_\beta \subseteq \partial H$ and $\mathrm{end}(H_\beta) = \{q, s\}$. Note that neither ∂H_α nor ∂H_β is contained in the other.

Suppose, for contradiction, that $\alpha \cap \beta = \emptyset$. Now neither H_α nor H_β is contained in the other, and so $H_\alpha \cap H_\beta = \emptyset$. But since $\mathrm{end}(H_\alpha) \cap \mathrm{end}(H_\beta) = \emptyset$, it follows easily that $\partial H_\alpha \cap \partial H_\beta = \emptyset$, giving a contradiction. ♢

Before discussing how we apply this, we give some general definitions that will be used again later.

Suppose (X, ρ) and (Y, d) are metric spaces, and let $\psi : X \longrightarrow Y$ be any map (not necessarily continuous). Let $F : [0, \infty) \longrightarrow [0, \infty)$ be an increasing function.

Definition : We say that ψ is *F-proper* if for all $x, y \in X$, $\rho(x, y) \leq F(d(\psi(x), \psi(y)))$. We say that ψ is *coarsely proper* if it is F-proper for some function F. (This is sometimes termed "uniformly proper" in the literature.) We say that ψ is quasi-isometric if it is F-proper of some linear function F.

The function F is referred to as the *parameter* of properness or quasi-isometry.

Given any subset $A \subseteq \mathbf{H}^3$, we define its *limit set* as the intersection of $\partial \mathbf{H}^3$ with the closure of A in $\mathbf{H}^3 \cup \partial \mathbf{H}^3$.

Lemma 3.3 : *Suppose that H is a complete path-metric space homeomorphic to a half-plane. Suppose that $\psi : H \longrightarrow \mathbf{H}^2$ is a quasi-isometric map. The limit set of $\psi(H)$ is a K-quasi-arc, where K depends only on the parameters of ψ.*

For our applications, we can assume that ψ is continuous, and that ψ restricted to $\mathrm{fr}(H)$ extends to a continuous map of the closed interval. This will eliminate a few technical details.

Proof : By the quasi-isometric invariance of hyperbolicity, we see that H is intrinsically hyperbolic, and ψ extends to a continuous map of $H \cup \partial H$ to $\mathbf{H}^3 \cup \partial \mathbf{H}^3$. The restriction of ψ to ∂H is a homeomorphism from ∂H to the limit set, γ, of $\psi(H)$. Thus, by Lemma 3.1, γ is a closed interval.

Now consider points, $x, y, z, w \in \gamma$, occurring in this order. These are, respectively, the ψ-images of points $p, q, r, s \in \partial H$. Let α and β be bi-infinite geodesics in H connecting p to r and q to s. By Lemma 3.2, $\alpha \cap \beta \neq \emptyset$. But $\psi(\alpha)$ is quasigeodesic in $\mathbf{H}^3$, and hence remains a bounded distance from the hyperbolic geodesic $[x, z]$. Similarly, $\psi(\beta)$ remains a bounded distance from $[y, w]$. It follows that $d([x, z], [y, w])$ is bounded in terms of the parameters of ψ. Thus γ is a quasi-arc as claimed. $\diamondsuit$

4. Simply degenerate ends.

Let Γ be a singly degenerate surface group with quotient $M = \mathbf{H}^3/\Gamma$. Write $Y \subseteq \mathbf{H}^3$ for the hyperbolic convex hull of $\Lambda\Gamma$, and write $\mathrm{core}(M) = Y/\Gamma \subseteq M$ for the convex core of M. We suppose that $\mathrm{inj}(M) > 0$. (This is sometimes termed "bounded geometry".)

In this case, Minsky [Min1,Min2], shows that Y is equivariantly quasi-isometric to a certain "singular Sol" model space having a natural singular foliation. (The dependence of the parameters on $\mathrm{inj}(M)$ and $\mathrm{genus}(\Sigma)$ is not addressed there, but could, in principle be extracted from the approach discussed in [Bow1,Bow2].) Miyachi [Miy] shows that leaves in this foliation give rise to quasi-isometrically embedded planes. However, his argument does not give uniformity of quasi-arcs. Here we describe another approach, which does give uniformity, and also bypasses much of the difficult part of [Min1,Min2] (namely relating the geometry of $\mathrm{core}(M)$ to Teichmüller geodesics, and hence to singular Sol geometry).

Throughout the rest of this paper, we describe a constant or function as "uniform" if it ultimately depends only on the initial data, namely $\mathrm{genus}(\Sigma)$ and $\mathrm{inj}(M)$. (It is for this reason that we have substituted the term "coarsely proper" for the more usual "uniformly proper".)

Now by simple degeneracy and interpolation of pleated surfaces [Bon,T], we can find a sequence of hyperbolic (i.e. constant curvature -1) metrics, ρ_i, on Σ and uniformly lipschitz homotopy equivalences, $f_i : \Sigma_i \longrightarrow \mathrm{core}(M)$, where $\Sigma_i = (\Sigma, \rho_i)$. The diameters of the images $f_i(\Sigma_i)$ are necessarily bounded, and we can assume, moreover, that each $f_i(\Sigma_i)$ separates $f_{i-1}(\Sigma_{i-1})$ from $f_{i+1}(\Sigma_{i+1})$ and that for all $i > 0$, $d(f_i(\Sigma_i), f_{i+1}(\Sigma_{i+1}))$ is bounded above and below by uniform positive constants. We may as well take the lower bound to be 1. We can also take f_0 to be an isometry from Σ_0 to the boundary of core(M). One can show that $\bigcup_{i=0}^{\infty} f_i(\Sigma_i)$ is unformly quasidense in core(M), i.e. each point of core(M) lies within a bounded distance of some $f_i(\Sigma_i)$.

Lifting to $\mathbf{H}^3$, we obtain a sequence of equivariant uniformly lipschitz maps, $\phi_i : X_i \longrightarrow \mathbf{H}^3$, where $X_i \cong \mathbf{H}^2$ is the universal cover of Σ_i. Moreover, it is shown in [Min1] that the maps ϕ_i are uniformly coarsely proper. Indeed, the parameters are computable in terms of genus(Σ) and inj(M) (see [Bow1]).

A simple consequence of this, in turn, is that if we choose $r > 0$ sufficiently large, then the relation, $\sim_r$, defined on $X_i \times X_{i+1}$ by $x \sim_r y$ if $d(\phi_i(x), \phi_i(y)) \leq r$ is a uniform (depending on r) quasi-isometry from X_i to X_{i+1}. By the stability of quasigeodesics, it follows that if $x \sim_r x'$ and $y \sim_r y'$ then $\phi_i([x, y]_i)$ is a bounded Hausdorff distance from $\phi_{i+1}([x', y']_{i+1})$, where $[x, y]_i$ denotes the geodesic in X_i from x to y. Indeed, there is some uniform $s > 0$ such that the relation $\sim_s$ restricted to $[x, y]_i \times [x', y']_{i+1}$ is a uniform quasi-isometry from $[x, y]_i$ to $[x', y']_{i+1}$.

Suppose that $x, y, z \in X_i$. We define $\mathrm{cent}_i(x, y, z) \in X_i$ to be the nearest point to z in $[x, y]_i$. Another consequence of the above remarks is that if $x, y, z \in X_i$ and $x', y', z' \in X_{i+1}$ with $x \sim_r x'$, $y \sim_r y'$ and $z \sim_r z'$, then $d(\phi_i(\mathrm{cent}_i(x, y, z)), \phi_{i+1}(\mathrm{cent}_{i+1}(x', y', z')))$ is uniformly bounded in terms of r.

5. Proof of the main theorem.

In this section, we prove Theorem 2.1 by constructing quasi-isometric maps of half-planes into $\mathbf{H}^3$. We begin with a couple of technical observations.

By a *riemannian rectangle* we shall mean a riemannian metric ρ on the square $[0, 1]^2$, which is isometric to a euclidean metric in a neighbourhood of the "horizontal sides" $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$. We refer to the intervals $\{t\} \times [0, 1]$ as "vertical".

Suppose that J and J' are compact real intervals (or path metric spaces homeomorphic, and hence isometric, to compact intervals). Suppose that $\sim \subseteq J \times J'$ is a quasi-isometry. We note:

Lemma 5.1 : *We can identify J and J' with the horizontal sides of a riemannian rectangle, R, so that each vertical interval has bounded length, so that the inclusions of J and J' into R are quasi-isometries, and so that the inclusion of J into R composed with the quasi-isometry $\sim$ agrees up to bounded distance with inclusion of J' into R.*

This can be deduced using the fact that a quasi-isometry of intervals agrees up to bounded distance with a diffeomorphism.

Now suppose that J and J' are real intervals, and that $\phi : J \longrightarrow \mathbf{H}^3$ and $\phi' : J' \longrightarrow \mathbf{H}^3$ are coarsely proper lipschitz maps. Let $\sim \subseteq J \times J'$ be the relation defined by $x \sim y$ if $d(\phi(x), \phi'(y)) \leq r$ for some fixed r. Suppose that $\sim$ is a quasi-isometry. (For example, if $\mathrm{HausDist}(\phi(J), \phi'(J')) \leq r$ where HausDist denotes Hausdorff distance.) Let R be a riemannian rectangle as given by Lemma 5.1. We can extend $\phi \sqcup \phi'$ to a map $\psi : R \longrightarrow \mathbf{H}^3$ by mapping each vertical interval linearly to a geodesic segment in $\mathbf{H}^3$. The resulting map ψ will be uniformly coarsely proper. It will also be uniformly coarsely lipschitz, in the sense that $d(\phi(x), \phi(y))$ is bounded above by a uniform linear function of $\rho(x, y)$. (Indeed by constructing the metric on R sensibly, one can assume it to be uniformly lipschitz.)

We shall apply this construction in the situation where we have a sequence of intervals, $(J_i)_{i \in \mathbf{N}}$ and uniformly proper maps $\phi_i : J_i \longrightarrow \mathbf{H}^3$, with $\mathrm{HausDist}(\phi_i(J_i), \phi_{i+1}(J_{i+1}))$ bounded. The above construction gives us riemannian rectangles, R_i, between J_i and J_{i+1} and maps $\psi_i : R_i \longrightarrow \mathbf{H}^3$ that are uniformly coarsely lipschitz and coarsely proper. Gluing these together, we get a riemannian half-space, $H = \bigcup_{i=0}^{\infty} R_i$, and a coarsely lipschitz map $\psi : H \longrightarrow \mathbf{H}^3$. In the case of interest which we describe below, ψ will be quasi-isometric.

We now return to the set-up described in Section 4, where we have a sequence of planes, $X_i \cong \mathbf{H}^2$, and uniformly coarsely proper lipschitz maps $\phi_i : X_i \longrightarrow \mathbf{H}^3$. It is notationally convenient to assemble the X_i into a disjoint union $\Xi = \bigsqcup_{i=0}^{\infty} X_i$. We refer to Ξ as a "stack" of "sheets" X_i. We write $\phi = \bigcup_i \phi_i$ for the map $\phi : \Xi \longrightarrow \mathbf{H}^3$. Thus $\phi(\Xi)$ is quasidense in the convex hull, Y, of the limit set $D = \Lambda\Gamma$.

By an *r-chain*, for $r \geq 0$, we mean a sequence $\underline{x} = (x_i)_{i \in \mathbf{N}}$, with $x_i \in X_i$ and $d(\phi_i(x_i), \phi_{i+1}(x_{i+1})) \leq r$ for all i. From the separation properties of the images $\phi_i(X_i)$, we see that $(\phi_i(x_i))_i$ is a quasigeodesic sequence in Y, and hence converges on some point, denoted $\pi(\underline{x})$, in D.

It is not hard to see that one can find some uniform $r_0 \geq 0$ such that each point of $\mathrm{cut}(D)$ has the form $\pi(\underline{x})$ for some r_0-chain. (In particular, it follows that $\mathrm{cut}(D)$ consists entirely on non-conical limit points, as observed in the introduction.) For details, see [Bow2].

Suppose $a, b \in \mathrm{cut}(D)$ are distinct. We can find r_0-chains, $\underline{a}$ and $\underline{b}$ such that $a = \pi(\underline{a})$ and $b = \pi(\underline{b})$. Let $J_i = [a_i, b_i]_i$ and write $\phi_i : J_i \longrightarrow \mathbf{H}^3$ for the restriction of ϕ_i to J_i. As observed in Section 4, $\mathrm{HausDist}(\phi_i(J_i), \phi_{i+1}(J_{i+1}))$ is uniformly bounded. Thus, we can embed the J_i in a riemannian half-plane, H, and construct a map $\psi : H \longrightarrow \mathbf{H}^3$ as above. From the construction, $\psi(H) \subseteq Y$. We denote the metric on H by ρ.

Lemma 5.2 : *The map $\psi : H \longrightarrow \mathbf{H}^3$ is uniformly quasi-isometric.*

Proof : We already know that ψ is uniformly coarsely lipschitz. Thus, given $x, y \in H$, we want to show that $d(x, y)$ is bounded above by a uniform linear function of $d(\psi(x), \psi(y))$. The argument follows that in [Mit].

Since $\bigcup_i J_i$ is uniformly quasidense in H, we can assume that $x, y \in \bigcup_i J_i$. Let $p = \psi(x) = \phi(x)$ and $q = \psi(y) = \phi(y)$. Let $\alpha = [p, q] \subseteq Y$ be the geodesic segment in $Y \subseteq \mathbf{H}^3$ connecting p to q. Let $p_0 = p, p_1, \ldots, p_n = q \in \alpha$ cut α into $n \leq d(p, q) + 1$ segments, each of length at most 1.

Now $\phi(\Xi)$ is quasidense in Y, so we can find points $x_i \in \Xi$ with $d(p_i, \phi(x_i))$ uniformly bounded. We can take $x_0 = x$ and $x_n = y$. We write $x_i \in X_{j(i)}$. Since $d(\phi_j(X_j), \phi_{j+1}(X_{j+1})) \geq 1$ for all j, we can choose the x_i so that $|j(i+1) - j(i)| \leq 1$ for all i.

Now, let $z_i = \mathrm{cent}_{j(i)}(a_i, b_i, x_i) \in J_{j(i)}$. Note that $z_0 = x_0 = x$ and $z_n = x_n = y$. Now, $d(\phi(a_{j(i)}), \phi(a_{j(i+1)})) \leq r_0$, $d(\phi(b_{j(i)}), \phi(b_{j(i+1)})) \leq r_0$, and by construction, $d(\phi(x_i), \phi(x_{i+1}))$ is uniformly bounded. Thus, as observed at the end of Section 5, $d(\phi(z_i), \phi(z_{i+1}))$ is uniformly bounded. By construction, $\phi(z_i) = \psi(z_i)$ and $\phi(z_{i+1}) = \psi(z_{i+1})$. Since

$|j(i+1) - j(i)| \leq 1$, z_i and z_{i+1} lie in the same rectangle, R_j, of H. Now ψ restricted to R_j is uniformly proper. Thus, $\rho(z_i, z_{i+1})$ is bounded above by a uniform constant, k, say. Thus, $\rho(x, y) = \rho(z_0, z_n) \leq kn \leq k(d(p, q) + 1) = k(d(\psi(x), \psi(y)) + 1)$ as required. $\diamondsuit$

Now, by Lemma 3.3, the limit set, γ, of $\psi(H)$ is a uniform quasi-arc with endpoints a and b. Since $\psi(H) \subseteq Y$, we see that $\gamma \subseteq D$.

We have shown that any pair of points of $\mathrm{cut}(D)$ are connected by a uniform quasi-arc in D. It follows that D is a uniform quasidendrite, thus proving Theorem 2.1.

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