Geometrical finiteness with variable negative curvature.

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0. Introduction.

A *Hadamard manifold* is a complete, simply-connected Riemannian manifold of non-positive curvature. By a *pinched Hadamard manifold*, we shall mean a Hadamard manifold of pinched negative curvature, i.e. all the sectional curvatures lie between two negative constants.

The aim of this paper is to describe a notion of "geometrical finiteness" for a discrete group, Γ, acting on a pinched Hadamard manifold X.

The notion of geometrical finiteness has been principally used in the case where X is 3-dimensional hyperbolic space $\mathbf{H}^3$. The original definition supposed that Γ should possess a finite-sided fundamental polyhedron. Under this hypothesis, Ahlfors showed that the limit set of Γ has either zero or full spherical Lebesgue measure [Ah]. Since that time, other definitions of geometrical finiteness have been given, notably by Marden [M], Beardon and Maskit [BeM] and Thurston [T], and the notion has become central to the study of 3-dimensional hyperbolic groups.

As an isolated object, a geometrically finite group is not particularly interesting. A major problem in 3-dimensional hyperbolic geometry is to understand finitely-generated discrete hyperbolic groups that are not geometrically finite. An important conjecture is that every such group is an "algebraic limit" of geometrically finite groups.

In 3 dimensions, Teichmüller theory together with ideas of Thurston have provided powerful tools for understanding hyperbolic groups. In higher dimensions, the theory is much less well developed, and has been some confusion in the literature as to the correct notion of geometrical finiteness in this context. The existence of finite sided fundamental polyhedra, without further qualification, becomes an inapropriate hypothesis. My previous paper [Bo1] was an attempt to clarify this matter.

It seems natural to wonder what happens if one generalises in another direction, by allowing variable curvature. The extra flexibility would potentially allow for more possibilities in the construction of exotic examples. The first step, however, is to clearly understand the "geometrically finite" groups. This paper is aimed in that direction.

Let us suppose that Γ is a discrete group of isometries of a pinched Hadamard manifold, X. We want to say what it means for Γ to be geometrically finite. Now, the description involving finite sided fundamental polyhedra falls apart altogether, and it is not clear how to give a new definition on this basis. In this paper, I will not have much to say about fundamental polyhedra. The subject of Dirichlet polyhedra for discrete groups acting on complex hyperbolic space has been explored by Goldman, Parker and Phillips

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[G,GP,Ph,Pa]. In particular, the example of a discrete parabolic group given in [GP] suggests that no elegant formulation of geometrical finiteness, along these lines exists.

However, the remaining definitions (as described in [Bo1]) all have a natural interpretation for pinched negative curvature. One of the principal aims of this paper, therefore, is to show the equivalence of these notions (Theorem 6.1). Many of the arguments will run parallel to those given [Bo1], though we lose some of the useful tools such as the existence of nice convex half-spaces.

We shall retain the term "geometrical finiteness" for the notion thus defined, although it is clearly less appropriate in this context. We shall see that geometrically finite groups are finitely generated (Proposition 5.5.1), have finitely many conjugacy classes of finite subgroups (Proposition 5.5.2), and have finitely many conjugacy classes of maximal parabolic subgroups (Corollary 6.5).

It seems reasonable to conjecture that geometrical finiteness implies topological finiteness (i.e. that X/Γ is orbifold-homeomorphic to the interior of a compact orbifold with boundary). The problem reduces to the case of discrete parabolic groups of isometries. In [Bo2], it was shown that such groups are finitely generated, though the question of topological finiteness, to my knowledge, remains open.

The four main definitions of geometrical finiteness we shall use may be outlined as follows. Each definition may be stated in more than one way, and we shall describe in Chapter 5 some of the variants.

The definition which we consider the central one, since in some sense it gives us the most information, is the generalisation of Marden's definition. To the orbifold X/Γ, we adjoin the quotient, Ω/Γ, of the discontinuity domain, Ω of the ideal sphere at infinity. We thus obtain an orbifold with boundary $M_C(\Gamma) = (X \cup \Omega)/\Gamma$. We say that Γ is "F1" (geometrically finite in the first sense) if $M_C(\Gamma)$ has only finitely many topological ends, and each such end is a "parabolic end". To say that an end is a "parabolic end" means essentially that it can identified with the end of $M_C(G)$, where G is a maximal parabolic subgroup of Γ.

The second definition, F2, demands that the limit set Λ of Γ should consist enirely of "conical limit points" and "bounded parabolic fixed points". This definition can be made intrinsic to the action of Γ on Λ. It is due, in constant curvature, to Beardon and Maskit.

In [Bo1], due to lack of foresight, the third definition was that of finite-sided fundamental polyhedra, so we shall call the remaining definitions F4 and F5. These are both due to Thurston in the constant curvature case. We leave F3 for someone else to define.

Property F4 says that the "thick part" of the "convex core" of X/Γ is compact. The "convex core" is the quotient, under Γ, of the (closed) convex hull of the limit set. In the case where Γ is torsion-free, so that X/Γ is a manifold, the "thick part" of the convex core is the set of points where the injectivity radius is greater than or equal to some small positive number. We give a definition of the thick part of an orbifold in Section 3.5.

Finally, F5 says that for some $\eta > 0$, the unform η-neighbourhood of the convex core has finite volume, and that there is a bound on the orders of finite subgroups of Γ. I suspect that this latter assumption is superfluous. Certainly, if X/Γ has finite volume, then there is necessarily such a bound, and in this case it follows that X/Γ is topologically finite as an orbifold (Proposition 6.6).

We made, at the beginning, the assumption that X has pinched negative curvature. The upper curvature bound (away from 0) is essential for the Toponogov comparison theorem (Proposition 1.1.2), the consequences of which are used throughout this paper. The lower curvature bound (away from $-\infty$) is needed for the Margulis Lemma (Proposition 3.5.1), and to give an upper bound on the the volumes of uniform balls (Proposition 1.2.2). The construction of convex sets (due to Anderson [An]), which we describe in Section 2.5, uses both curvature bounds, though for most purposes one could make do with some notion of quasiconvexity, which would only require a bound away from 0. Both bounds are also used in [Bo2], as quoted in Chapter 4 — see Proposition 4.1.

It will be assumed throughout this paper that X has curvature at most -1. The additional assumption of a lower curvature bound ($-\kappa^2$) will be made in Sections 1.2, 2.5 and 3.5, and throughout Chapters 4, 5 and 6. Results given in these places should be assumed to take this as a hypothesis, though we will not always say so explicitly.

The structure of this paper, in outline, is as follows. In Chapter 1, we collect together the basic facts about Hadamard manifolds which we shall need. Chapter 2 is discussion of convexity and quasiconvexity. In Chapter 3, we describe some constructions relating to discrete group actions. In Chapter 4, we say something of the geometry of discrete parabolic groups. In Chapter 5, we give in detail the various definitions of geometric finiteness, and show some basic group-theoretic properties. Finally, in Chapter 6, we complete the proofs of equivalence of the four definitions F1, F2, F4 and F5.

1. Review of negative curvature.

The purpose of this chapter is introduce some terminology and notation, and to summarise some basic results about Hadamard manifolds which we shall need. A good reference for such manifolds is [BaGS].

A basic fact about Hadamard manifolds in general is that the exponential map based at any point is injective. Thus, any such manifold, X, is diffeomorphic to $\mathbf{R}^n$. In fact, X can be naturally compactified by adjoining an *ideal sphere* X_I to X. Thus, $X_C = X \cup X_I$ is homeomorphic to a closed n-dimensional ball.

Much of theory of Hadamard manifolds can be refined or simplified when the curvature is bounded away from 0. In this case, we can always scale the metric so that all sectional curvatures are at most -1. This will be assumed throughout this paper.

In Section 1.1, we give some properties of X under this assumption. In Section 1.2, we describe some additional properties when there is also a lower curvature bound.

1.1. Curvature bounded away from 0.

Suppose X has all sectional curvatures at most -1. Let d be the Riemannian path-metric on X.

In this case X is a *visibility manifold* i.e. any two points $x, y \in X_C$ can be joined by a unique geodesic, which we denote by $[x, y]$. The geodesic $[x, y]$ depends continuously on its endpoints x and y. We shall usually think of $[x, y]$ as a closed subset of X_C. When we speak of geodesics as paths, they will always be assumed parameterised by arc-length. If $x, y \in X$, we call $[x, y]$ a *geodesic segment*. If $x \in X$ and $y \in X_I$, we call $[x, y]$ a *geodesic ray* based at x, and tending to y. If $x, y \in X_I$, we call $[x, y]$ a *bi-infinite geodesic*.

Any two geodesic rays, $[a, y]$ and $[b, y]$, tending to the same ideal point $y \in X_I$ "converge exponentially". This will be made precise by Proposition 1.1.11. For the moment, we just note that we can find sequences $a_n \in [a, y] \cap X$, and $b_n \in [b, y] \cap X$, both tending to y, with $d(a_n, b_n) \to 0$. We can regard X_I as the set of equivalence classes of geodesic rays, where equivalence is defined by convergence of rays.

We shall write $T_x X$ and $T^1_x X$, respectively, for the tangent space and unit-tangent space to X at x. Given $x \in X$, and $y \in X_C \setminus \{x\}$, we shall write $\overrightarrow{xy} \in T^1_x X$ for the unit tangent vector based at x in the direction of y, i.e. $\overrightarrow{xy}$ is the derivative $\alpha'(0)$, where $\alpha : [0, d(x, y)] \longrightarrow X_C$ is the geodesic $[x, y]$. If $x \in X$, and $y, z \in X_C \setminus \{x\}$, we write $y\hat{x}z$ for the angle $\angle(\overrightarrow{xy}, \overrightarrow{xz})$ between $\overrightarrow{xy}$ and $\overrightarrow{xz}$.

If Q is a closed subset of X_C, and $r \geq 0$, let $N^C_r(Q)$ be the closure, in X_C, of the set $\{y \in X \mid d(y, Q \cap X) \leq r\}$. Set $N_r(Q) = Q \cup N^C_r(Q)$. We call $N_r(Q)$ the *uniform r-neighbourhood* of Q. We will only be interested in the case where $Q \cap X$ is dense in Q, and so $Q \subseteq N^C_r(Q)$. If Q_1 and Q_2 are both closed in X_C, with $Q_1 \cap X$ and $Q_2 \cap X$ dense in Q_1 and Q_2 respectively, we shall write $d(Q_1, Q_2) = d(Q_1 \cap X, Q_2 \cap X) = \inf\{\mathrm{d(x, y)} \mid \mathrm{x} \in \mathrm{Q_1 \cap X, y \in Q_2 \cap X}\}$.

Suppose $x \in X$. We choose an identification of the tangent space $T_x X$ with $\mathbf{R}^n$, so that the standard inner-product on $\mathbf{R}^n$ induces the Riemannian inner-product on $T_x X$. This defines an exponential map $\exp(X, x) : \mathbf{R}^n \longrightarrow X$. We have that $\exp(X, x)$ is a diffeomorphism from $\mathbf{R}^n$ to X.

Let $\mathbf{H}^n$ be n-dimensional hyperbolic space. Fix some basepoint $a_0 \in \mathbf{H}^n$, and an identification of $\mathbf{R}^n$ with $T_{a_0}\mathbf{H}^n$. The map

$$e = e(X, x) = \exp(\mathbf{H}^n, a_0) \circ \exp(X, x)^{-1} : X \longrightarrow \mathbf{H}^n$$

is a diffeomorphism with $a_0 = e(x)$. It follows from the Rauch comparison theorem [CE], that:

Proposition 1.1.1 : *The map $e : X \longrightarrow \mathbf{H}^n$ is distance non-increasing.* ♢

One simple consequence, is the following version of Toponogov's comparison theorem, which is the basis of most of the results of this section [CE].

Proposition 1.1.2 : *Suppose x, y, z are any three distinct points in X. Let x', y', z' be three points in the hyperbolic plane $(\mathbf{H}^2, d')$ satisfying $d'(x', y') = d(x, y)$, $d'(y', z') = d(y, z)$ and $d'(z', x') = d(z, x)$. Then $x\hat{y}z \leq x'\hat{y}'z'$, $y\hat{z}x \leq y'\hat{z}'x'$ and $z\hat{x}y \leq z'\hat{x}'y'$.* ♢

We call $x'y'z'$ a *comparison triangle* for xyz.

Corollary 1.1.3 : *Suppose $x \in X$, and $y, z \in X_C \setminus \{x\}$ are distinct. Let $\theta = y\hat{x}z$, and $r = d(x, [y, z])$. Then*

$$\sin(\theta/2) \leq \operatorname{sech} r.$$

Proof : The inequality comes from the following formula of hyperbolic trigonometry.

Suppose a and b are distinct points of the hyperbolic plane $(\mathbf{H}^2, d')$. Suppose $c \in \mathbf{H}^2_I$ with $a\hat{b}c = \pi/2$, $b\hat{a}c = \phi$ and $d'(a, b) = R$. Then $\sin\phi \cosh R = 1$. From this, one deduces easily that in the more general situation where $c \in X_C \setminus \{a, b\}$, and $a\hat{b}c \geq \pi/2$, we have $\sin\phi \leq \operatorname{sech} R$.

The result now follows by letting $w \in [x, y]$ be any (in fact the unique) nearest point to x, and applying Proposition 1.1.2 to the triangles xwz and/or xwy. ♢

By similar arguments, one may show:

Corollary 1.1.4 : *Suppose that $x, y, z \in X$ are distinct, with $x\hat{y}z \geq \pi/2$. Let $r = d(y, z)$. Then,*

$$d(x, z) \geq d(x, y) + \log(\cosh r).$$

♢

Another comparison theorem central to the study of negative curvature is the following "CAT(−1)" inequality (See for example [Br]).

Proposition 1.1.5 (CAT(−1)) : *Suppose $x, y, z \in X$, and $u \in [x, y]$ and $v \in [x, z]$. Let $x'y'z'$ be a comparison triangle for xyz in the hyperbolic plane $(\mathbf{H}^2, d')$, i.e. $d'(x', y') = d(x, y)$, $d'(y', z') = d(y, z)$ and $d'(z', x') = d(z, x)$. Let $u' \in [x', y']$ and $v' \in [x', z']$ be the points with $d'(x', u') = d(x, u)$ and $d'(x', v') = d(x, v)$. Then $d(u, v) \leq d'(u', v')$.*

Proof : By applying Proposition 1.1.2, first to the triangles xuv and yuv, and then to the triangles xvy and zvy. ♢

Corollary 1.1.6 : *If $x, y, z \in X_C$, then $[y, z] \subseteq N_{\lambda_0}([x, y] \cup [x, z])$, where $\lambda_0 = \cosh^{-1}\sqrt{2}$.*

Proof : By continuity, we can suppose that $x, y, z \in X$. By Proposition 1.1.5, it is enough to prove the result for a triangle in the hyperbolic plane. This is an exercise in hyperbolic trigonometry. ♢

Given $a \in X$, define $h_a : X \times X \longrightarrow \mathbf{R}$ by

$$h_a(x, y) = d(x, a) - d(x, y).$$

Let $\Delta(X_I) \subseteq X_C \times X_C$ be the diagonal

$$\Delta(X_I) = \{(x, x) \mid x \in X_I\}.$$

Proposition 1.1.7 : *For all $a \in X$, the map h_a extends uniquely to a continuous map*

$$h_a : (X_C \times X_C) \setminus \Delta(X_I) \longrightarrow [-\infty, \infty),$$

with $h_a(x, y) = -\infty$ if and only if $y \in X_I$.

Proof : Suppose $(x, y) \in (X_C \times X_C) \setminus \Delta(X_I)$. It is enough to show that for any sequences $x_n \to x$ and $y_n \to y$ with $x_n, y_n \in X$ we have $h_a(x_n, y_n)$ convergent in $[-\infty, \infty)$, i.e. either $h_a(x_n, y_n)$ is Cauchy, or $h_a(x_n, y_n) \to -\infty$.

Clearly h_a is continuous on $X \times X$, so we can suppose that $(x, y) \notin X \times X$. If $x \in X$ and $y \in X_I$, then $h_a(x_n, y_n) \to -\infty$. Thus, we can suppose that $x \in X_I$.

Given any $\epsilon > 0$, we can find, by the convergence of geodesic rays, points $u \in [a, x]$ and $v \in [y, x]$ with $d(u, v) \le \epsilon$. Now, $d(u, [a, x_n]) \to 0$ and $d(v, [a, y_n]) \to 0$ as $n \to \infty$. Thus, for some n_0, we have points $u_n \in [a, x_n]$ and $v_n \in [y_n, x_n]$ with $d(u, u_n) \le \epsilon$ and $d(v, v_n) \le \epsilon$, for all $n \ge n_0$. It follows that $|h_a(x_n, y_n) - h_a(u, y_n)| \le 6\epsilon$ for all $n \ge n_0$.

If $y \in X_I$, then clearly $h_a(u, y_n) \to -\infty$, and so $h_a(x_n, y_n) \to -\infty$.

Suppose $y \in X$. If $m, n \ge n_0$, then $|h_a(x_n, y_n) - h_a(x_m, y_m)| \le |h_a(u, y_n) - h_a(u, y_m)| + 12\epsilon \le d(y_n, y_m) + 12\epsilon$. It follows, in this case, that $h_a(x_n, y_n)$ is a Cauchy sequence. $\diamondsuit$

Th last result is needed for our discussion of projection to quasiconvex sets, in Chapter 2. It also gives us the basic properties of "Busemann functions". Note that if $a, b, x, y \in X$, then $h_b(x, y) - h_a(x, y) = h_b(x, a)$. Thus Proposition 1.1.7 tells us that:

Corollary 1.1.8 : *For all $a, b \in X$, and $(x, y) \in (X_C \times X_C) \setminus \Delta(X_I)$, we have $h_b(x, y) - h_a(x, y) = h_b(x, a)$.* $\diamondsuit$

Corollary 1.1.9 : *Suppose $a, y \in X$, and $x \in X_C$. If $z \in [a, x]$, then $h_a(z, y) \le h_a(x, y)$.*

Proof : The case where $x \in X$ is just the triangle inequality. The case where $x \in X_I$ follows by continuity (Proposition 1.1.7), after taking a sequence $x_n \in [z, x] \cap X$ tending to x. $\diamondsuit$

If $x \in X_I$, a function of the form $[y \mapsto h_a(x, y)]$ is called a *Busemann function* about x. Corollary 1.1.8 tells us that any two Busemann functions about x differ by a constant. By convention, we take the value of a Busemann function at x itself to be $+\infty$. It turns out that any Busemann function, h, is C^2 [HI], and the norm of its gradient is everywhere equal to 1. The level sets of h are called *horospheres* about x. Thus the horospheres form a codimension-1 foliation of X, orthogonal to the foliation by bi-infinite geodesics having one endpoint at x.

A set of the form $h^{-1}[r, \infty)$ for $r \in \mathbf{R}$ is called a *horoball* about x. Using Corollary 1.1.9, such a horoball may alternatively be described as the closure of the set $\bigcup\{N_t(\beta(t + u)) \mid t \in [0, \infty)\}$, where $\beta : [0, \infty)$ is a geodesic ray tending to x, with $\beta(0) \in h^{-1}r$. In particular, horoballs are convex.

Applying Propositon 1.1.7, we may extend Corollary 1.1.4 as follows:

Proposition 1.1.10 : *Suppose $x \in X_I$, and that h is a Busemann function about x. Suppose $y, z \in X$ with $x\hat{y}z \ge \pi/2$, and $d(x, y) = r$. Then,*

$$h(y) - h(z) \ge \log \cosh r.$$

$\diamondsuit$

We introduce the following (non-standard) notation. Suppose $x \in X_I$, and let $h : X_C \longrightarrow [-\infty, \infty]$ be a Busemann function about x. Let ϕ_t be the gradient flow for $-h$. Given $y \in X$, we shall write $y+t$ for $\phi_t(x)$. Thus, if y lies in the bi-infinite geodesic $[z,x]$ and $t \geq 0$, then $y+t$ and $y-t$ are the points in $[y,x]$ and $[z,y]$ respectively, at distance t from y. We shall write $y-\infty$ for z, and $y+\infty$ for x.

The following describes the exponential convergence of geodesic rays tending to the same ideal point. It may be deduced from the constant curvature case using the CAT(-1) inequality (Poposition 1.1.5).

Proposition 1.1.11 :
(1) Given any $y, z \in X$, $d(y+t, z+t)$ is monotonically decreasing in t.
(2) For all r, there exists R, such that if $y, z \in X$ satisfy $h(y) = h(z)$ and $d(y,z) \leq r$, then $d(y+t, z+t) \leq Re^{-t}$ for all t. ♢

Finally, we find a second direct application of Proposition 1.1.1 to give a lower bound on the volumes of uniform balls in X.

Let $V(r,n)$ be the volume of the uniform r-ball in $\mathbf{H}^n$. Then:

Proposition 1.1.12 : *For any $x \in X$ and $r \geq 0$, the volume of the uniform r-ball, $N_r(x)$, is at least $V(r,n)$.*

Proof : Let $e = e(X,x) : X \longrightarrow \mathbf{H}^n$ be as in Proposition 1.1.1. Then $e(N_r(x))$ is the uniform r-ball in $\mathbf{H}^n$ about $e(x)$. The result follows since e is distance non-increasing. ♢

1.2. Curvature bounded away from $-\infty$.

In this section, we suppose that all the sectional curvatures of X are at least $-\kappa^2$, where $\kappa \geq 1$.

Let $\mathbf{H}^n(\kappa)$ be the Hadamard manifold of constant curvature $-\kappa^2$. (Thus $\mathbf{H}^n(1) = \mathbf{H}^n$.) Suppose that $a_0 \in \mathbf{H}^n(\kappa)$ is some fixed basepoint. Given any $x \in X$, we may define, as with Proposition 1.1.1, the map

$$e_\kappa = e_\kappa(X,x) : X \longrightarrow \mathbf{H}^n(\kappa)$$

by $e_\kappa(X,x) = \exp(\mathbf{H}^n(\kappa), a_0) \circ \exp(X,x)^{-1}$. Thus, as before, e_κ is a diffeomorphism. Again, by the Rauch comparison theorem we have:

Proposition 1.2.1 : *The map $e_\kappa : X \longrightarrow \mathbf{H}^n(\kappa)$ is distance non-decreasing.* ♢

The corresponding version of Toponogov's comparison theorem now gives a lower bound on angles (c.f. Proposition 1.1.2):

Proposition 1.2.2 : *Suppose x, y, z are distinct points of X. Let x_0, y_0, z_0 be three points in $(\mathbf{H}^2(\kappa), d_0)$ satisfying $d_0(x_0, y_0) = d(x, y)$, $d_0(y_0, z_0) = d(y, z)$ and $d_0(z_0, x_0) = d(z, x)$. Then, $x\hat{y}z \geq x_0\hat{y}_0z_0$, $y\hat{z}x \geq y_0\hat{z}_0x_0$ and $z\hat{x}y \geq z_0\hat{x}_0y_0$.* $\diamondsuit$

Corollary 1.2.3 : *Suppose $x, y \in X$ are distinct and $z \in X_I$. Let $r = d(x, y)$ and $\theta = y\hat{x}z$. If $x\hat{y}z \leq \pi/2$, then $\sin\theta \geq \operatorname{sech}(\kappa r)$.*

Proof : Suppose a, b are two points in $(\mathbf{H}^2(\kappa), d_0)$ and c an ideal point with $a\hat{b}c = \pi/2$. If $R = d_0(a, b)$, then $b\hat{a}c = \sin^{-1}\operatorname{sech}(\kappa r)$ (c.f. Corollary 1.1.3). We can deduce that if $a, b, c \in \mathbf{H}^2(\kappa)$ with $d(a, b) = R$, $d(b, c) = h$, and $a\hat{b}c \leq \pi/2$, then $b\hat{a}c \geq \sin^{-1}\operatorname{sech}(\kappa R) - \epsilon_{\kappa,R}(h)$, where $\epsilon_{\kappa,R} \to 0$ as $h \to \infty$.

Suppose now that x, y, z, r, θ are as in the hypotheses. Choose a sequence of points $z_i \in [y, z] \cap X$ with $z_i \to z$. Let $x_0y_0z_{i0}$ be a comparison triangle in $\mathbf{H}^2(\kappa)$ for xyz_i. Thus, by Proposition 1.2.3, we have $x_0\hat{y}_0z_{i0} \leq x\hat{y}z_i = x\hat{y}z \leq \pi/2$. Thus, again by Proposition 1.2.3, we have $y\hat{x}z_i \geq y_0\hat{x}_0z_{i0} \geq \sin^{-1}\operatorname{sech}(\kappa r) - \epsilon_{\kappa,r}(d(y, z_i))$. As $i \to \infty$, $y\hat{x}z_i \to \theta$, and $d(y, z_i) \to \infty$ so $\epsilon_{\kappa,r}(d(y, z_i)) \to 0$. Thus, $\theta \geq \sin^{-1}\operatorname{sech}(\kappa r)$ as required. $\diamondsuit$

Another consequence of Propositon 1.2.1 is an upper bound on the volumes of uniform balls (c.f. Proposition 1.1.12). Note that the volume of a uniform r-ball in $\mathbf{H}^n(\kappa)$ is $V(\kappa r, n)/\kappa^n$.

Proposition 1.2.4 : *If $x \in X$ and $r \geq 0$, then the volume of the uniform r-ball $N_r(x)$ is at most $V(\kappa r, n)/\kappa^n$.* $\diamondsuit$

2. Convexity.

Let X be a Hadamard manifold. For Sections 2.1–2.4, we assume only an upper curvature bound, -1, for X. For Section 2.5, we need also a lower curvature bound.

A subset Q of X_C is *convex* if $[x, y] \subseteq Q$ for all $x, y \in Q$. The lack of a good notion of half-space in a variably curved manifold means that convex sets are difficult to construct and work with. A construction, due to Anderson, for pinched Hadamard manifolds will be described in Section 2.5. For most purposes, however, one could make do with some notion of quasiconvexity, as we describe in Section 2.2. We begin with a general discussion of projection to closed sets. A detailed discussion of the constant curvature case is given in [EM].

2.1. Projection.

Suppose that $Q \subseteq X_C$ is closed. Let

$$\operatorname{proj}^0_Q = \{(x, y) \in X \times (Q \cap X) \mid d(x, y) = d(x, Q)\}.$$

In other words, $\{y \mid (x,y) \in \mathrm{proj}^0_Q\}$ is the set of nearest points of Q to x. Clearly, proj^0_Q is a closed subset of $X \times X$. Let proj^C_Q be the closure of proj^0_Q in $X_C \times X_C$, and set

$$\mathrm{proj}_Q = \Delta(Q) \cup \mathrm{proj}^C_Q,$$

where $\Delta(Q)$ is the diagonal $\{(x,x) \mid x \in Q\}$. We shall only be interested in cases where $Q \cap X$ is dense in Q, and so $\Delta(Q) \subseteq \mathrm{proj}^C_Q$.

Given $x \in X_C$, we write

$$\mathrm{proj}_Q(x) = \{y \in X_C \mid (x,y) \in \mathrm{proj}_Q\}.$$

Clearly, $\mathrm{proj}_Q(x) \subseteq Q$.

Suppose $Q \cap X \neq \emptyset$. If $x \in X$, we have $\mathrm{proj}_Q(x) \subseteq X$, which we have already described as the set of nearest points to x. We want to describe $\mathrm{proj}_Q(x)$ in the case where $x \in X_I$.

Suppose $x \in X_I$, and let $h : X_C \longrightarrow [-\infty, \infty]$ be a Busemann function about x. Let

$$m(x) = \{y \in X \mid h(z) \leq h(y) \text{ for all } z \in Q \cap X\},$$

i.e. $m(x)$ is the set of those $y \in Q$ which maximise $h(y)$. (Perhaps $m(x) = \emptyset$ if $x \in Q \cap X_I$.) Note that $m(x)$ is defined independently of the choice of h.

Proposition 2.1.1 : *Suppose $Q \subseteq X_C$ is closed, and $Q \cap X \neq \emptyset$.*
If $x \in Q \cap X_I$, then $\mathrm{proj}_Q(x) = \{x\} \cup m(x)$.
If $x \in X_I \setminus Q$, then $\mathrm{proj}_Q(x) = m(x)$.

Proof : We choose a Busemann function $h = [y \mapsto h_a(x,y)]$ for some $a \in X$.

Note that since $\mathrm{proj}_Q(x) \subseteq Q$, we have $x \in \mathrm{proj}_Q(x)$ if and only if $x \in X_I$. We claim that $m(x) \subseteq \mathrm{proj}_Q(x) \subseteq \{x\} \cup m(x)$.

First, we show $\mathrm{proj}_Q(x) \subseteq \{x\} \cup m(x)$. Suppose that $y \in \mathrm{proj}_Q(x) \setminus \{x\}$. Then $(x,y) \in \mathrm{proj}^C_Q$, and so there is a sequence $(x_n, y_n) \in \mathrm{proj}^0_Q$, with $(x_n, y_n) \to (x,y)$. Suppose $z \in Q \cap X$. By the definition of proj^0_Q, we have $d(x_n, z) \geq d(x_n, y_n)$. Thus $h_a(x_n, y_n) - h_a(x_n, z) \geq 0$ for all n. By continuity of h_a (Proposition 1.1.7), we have that $h_a(x_n, y_n) - h_a(x_n, z) \to h_a(x,y) - h_a(x,z) = h(y) - h(z)$. Thus $h(z) \leq h(y)$. This shows that $y \in m(x)$.

It remains to see that $m(x) \subseteq \mathrm{proj}_Q(x)$. Suppose $y \in m(x) \subseteq Q$. Let $x_n \in [y,x] \cap X$ be a sequence of points tending to x. If $z \in Q \cap X$, then $h(y) \geq h(z)$. By Corollary 1.1.8, we have $h_y(x,z) = h_a(x,z) - h_a(x,y) = h(z) - h(y) \leq 0$. By Lemma 1.1.9, we have $h_y(x_n, z) \leq h_y(x,z) \leq 0$ for all n. Thus $d(x_n, y) \leq d(x_n, z)$. We conclude that $(x_n, y) \in \mathrm{proj}^0_Q$ for all n. But $(x_n, y) \to (x,y)$ and so $(x,y) \in \mathrm{proj}_Q$, i.e. $y \in \mathrm{proj}_Q(x)$. $\diamondsuit$

2.2. Quasiconvexity.

Recall the definition of uniform neighbourhoods $N_r(Q)$ from Chapter 1.

Definition : A closed subset $Q \subseteq X_C$ is λ-*quasiconvex* if $[x,y] \subseteq N_\lambda(Q)$ for all $x, y \in Q$.

We say that a set is *quasiconvex* if it is λ-quasiconvex for some $\lambda \in [0,\infty)$. Note that if Q is quasiconvex and contains more than one point, then Q meets X. In fact, $Q \cap X$ is dense in Q.

Definition : A closed subset $Q \subseteq X_C$ is *starlike* about $x \in X_C$ if $[x,y] \subseteq Q$ for all $y \in Q$.

Corollary 1.1.6 shows that any starlike set is λ_0-quasiconvex, where $\lambda_0 = \cosh^{-1}\sqrt{2}$. The following lemma provides more examples of quasiconvex sets.

Lemma 2.2.1 : *Suppose $x_0, x_1, \ldots, x_n \in X_C$ are $n+1$ points, then*

$$[x_0, x_n] \subseteq N_\lambda([x_0,x_1] \cup [x_1,x_2] \cup \cdots \cup [x_{n-1},x_n]),$$

where $\lambda = \lambda_0 \lceil \log_2 n \rceil$, where $\lambda_0 = \cosh^{-1}\sqrt{2}$, and $\lceil \log_2 n \rceil$ is the smallest integer greater than or equal to $\log_2 n$.

Proof : We can assume that $n = 2^r$ for some $r \in \mathbf{N}$. Let $m = \frac{n}{2} = 2^{r-1}$. By Corollary 1.1.6, we have

$$[x_0, x_n] \subseteq N_{\lambda_0}([x_0, x_m] \cup [x_m, x_n]).$$

The result follows by induction on r. ◊

Thus, any set Q in which any two points can be joined by a piecewise geodesic path with a bounded number of segments is quasiconvex.

Given any closed subset $Q \subseteq X_C$, we define

$$\text{join}(Q) = \bigcup \{[x,y] \mid x, y \in Q\}.$$

Thus, join(Q) will be a first approximation to the convex hull of Q. By Lemma 2.2.1, see that join(Q) is $(2\lambda_0)$-quasiconvex for any closed set $Q \subseteq X_C$. We also have that join$(Q) \cap X_I = Q \cap X_I$. Note that to say that Q is λ-quasiconvex means precisely that join$(Q) \subseteq N_\lambda(Q)$.

Definition : Given two closed subsets $Q_1, Q_2 \subseteq X_C$, the *Hausdorff distance* between Q_1 and Q_2 is the minimal $r \in [0,\infty]$ such that both $Q_1 \subseteq N_r(Q_2)$ and $Q_2 \subseteq N_r(Q_1)$.

We write hd(Q_1, Q_2) for the Hausdorff distance. Note that if hd$(Q_1,Q_2) < \infty$, then $Q_1 \cap X_I = Q_2 \cap X_I$.

If hd$(Q_1, Q_2) = r < \infty$ and Q_1 is λ-quasiconvex, then Q_2 is $(2r+\lambda)$-quasiconvex. (This uses the CAT(-1) inequality.) We see that quasiconvexity is invariant under the equivalence relation of having finite Hausdorff distance.

We next want to consider projection to quasiconvex sets.

Proposition 2.2.2 : Suppose $Q_1, Q_2 \subseteq X_C$ are both closed and λ-quasiconvex, and that both meet X. Suppose that $\mathrm{hd}(Q_1, Q_2) = r < \infty$ (so that $Q_1 \cap X_I = Q_2 \cap X_I$). If $x \in X_C \setminus (Q_1 \cap X_I)$, then

$$\mathrm{diam}\big(\mathrm{proj}_{Q_1}(x) \cup \mathrm{proj}_{Q_2}(x)\big) \leq r + \cosh^{-1} e^{\lambda} + \cosh^{-1} e^{\lambda + r}.$$

◊

Proof : Suppose first, that $x \in X$.

Let $y, z \in \mathrm{proj}_{Q_1}(x) \cup \mathrm{proj}_{Q_2}(x)$. Without loss of generality, we can suppose that $d(x, y) \geq d(x, z)$ and that $y \in \mathrm{proj}_{Q_1}(x)$. Since $\mathrm{hd}(Q_1, Q_2) \leq r$, there is some $w \in Q_1$ with $d(z, w) \leq r$ (Figure 2a).

Figure 2a.

Let u be a nearest point on $[y, w]$ to x (in fact the unique nearest point). Since Q_1 is λ-quasiconvex, $u \in N_\lambda(Q)$. Since $y \in \mathrm{proj}_{Q_1}(x)$, we must have

$$d(x, y) \leq d(x, u) + \lambda.$$

Also,

$$\begin{aligned} d(x, w) &\leq d(x, z) + r \leq d(x, y) + r \\ &\leq d(x, u) + \lambda + r. \end{aligned}$$

If $u \neq y$, then $x\hat{u}y \geq \pi/2$. Since $d(x, y) - d(x, u) \leq \lambda$, applying Corollary 1.1.4, we find that $\log\cosh(d(u, y)) \leq \lambda$, thus

$$d(u, y) \leq \cosh^{-1} e^{\lambda}.$$

Similarly, we get that

$$d(u, w) \leq \cosh^{-1} e^{\lambda + r}.$$

We conclude that

$$d(y, z) \leq r + \cosh^{-1} e^{\lambda} + \cosh^{-1} e^{\lambda + r}$$

as required.

The case where $x \in X_I$ follows similarly, applying Lemma 1.1.10 in place of Corollary 1.1.4, and using the description of $\mathrm{proj}_{Q_i}(x)$ given by Proposition 2.1.1. In this case, we take u to be a point (in fact the unique point) on $[y, w]$ maximising a Busemann function.

◊

Corollary 2.2.3 : *Suppose $Q \subseteq X_C$ is closed, λ-quasiconvex, and meets X. If $x \in X_C \setminus (Q \cap X_I)$, then*

$$\mathrm{diam}\ \mathrm{proj}_Q(x) \leq 2\cosh^{-1} e^{\lambda}.$$

Proof : Set $Q = Q_1 = Q_2$ in Proposition 2.2.2. $\diamondsuit$

It is easy to check that if Q is quasiconvex, and $x \in Q \cap X_I$ then $\mathrm{proj}_Q(x) = \{x\}$.

In particular, we see that if Q is convex (i.e. 0-quasiconvex), then $\mathrm{proj}_Q(x)$ consists of a single point of Q, for each $x \in X_C$. Thus, in this case, we may think of proj_Q as a map from X_C to Q. Since the graph is closed, by definition, this map is continuous. Thus, proj_Q is a retraction of X_C onto Q.

Lemma 2.2.4 : *Suppose $Q \subseteq X_C$ is convex.*
(1) Suppose $x \in X$, and that $y \in Q \cap X$ locally minimises the function $[z \mapsto d(x,z)]$ on $Q \cap X$. Then, $y = \mathrm{proj}_Q x$.
(2) Suppose $x \in X_I$. Let h be a Busemann function about x. Suppose $y \in Q \cap X$ locally maximises h on $H \cap X$. Then $y = \mathrm{proj}_Q x$.

Proof : Suppose that $y \neq z = \mathrm{proj}_Q x$. Then $x\hat{y}z < \pi/2$ (by Toponogov's comparison theorem, if $x \in X$, and so by continuity if $x \in X_I$). But $[y,z] \subseteq Q$, which contradicts the hypothesis on y. $\diamondsuit$

Note that one can generalise to the case where Q is quasiconvex. Thus, if $y \in Q \cap X$ minimises the distance to x over a sufficiently large part of $Q \cap X$, then $y \in \mathrm{proj}_Q x$.

2.3. Pseudoconvexity.

This section is a digression. We shall not refer to it again in the rest of this paper.

The definition of quasiconvexity we have just given is a standard notion. It is somewhat unfortunate that the intersection of two quasiconvex sets need not be quasiconvex. It thus does not make much sense to speak of "quasiconvex hulls" of arbitrary sets. However, in the context in which we are working, we could replace quasiconvexity by the following notion of "pseudoconvexity". Given $\mu \in [0,\infty)$, we say that $Q \subseteq X_C$ is *μ-pseudoconvex* if $[x,y] \subseteq Q$ whenever $x,y \in Q$ and $d(x,y) > \mu$. This is clearly closed under intersection. Given $X \subseteq X_C$, we define $\mathrm{hull}_\mu(Q)$ as the smallest μ-pseudoconvex set containing Q.

It is not hard to see that, given any $\mu > 0$, there is some $R(\mu) > 0$, so that for any $Q \subseteq X_C$, $N_{R(\mu)}(\mathrm{join}(Q))$ is μ-pseudoconvex. In particular, if Q is λ-quasiconvex, then $\mathrm{hull}_\mu(Q) \subseteq N_{\lambda+R(\mu)}(Q)$.

Most of the discussion involving convex hulls in this paper can be interpreted for μ-pseudoconvex hulls, though we shall make no explicit mention of this. It is not clear what the boundary of a pseudoconvex hull looks like in general. Note that as $\mu \to 0$, then the quantity $R(\mu)$ will necessarily tend to ∞. Thus, this does not give rise to a construction of convex sets. For this, we need to assume a lower curvature bound (Section 2.5).

2.4. Cones and visual radii.

Let ξ be a unit tangent vector based at $x \in X$. Given any $\theta \in [0,\pi]$, write

$$\mathrm{cone}(\xi,\theta) = \{y \in X_C \mid \angle(\xi, \overrightarrow{xy}) \leq \theta\}.$$

Recall that $\overrightarrow{xy}$ is the unit tangent vector at x in the direction of y. We use $\angle(\ ,\ )$ for the angle between two tangent vectors. We call cone(ξ, θ) the *cone of angle* θ *about* ξ. It is the closure, in X_C, of the image of a spherical cone in $\mathbf{R}^n$ under the the exponential map based at x. If $\theta = \pi/2$, we shall refer to cone$(\xi, \pi/2)$ as the *half-space* about ξ.

Suppose that $y, z \in X_C$ are distinct. Then $[y, z] \subseteq X_C$ is convex. Thus, as remarked at the end of Section 2.2, $\mathrm{proj}_{[y,z]}$ is a retraction of X_C onto $[y, z]$. By applying Toponogov's comparison theorem (Propostion 1.1.2), we arrive at the following alternative description of half-spaces:

Lemma 2.4.1 : *Suppose $\xi \in T_x^1(X)$. Suppose that y and z are distinct points of X_C with $x \in [y, z] \setminus \{y, z\}$ and $\xi = \overrightarrow{xy}$. Then*

$$\mathrm{cone}(\xi, \pi/2) = \mathrm{proj}_{[y,z]}^{-1}[x, y].$$

◊

Given any closed set $Q \subseteq X_C$, and $x \in X \setminus Q$, we define

$$\mathrm{vr}(Q, x) = \min_{y \in X_I} \max_{z \in Q} y\hat{x}z.$$

We call vr(Q, x) the *visual radius* of Q at x. In other words, it is the smallest $\theta > 0$ such that Q lies inside some cone of angle θ based at x.

It is not hard to see that the map $[x \mapsto \mathrm{vr}(Q, x)]$ is continuous on $X \setminus Q$. Setting $\mathrm{vr}(Q, x) = \pi$ for $x \in Q$, it becomes upper-semicontinuous on the whole of X.

Given any $\theta \in (0, \pi)$ we write

$$V_\theta^0(Q) = \{x \in X \mid \mathrm{vr}(Q) \geq \theta\}.$$

We see that $V_\theta^0(Q)$ is a closed subset of X. Let $V_\theta^C(Q)$ be the closure of $V_\theta^0(Q)$ in X_C, and set $V_\theta(Q) = Q \cup V_\theta^C(Q)$.

We remark that, given $0 < \phi < \theta \leq \pi/2$, then there is some r such that for any Q, we have $V_\phi \subseteq N_r V_\theta(Q)$. We have no explicit use for this result, however we shall need the following.

Lemma 2.4.2 : Given any $\theta \in (0, \pi/2]$ and $\lambda \in [0, \infty)$, there is some $r = r(\theta, \lambda)$, such that if $Q \subseteq X_C$ is closed and λ-quasiconvex, then

$$V_\theta(Q) \subseteq N_r(Q).$$

Proof : Let $r = \lambda + \operatorname{sech}^{-1} \sin(\theta/2)$.

Suppose that $x \in X$ with $d(x, Q) > r$. If $y, z \in Q$, then $d(x, [y, z]) > r - \lambda = \operatorname{sech}^{-1} \sin(\theta/2)$. By Corollary 1.1.3, we find that $y\hat{x}z < \theta$, so certainly, $\operatorname{vr}(Q, x) < \theta$. Thus $x \notin V_\theta(Q)$. This shows that $V_\theta^0(Q) = V_\theta(Q) \cap X \subseteq N_r(Q)$.

But $N_r(Q)$ is, by definition, closed in X_C, and contains Q. Thus $V_\theta(Q) \subseteq N_r(Q)$. $\diamondsuit$

Now, if $Q \subseteq X_C$ is any closed set, we know that $\operatorname{join} Q$ is $(2\lambda_0)$-quasiconvex. Thus, by Lemma 2.4.2, we have

$$V_\theta(Q) \subseteq V_\theta(\operatorname{join} Q) \subseteq N_r(\operatorname{join} Q),$$

for some r depending only on θ. Note that $N_r(\operatorname{join} Q) \cap X_I = (\operatorname{join} Q) \cap X_I = Q \cap X_I$. Thus:

Corollary 2.4.3 : *Given any closed set $Q \subseteq X_C$ and any $\theta \in (0, \pi/2]$ we have*

$$V_\theta(Q) \cap X_I = Q \cap X_I.$$

Another way to say this is that if (x_n) is a sequence of points tending to a point of $X_I \setminus Q$, then $\operatorname{vr}(Q, x_n)$ tends to 0.

2.5. Construction of convex sets.

In this section, we assume that all the sectional curvatures of X lie between $-\kappa^2$ and -1.

Given any closed set $Q \subseteq X_C$, we write $\operatorname{hull}(Q)$ for the (closed) convex hull of Q, i.e. the smallest closed convex set containing Q. One can show that $\operatorname{hull}(Q)$ varies continuously with Q, in the Hausdorff topology [Bo3], though we shall not need this fact here.

In [An], Anderson gives a means of constructing convex sets in X. We state the result in the following form.

Proposition 2.5.1 : *For any $\kappa \geq 1$, there is some $\theta_0 = \theta_0(\kappa)$ such that if ξ is a unit tangent vector to X, then*

$$\operatorname{hull} \operatorname{cone}(\xi, \theta_0) \subseteq \operatorname{cone}(\xi, \pi/2).$$

Proof : Fix, for the moment, some $R > 0$.

Suppose $\xi \in T^1_x X$, and let $y \in X$ be the point with $\overrightarrow{xy} = \xi$ and $d(x,y) = R$. In [An], Anderson constructs a convex set $H \subseteq X_C$ with smooth boundary, ∂H, and ξ as the inward-pointing normal to ∂H at x. It follows that $H \subseteq \mathrm{cone}(\xi, \pi/2)$. [An, Lemma 2.4] says that there is some $\phi(R,\kappa) \in [0,\pi]$ such that if $z \in X_I \setminus H$, then $x\hat{y}z \leq \phi(R,\kappa)$ (Figure 2b).

Figure 2b.

Moreover, from the description given of $\phi(R,\kappa)$, it is clear that $\phi(R,\kappa) \to 0$ as $R \to \infty$. Thus, we can choose $R = R(\kappa)$ so that $\phi(R,\kappa) \leq \pi/2$. In other words, if $z \in X_I \setminus H$, then $x\hat{y}z \leq \pi/2$. It follows by Corollary 1.2.3, that $y\hat{x}z \geq \theta_0$ where $\theta_0 = \theta_0(\kappa) = \sin^{-1} \mathrm{sech}(\kappa R(\kappa))$. We see that $\mathrm{cone}(\xi, \theta_0) \cap X_I \subseteq H$. Since H is convex and contains x, we have $\mathrm{cone}(\xi,\theta_0) \subseteq H$. Thus $\mathrm{hull}\,\mathrm{cone}(\xi,\theta_0) \subseteq H \subseteq \mathrm{cone}(\xi,\pi/2)$. ♢

Proposition 2.5.2 : *If $Q \subseteq X_C$ is closed, then*

$$\mathrm{hull}(Q) \subseteq V_{\theta_0}(Q).$$

Proof : Suppose $x \in X \setminus V_{\theta_0}(Q)$. Thus $Q \subseteq \mathrm{cone}(\xi,\theta)$, where $\theta < \theta_0$, and ξ is a unit tangent vector at x. Let $z \in X_I$ be the ideal point with $\overrightarrow{xz} = \xi$. If $y \in [x,z] \setminus \{x\}$ is sufficiently close to x, then clearly $Q \subseteq \mathrm{cone}(\overrightarrow{yz}, \theta_0)$. Thus,

$$\mathrm{hull}(Q) \subseteq \mathrm{hull}\,\mathrm{cone}(\overrightarrow{yz},\theta_0) \subseteq \mathrm{cone}(\overrightarrow{yz},\pi/2).$$

But $x \notin \mathrm{cone}(\overrightarrow{yz},\pi/2)$, and so $x \notin \mathrm{hull}(Q)$.

Suppose $x \in X_I \setminus V_{\theta_0}(Q)$. Choose any $z \in Q$. By Corollary 2.4.3, $V_{\theta_0/2}(Q) \cap X_I = Q \cap X_I = V_{\theta_0}(Q) \cap X_I$, and so we can find $y \in [x,z] \cap X \setminus V_{\theta_0/2}(Q)$. Since $\mathrm{vr}(Q,y) \leq \theta_0/2$, and $z \in Q$, we must have $Q \subseteq \mathrm{cone}(\overrightarrow{yz},\theta_0)$. As in the first part, we see that $x \notin \mathrm{hull}(Q)$. ♢

Since (by Corollary 2.4.3), $Q \cap X_I = V_{\theta_0}(Q) \cap X_I$, we have as a corollary, the result of Anderson:

Corollary 2.5.3 : *If $Q \subseteq X_C$ is closed, then*

$$\mathrm{hull}(Q) \cap X_I = Q \cap X_I.$$

Also, applying Lemma 2.4.2, we get:

Proposition 2.5.4 : *Given $\kappa \geq 1$, and $\lambda \in [0, \infty)$, there is some $r \in [0, \infty)$ such that if $Q \subseteq X_C$ is closed and λ-quasiconvex, then*

$$\mathrm{hull}(Q) \subseteq N_r(Q).$$

3. Groups of isometries.

In this Chapter, we shall describe some constructions relating to discrete group actions. We assume throughout that X has curvature at most -1. In Section 3.5, we assume that X has curvature at least $-\kappa^2$.

3.1. Elementary and nilpotent groups.

By a *subspace*, Y, of X_C, we shall mean a totally geodesic subset which meets X (i.e. $Y \cap X \neq \emptyset$), such that if $x, y \in Y$ are distinct, then the bi-infinite geodesic through x and y lies in Y. (Thus, a single point of X is a subspace, but a single point of X_I is not.) Such a set is necessarily closed in X_C, and has, itself, the structure of a compactified Hadamard manifold, with the same curvature bounds. We say that Y is *proper* if $Y \neq X_C$.

Any isometry g of X extends to a homeomorphism of X_C, which we shall also denote by g. We shall write $\mathrm{fix}\, g$ for the set of fixed points of g in X_C. We have the following classification of isometries.

Any isometry g of X is of precisely one of the following types.

(0) g is the identity.

(1) g is *elliptic*. Thus $\mathrm{fix}\, g$ is a proper non-empty subspace of X_C.

(2) g is *parabolic*. Thus, $\mathrm{fix}\, g$ consists of a single point $p \in X_I$, and g preserves setwise each horosphere about p.

(3) g is *loxodromic*. Thus, $\mathrm{fix}\, g = \{p, q\}$ where p and q are distinct points of X_I. For all $x \in X_C \setminus \{p, q\}$, we have $g^n x \to p$ and $g^{-n} x \to q$. We call p the *attracting fixed point* of g, and q the *repelling fixed point* of g. The bi-infinite geodesic $[p, q]$ is called the *loxodromic axis*. It is preserved setwise by g.

Suppose G is a group of isometries acting on X. We write $\mathrm{fix}\, G$ for the set of all fixed points of G in X_C, i.e. $\mathrm{fix}\, G = \bigcap_{g \in G} \mathrm{fix}\, g$.

Definition : The group G is *elementary* if either $\mathrm{fix}\, G \neq \emptyset$, or else if G preserves setwise some bi-infinite geodesic in X_C.

It is not hard to separate elementary groups into three mutually exclusive classes.

Case(1): $\mathrm{fix}\, G$ is a non-empty subspace of X_C.

Case(2): $\mathrm{fix}\, G$ consists of a single point of X_I.

Case(3): G has no fixed point in X, and G preserves setwise a unique bi-infinite geodesic in X.

Proposition 3.1.1 : *Any virtually nilpotent group of isometries of X is elementary.*

Proof : First, we deal with the case where our group G is abelian. Choose any non-trivial $g \in G$. If g is elliptic, then $\mathrm{fix}\, g$ is a proper G-invariant subspace of X_C, and so we can use induction on dimension. If g is parabolic, then its fixed point must be fixed by G. If g is loxodromic, then G preserves setwise the loxodromic axis. We find that G is elementary.

Now, suppose that G is nilpotent. Let Z be the centre of G. If $\mathrm{fix}\, Z$ is a non-empty subspace of X_C (Case(1)), then G/Z acts on $\mathrm{fix}\, Z$, and the result follows by induction on dimension. If $\mathrm{fix}\, Z$ consists of a single point $p \in X_I$ (Case(2)), then p is fixed by G. If $X \cap \mathrm{fix}\, Z = \emptyset$ and Z preserves a unique bi-infinite geodesic (Case(3)), then this geodesic is preserved by G.

Finally, suppose that G has a nilpotent subgroup H of finite index. We can assume that H is normal in G (by intersecting its conjugates). If $\mathrm{fix}\, H$ is a non-empty subspace of X_C, then $\mathrm{fix}\, H$ is G-invariant, and G/H is finite and acts on $\mathrm{fix}\, H$. It follows that G/H, and hence G, has a fixed point in $X \cap \mathrm{fix}\, H$. (See the beginning of Section 3.5). If $\mathrm{fix}\, H$ consists of a single point of X_I, then this point is fixed by G. If H preserves setwise a unique bi-infinite geodesic, then this geodesic is preserved also by G. ♢

The interest in virtually nilpotent groups arises from the Margulis Lemma (3.5.1).

We are particularly interested in the following two types of elementary groups.

Definition : A group of isometries, G, of X is *parabolic* if $\mathrm{fix}\, G$ consists of a single point $p \in X_I$, and if G preserves setwise some (and hence every) horosphere about p.

Definition : A group of isometries, G, of X is *loxodromic* if G contains a loxodromic element and preserves setwise its axis.

It is easily seen that a loxodromic group is precisely what is described by Case(3) above. There are two types of loxodromic group. Either $\mathrm{fix}\, G = \{x, y\}$, where x and y are the endpoints of the loxodromic axis, or else there is some element of G which swaps x and y, in which case $\mathrm{fix}\, G = \emptyset$. We call these situations *loxodromic* of the *first* and *second type* respectively.

3.2. Discrete isometry groups.

We say that a group Γ acts *properly discontinuously* on a locally compact topological space if each compact set meets only finitely many images of itself under Γ. Alternatively, we may say that Γ acts properly discontinuously if, for each compact set K, the set of images, ΓK, of K under Γ is locally finite. (We take this to imply that $\mathrm{stab}_\Gamma K$ is finite.) It is a simple exercise that these definitions are equivalent.

We write $\mathrm{Isom}\, X$ for the set of isometries of X. Thus, $\mathrm{Isom}\, X$ has naturally the structure of a locally compact topological group. A subgroup Γ of $\mathrm{Isom}\, X$ is discrete as a subgroup if and only if it acts properly discontinuously on X. In such a case, the torsion elements of Γ (i.e. the non-trivial elements of finite order) are precisely the elliptic ones. In fact, any finite subgroup of $\operatorname{Isom} X$ must have a fixed point in X.

It's not hard to see that if G is a discrete subgroup of $\operatorname{Isom} X$ with $\operatorname{fix} G \neq \emptyset$, then G is finite, parabolic or loxodromic of the first type. As a corollary, we have:

Proposition 3.2.1 : *A discrete elementary subgroup of* $\operatorname{Isom} X$ *is finite, parabolic or loxodromic.*

Note that these three cases are mutually exclusive.

Suppose that $\Gamma \subseteq \operatorname{Isom} X$ is discrete. Given any subset $Q \subseteq X_C$, we shall write

$$\operatorname{stab}_\Gamma Q = \{\gamma \in \Gamma \mid \gamma Q = Q\}$$

for the setwise stabliser of Q.

Suppose $G \subseteq \Gamma$ is parabolic with fixed point p. By Proposition 3.2.1, we see that $\operatorname{stab}_\Gamma p$ is also parabolic. In fact, we see easily that $\operatorname{stab}_\Gamma p$ is a maximal parabolic subgroup of Γ.

Definition : We call $p \in X_I$ a *parabolic fixed point* of Γ if $\operatorname{stab}_\Gamma p$ is parabolic.

Note that there is a bijective correspondence between orbits of parabolic fixed points and conjugacy classes of maximal parabolic subgroups.

Suppose $G \subseteq \Gamma$ is loxodromic with axis β. Again using Proposition 3.2.1, we see that $\operatorname{stab}_\Gamma \beta$ is a maximal loxodromic subgroup of Γ. We have shown:

Lemma 3.2.2 : *Suppose* $\Gamma \subseteq \operatorname{Isom} X$ *is discrete. Every infinite elementary subgroup of* Γ *is contained in a unique maximal elementary subgroup.*

Clearly, every subgroup of an elementary group is elementary, so we have:

Corollary 3.2.3 : *If* $G, G' \subseteq \Gamma$ *are maximal elementary, and intersect in an infinite subgroup, then* $G = G'$.

Given $G \subseteq \Gamma$, we write

$$N_\Gamma(G) = \{\gamma \in \Gamma \mid \gamma G \gamma^{-1} = G\}$$

for the normaliser of G in Γ.

Lemma 3.2.4 : *If* $G \subseteq \Gamma$ *is infinite maximal elementary, then* $N_\Gamma(G) = G$.

Proof : Supppose G is parabolic with fixed point p. If $\gamma \in N_\Gamma(G)$, then $\{\gamma p\} = \gamma \mathrm{fix}\, G = \mathrm{fix}\, \gamma G \gamma^{-1} = \mathrm{fix}\, G = \{p\}$. Thus $p \in \mathrm{fix}\, N_\Gamma(G)$. By proposition 3.2.1, we see that $N_\Gamma(G)$ is parabolic. Thus $N_\Gamma(G) = G$.

If G is loxodromic, then $N_\Gamma(G)$ preserves the loxodromic axis. Again $N_\Gamma(G) = G$. $\diamondsuit$

We remark that a discrete loxodromic group of the first type is group-theoretically just an infinite-cyclic extention of a finite subgroup of orthogonal group $O(n-1)$. (For a description of discrete parabolic groups in the case where X has a lower curvature bound, see Chapter 4.)

Suppose that $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete. Given any $x \in X_C$, we write Γx for the orbit of x under Γ. For any $x \in X$, we write $\Lambda = \Lambda(\Gamma) \subseteq X_I$ for the set of accumulation points of Γx. Thus, Λ is closed and Γ-invariant. Also, Λ is defined independently of the choice of $x \in X$. (If $x_n \to z \in X_I$ and $d(x_n, y_n)$ is bounded, then $y_n \to z$.) In fact, we shall see that, unless Γ is loxodromic of the first type and x is a fixed point of Γ, then Λ is the set of accumulation points of Γx for any $x \in X_I$. We call Λ the *limit set* of Γ. The complement $\Omega = \Omega(\Gamma) = X_I \setminus \Lambda$ is called the *discontinuity domain*.

It is easily verified that $\Lambda(\Gamma) = \emptyset$ if and only if Γ is finite, that $\Lambda(\Gamma)$ consists of a single point if and only if Γ is parabolic, and that $\Lambda(\Gamma)$ consists of two points if and only if Γ is loxodromic.

Lemma 3.2.5 : *Suppose that $Q \subseteq X_I$ is closed and Γ-invariant and contains at least two points. Then, Γ acts properly discontinuously of $X_C \setminus Q$.*

Proof : Let $J = \mathrm{join}(Q)$, as defined in Section 2.2. Then $J \cap X$ is dense in J. Suppose that K is a compact subset of $X_C \setminus Q$. Then, $\mathrm{proj}_Q K = \bigcup_{x \in K} \mathrm{proj}_Q(x)$ is a compact subset of X (since $\mathrm{proj}_Q \subseteq X_C \times X_C$ is closed). Now, proj_Q is Γ-equivariant, and so if $\gamma K \cap K \neq \emptyset$ for some $\gamma \in \Gamma$, then $\gamma\, \mathrm{proj}_Q K \cap \mathrm{proj}_Q K \neq \emptyset$. Since Γ acts properly discontinuously on X, there are only finitely many such γ. $\diamondsuit$

If Γ is any discrete group, then clearly Γ does not act properly discontinuously at any point of the limit set, Λ. By Lemma 3.2.5, we see that if Γ is not loxodromic, then Λ is the minimal non-empty closed Γ-invariant subset. Note that the set if accumulation points of Γx for $x \in X_I$ is such a subset, and thus equal to Λ. In fact, the only exceptional case is when Γ is loxodromic of the first type, and x is a fixed point of Γ.

Proposition 3.2.6 : *Suppose $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete, and $\Omega = \Omega(\Gamma)$ is the discontinuity domain. Then, Γ acts properly discontinuously on $X \cup \Omega$.*

Proof : By Lemma 3.2.5, we need only verify the cases where Γ is finite or parabolic. If Γ is finite, the statement is trivial. If Γ is parabolic, we can use the argument of Lemma 3.2.5, with J replaced by a horoball about the fixed point. $\diamondsuit$

We have already remarked that the torsion elements of a discrete group Γ are precisely the elliptic ones. Let

$$\mathcal{S} = \{\mathrm{fix}\,\gamma \mid \gamma \in \Gamma \text{ is elliptic}\}.$$

Let $\Sigma = \bigcup \mathcal{S}$. Thus $\Sigma \cap X$ is the set of points, x, of X for which $\mathrm{stab}_\Gamma\, x$ is non-trivial.

Lemma 3.2.7 : *$\mathcal{S}$ is locally finite on $X \cup \Omega$.*

Proof : Suppose, to the contrary, that there is some sequence (γ_n) of distinct elements of Γ with the sets $\mathrm{fix}\,\gamma_n$ accumulating on some point $x \in X \cup \Omega$. We can find points $x_n \in X$ with $\gamma_n x_n = x_n$ and $x_n \to x$. Thus, if K is any compact neighbourhood of x, disjoint from Λ, we have $\gamma_n K \cap K \neq \emptyset$ for all sufficiently large n. This contradicts the proper discontinuity of Γ on $X \cup \Omega$. (Proposition 3.2.6). $\diamondsuit$

If $G \subseteq \Gamma$ is finite then $\mathrm{fix}\,G$ is a subspace of Γ. By an induction on dimension, we see that every finite subgroup of Γ is contained in some (possibly many) maximal finite subgroup. If $G \subseteq \Gamma$ is maximal finite, then clearly $\mathrm{fix}\,G$ determines G. Let

$$\mathcal{G} = \{\mathrm{fix}\,G \mid G \subseteq \Gamma \text{ is maximal finite}\}.$$

Thus $\mathcal{G}$ is a disjoint collection of subsets of X_C. Each element of $\mathcal{G}$ is a finite intersection of elements of $\mathcal{S}$, and so:

Corollary 3.2.8 : *$\mathcal{G}$ is locally finite on $X \cup \Omega$.*

Given the discrete group Γ, we write

$$M_C = M_C(\Gamma) = (X \cup \Omega)/\Gamma.$$

Thus, $M_C = M \cup M_I$, where $M = M(\Gamma) = X/\Gamma$ and $M_I = M_I(\Gamma) = \Omega/\Gamma$. Note that $\Sigma \cap (X \cup \Omega)$ descends to a closed subset, $\hat{\Sigma}$ of M_C which we call the *singular set* of M_C. We shall say more about this in Section 3.4.

Suppose $G \subseteq \Gamma$ is maximal finite, so that $\mathrm{fix}\,G \in \mathcal{G}$. If $\gamma \in \Gamma$, then $\gamma \mathrm{fix}\,G = \mathrm{fix}\,\gamma G \gamma^{-1}$, and so the set $\bigcup \Gamma \mathrm{fix}\,G = \bigcup_{\gamma \in \Gamma} \gamma \mathrm{fix}\,G$ corresponds to the conjugacy class of G in Γ. Let

$$\hat{\mathcal{G}} = \{\pi(\bigcup \Gamma F \setminus \Lambda) \mid F \in \mathcal{G}\},$$

where $\pi : X \cup \Omega \longrightarrow M_C$ is the projection. Thus, $\hat{\mathcal{G}}$ is a locally finite collection of disjoint closed subsets of M_C. The elements of $\hat{\mathcal{G}}$ are in bijective correspondence with the conjugacy classes of maximal finite subgroups of Γ.

3.3. Dirichlet domains.

Suppose $\Gamma \subseteq \mathrm{Isom}\,X$ is discrete, with discontinuity domain $\Omega \subseteq X_I$.

Proposition 3.3.1 : *Suppose $Q \subseteq X_C$ is quasiconvex, and that $\Gamma Q = \{\gamma Q \mid \gamma \in \Gamma\}$ is locally finite on X. Then ΓQ is locally finite on $X \cup \Omega$.*

Proof : Suppose $x \in \Omega$. Let $V_1, V_2, V_3 \subseteq X \cup \Omega$ be three compact neighbourhoods of x, with V_1 contained in the interior of V_2, and V_2 contained in the interior of V_3. Let W be the closure, in X_C, of $X_C \setminus V_3$. Thus, $(V_1 \cup W) \cap \partial V_2 = \emptyset$. Let $J = \mathrm{join}(V_1 \cup W)$, so that $J \cap X_I \subseteq V_1 \cup W$. We see that J meets ∂V_2 in a compact subset, K, of X. Any geodesic from point in V_1 to a point in W must meet K. (Figure 3a.)

Figure 3a.

Let Q be as in the hypothesis. Thus Q is λ-quasiconvex for some $\lambda \geq 0$. We claim that V_1 can meet only finitely many images of Q under Γ. Since $V_3 \subseteq X \cup \Omega$, certainly V_3 can only contain finitely many such images. Suppose γQ meets V_1 but is mot contained in V_3. Then we can find $y \in V_1 \cap \gamma Q$ and $z \in W \cap \gamma Q$. Now, $[y, z] \subseteq N_\lambda(\gamma Q)$ and $[y, z] \cap K \neq \emptyset$. Thus γQ meets the compact set $N_\lambda(K) \subseteq X$. Since ΓQ is locally finite on X, this can happen for only finitely many $\gamma \in \Gamma$. ♢

Proposition 3.3.1 will be used in the discussion of conical limit points in Chapter 5. Another application is to Dirichlet domains for Γ.

Suppose $A \subseteq X$ is a discrete subset and $a \in A$. We write $D(a, A)$ for the closure in X_C of $\{x \in X \mid d(x, a) \leq d(x, A)\}$. It is easy to see that $D(a, A)$ is starlike about a, and hence λ_0-quasiconvex. Moreover, the collection $\mathcal{D}(A) = \{D(a, A) \mid a \in A\}$ is locally finite on X and covers X.

Of particular interest is the case where $A = \Gamma a$ is an orbit under the discrete group Γ. We call $D(a, \Gamma a)$ a *Dirichlet domain.* Note that the stabliser $\mathrm{stab}_\Gamma D(a, \Gamma a) = \mathrm{stab}_\Gamma a$ is finite. Since $\mathcal{D}(\Gamma a) = \Gamma D(a, \Gamma a)$, applying Proposition 3.3.1, we get:

Corollary 3.3.2 : *$\mathcal{D}(\Gamma a)$ is locally finite on $X \cup \Omega$.*

Note that it follows that $\bigcup \mathcal{D}(\Gamma a) \cap (X \cup \Omega) = \bigcup_{\gamma \in \Gamma} \gamma D(a, \Gamma a) \cap (X \cup \Omega)$ is closed in $X \cup \Omega$. Thus $X \cup \Omega \subseteq \bigcup \mathcal{D}(\Gamma a)$.

3.4. Orbifolds.

For the definitions of geometrical finiteness, we shall need to refer to orbifold having both ideal and metric boundaries. In this section, we clarify what is meant by this.

The notion of an orbifold was defined by Thurston [T] as a generalisation of a manifold. A typical example of an orbifold is the quotient of a manifold by a group action which is properly discontinuous though not necessarily free. (However not every orbifold is obtained in this way.) Thus an orbifold is locally modelled on $\mathbf{R}^n$ quotiented out by a finite subgroup of the orthogonal group $O(n)$. These subgroups are considered part of the structure of the orbifold. Thus we may define orbifold homeomorphism. There are also notions of covering spaces, universal cover and fundamental group for an orbifold. A *good* orbifold is one which is covered by a manifold. Thus a good orbifold is the quotient of a simply connected manifold (its universal cover) by the action of its fundamental group. For details see [T]. We shall only be interested in good orbifolds.

We can also speak of an orbifold with boundary. Boundary points are locally modelled on a quotient of $\mathbf{R}^{n-1} \times [0, \infty)$ by a finite subgroup of $O(n-1)$ (acting on the $\mathbf{R}^{n-1}$ factor). We can also define a (codimension-0) suborbifold, N', of an orbifold with boundary, N. At a point of the topological boundary of N' in N, the pair (N, N') is locally modelled on either $(\mathbf{R}^n, \mathbf{R}^{n-1} \times [0, \infty))$ quotiented by a finite subgroup of $O(n-1)$, or else $(\mathbf{R}^{n-1} \times [0, \infty), \mathbf{R}^{n-2} \times [0, \infty)^2)$ quotiented by a finite subgroup of $O(n-2)$. Thus a suborbifold of an orbifold with boundary is itself an orbifold with boundary.

Suppose Γ is a discrete subgroup of $\mathrm{Isom}\, X$. Then $M = M(\Gamma) = X/\Gamma$ is a (good) orbifold. M also has a metric structure induced from X. This is a Riemannian metric away from the singular set $\hat{\Sigma} \cap M$, as defined is Section 3.2. Clearly X and Γ can be completely recovered from the metric and orbifold structures on M. (In fact, the orbifold structure of M, i.e. the system of subgroups of $O(n)$, is completely determined just by the metric structure.)

Since Γ acts properly discontinuously on $X \cup \Omega$, we can define

$$M_C = M_C(\Gamma) = (X \cup \Omega)/\Gamma.$$

Thus, M_C is an orbifold with boundary. We have $\Gamma = \pi_1 M = \pi_1 M_C$, where π_1 is the orbifold fundamental group.

We shall want to speak about negatively curved orbifolds with convex boundary. This may be defined intrinsically, though since we are only interested in good orbifolds, it is most simply done with reference to the universal cover.

Suppose that Y is a metrically complete simply connected Riemannian manifold, with convex boundary ∂Y, all of whose sectional curvatures (in the interior) are at most -1. As with X, we may define the ideal boundary, Y_I, of Y as a set of equivalence classes of geodesic rays in Y. Thus, $Y_C = Y \cup Y_I$ is compact, in fact homeomorphic to an n-ball. If it happens that $Y \subseteq X$, then we may identify Y_C as the closure of Y in X_C.

If Γ is a group acting faithfully and properly discontinuously on Y, we may define the discontinuity domain $\Omega^Y \subseteq Y_I$, just as for X. Let $M^Y = M^Y(\Gamma) = Y/\Gamma$ and $M_C^Y = M_C^Y(\Gamma) = (Y \cup \Omega^Y)/\Gamma$. Thus, M_C^Y is an orbifold with boundary. The orbifold boundary of M_C^Y is the union of the ideal boundary Y_I/Γ and the convex boundary $\partial Y/\Gamma$. We have $\Gamma = \pi_1 M^Y = \pi_1 M_C^Y$. In the case where Γ is a discrete subgroup of $\mathrm{Isom}\, X$, and Y is a closed convex Γ-invariant subset of X, we will have $\Omega^Y = Y_I \cap \Omega$, where Ω is the discontinuity domain in X_I, and where we have identified Y_I as a subset of X_I. The topological boundary of M_C^Y in M_C is the closure of the convex boundary.

Suppose, more generally that we have groups Γ and G acting faithfully and properly discontinuously on X and Y respectively. Suppose that $M_C^Y(G)$ may be identified as a "convex suborbifold" of $M_C(\Gamma)$ i.e. a suborbifold with convex boundary. From the orbifold definitions, it follows that the inclusion $M^Y(G) \hookrightarrow M_C^Y(\Gamma)$ lifts to a map of universal covers $Y \longrightarrow X$, which is a local isometry. Since Y has convex boundary, this map is injective, and so we can identify Y has a convex subset of X, and Y_C as a subset of X_C. It follows that that $G = \pi_1 M^Y(G)$ injects into $\pi_1 M(\Gamma) = \Gamma$. In other words, we can identify G as a subgroup of Γ. In summary, $Y_C \subseteq X_C$ is closed convex and G-invariant, where $G \subseteq \Gamma$. For our purposes, we can take this conclusion as the definition of "convex suborbifold".

The convex suborbifolds in which we will be interested are neighbourhoods of end of quotients of parabolic groups (Chapter 4), and uniform neigbourhoods of convex cores (Section 5.3).

3.5. The Margulis Lemma and thick-thin decomposition.

The thick-thin decomposition is central to the study of negatively curved manifolds [BaGS]. Here we shall need a generalisation to the orbifold case, for which I know of no written account. We set out in this section the basic facts we shall need. We shall assume that all the sectional curvatures of X lie between $-\kappa^2$ and -1.

Given $x \in X$, and $\epsilon > 0$, write

$$\mathcal{I}_\epsilon(x) = \{\gamma \in \operatorname{Isom} X \mid d(x, \gamma x) \leq \epsilon\}.$$

If $\Gamma \subseteq \operatorname{Isom} X$ is discrete, we write

$$\Gamma_\epsilon(x) = \langle \Gamma \cap \mathcal{I}_\epsilon(x) \rangle,$$

i.e. $\Gamma_\epsilon(x)$ is generated by those elements of Γ which move the point x a distance at most ϵ.

Proposition 3.5.1. (Margulis Lemma) : *There is a constant $\epsilon(n, \kappa) > 0$ such that if $\Gamma \subseteq \operatorname{Isom} X$ is discrete, and $x \in X$, then $\Gamma_\epsilon(x)$ is virtually nilpotent for all $\epsilon \leq \epsilon(n, \kappa)$. Here, $\epsilon(n, \kappa)$ depends only on the dimension, n, of X, and the lower curvature bound, $-\kappa^2$.*

Proof : See, for example, [BaGS]. ♢

We call $\epsilon(n, \kappa)$ the *Margulis constant*.

Given a discrete subgroup, Γ, of $\operatorname{Isom} X$, we write

$$T_\epsilon(\Gamma) = \{x \in X \mid \Gamma_\epsilon(x) \text{ is infinite}\}.$$

Thus, $T_\epsilon(\Gamma)$ is a closed Γ-invariant subset of X. Note that, since Γ acts properly discontinuously on $X \cup \Omega$, the closure of $T_\epsilon(G)$ in X_C is a subset of $X \cup \Lambda$. First, we describe $T_\epsilon(G)$ for an elementary group G. Clearly $T_\epsilon(G) = \emptyset$ if G is finite.

Proposition 3.5.2 : *Suppose $G \subseteq \operatorname{Isom} X$ is discrete parabolic, with fixed point p, and suppose $\epsilon > 0$. Then, $T_\epsilon(G)$ is connected. Moreover, if $x \in X_C \setminus \{p\}$, then $[x, p]$ meets $T_\epsilon(G)$ in a non-empty ray tending to p. (Thus $T_\epsilon(G) \cup \{p\}$ is closed in X_C, and starlike about p.)*

Proof : We shall need the fact (Proposition 4.2) that G contains a parabolic element γ.

Suppose $x \in X$, and $y \in [x,p] \setminus \{p\}$. Given any $g \in G$, then, applying Proposition 1.1.11(1), we see that $d(y, gy) \leq d(x, gx)$. Thus, $G_\epsilon(x) \subseteq G_\epsilon(y)$. It follows that if $x \in T_\epsilon(G)$ then $[x,p] \subseteq T_\epsilon(G) \cup \{p\}$.

Suppose now that x and y are any points of $T_\epsilon(G)$. Let $\alpha : [0.1] \longrightarrow X$ be any path with $\alpha(0) = x$ and $\alpha(1) = y$. Set $r = \max\{d(\alpha(u), \gamma\alpha(u)) \mid u \in [0,1]\}$. Let R be the constant given by Proposition 1.1.11(2), and set $t = \max(\log_e(R/\epsilon), 0)$. Thus, for all $u \in [0,1]$ we have $d(\alpha(u) + t, \gamma(\alpha(u) + t)) \leq \epsilon$. It follows that $\gamma \in G_\epsilon(\alpha(u) + t)$, and so $\alpha(u) + t \in T_\epsilon(G)$. We see that the path $[u \mapsto \alpha(u) + t]$ joins $x + t$ to $y + t$ in $T_\epsilon(G)$. We already know that $[x, x+t] \subseteq T_\epsilon(G)$ and $[y, y+t] \subseteq T_\epsilon(G)$. Thus we have shown that $T_\epsilon(G)$ is connected.

It remains to see that for any $x \in X_I \setminus \{p\}$, both $[x,p] \cap T_\epsilon(G)$ and $X \cap [x,p] \setminus T_\epsilon(G)$ are non-empty. Again, this follows easily from Proposition 1.1.11(2). ♢

The situation for loxodromic groups is a little more complicated. Suppose $G \subseteq \operatorname{Isom} X$ is discrete loxodromic with axis β. Suppose $x \in X$, and let $z = \operatorname{proj}_\beta x$. If $y \in [x,z]$, and $g \in G$, then using the CAT(-1) inequality, we see that $d(y, gy) \leq d(x, gx)$. Thus $G_\epsilon(x) \subseteq G_\epsilon(y)$, and so if $x \in T_\epsilon(G)$ then $[x,z] \subseteq T_\epsilon(G)$. We see that $T_\epsilon(G)$ retracts onto $\beta \cap T_\epsilon(G)$. Let $\gamma \in G$ be a loxodromic element of minimal translation distance, μ, on β. Suppose first that G is loxodromic of the first type (i.e. G respects the orientation of β). In this case, we see that if $\mu \leq \epsilon$, then $\beta \cap X \subseteq T_\epsilon(G)$, whereas if $\mu > \epsilon$, then $\beta \cap T_\epsilon(G) = \emptyset$, and so $T_\epsilon(G) = \emptyset$. Thus $T_\epsilon(G)$ is connected (or empty). Suppose now that G is of the second type (i.e. there is an element of G which swaps the two endpoints of β). This time, there are three possibilities. If $\mu \leq \epsilon$, again $\beta \cap X \subseteq T_\epsilon(G)$, and $T_\epsilon(G)$ is connected. If $\epsilon < \mu \leq 2\epsilon$ then $\beta \cap T_\epsilon(G)$ consists of a countable disjoint union of closed intervals (or points if $\mu = 2\epsilon$). These are the images under G of a single interval, and so $(\beta \cap T_\epsilon(G))/G$ and thus $T_\epsilon(G)/G$ are connected. Finally, if $\mu > 2\epsilon$, then $T_\epsilon(G) = \emptyset$.

Note that we have shown:

Proposition 3.5.3 : *If $G \subseteq \operatorname{Isom} X$ is discrete elementary, and $\epsilon > 0$, then $T_\epsilon(G)/G$ is connected (or empty).* ♢

Suppose now that $\Gamma \subseteq \operatorname{Isom} X$ is any discrete group. Suppose that $x \in X$, that $\epsilon \leq \epsilon(n,\kappa)$, where $\epsilon(n,\kappa)$ is the Margulis constant. By the Margulis Lemma (Proposition 3.5.1), we have that $\Gamma_\epsilon(x)$ is virtually nilpotent, and so by Proposition 3.1.1, $\Gamma_\epsilon(x)$ is elementary. If $x \in T_\epsilon(\Gamma)$, then $\Gamma_\epsilon(x)$ is infinite, and so, by Lemma 3.2.2, $\Gamma_\epsilon(x)$ is contained in a unique maximal elementary subgroup of G of Γ. Clearly $x \in T_\epsilon(G) \subseteq T_\epsilon(\Gamma)$. We have shown that $T_\epsilon(\Gamma)$ is a union of $T_\epsilon(G)$ as G varies over all maximal infinite elementary subgroups of Γ. We also have:

Proposition 3.5.4 : *Suppose $\epsilon < \epsilon(n,\kappa)$. Let $\delta > 0$ be such that $\epsilon + 2\delta \leq \epsilon(n,\kappa)$. Suppose $\Gamma \subseteq \operatorname{Isom} X$ is discrete, and that G and G' are two distinct maximal elementary subgroup of Γ. Then $d(T_\epsilon(G), T_\epsilon(G')) \geq \delta$. (Of course one or both of $T_\epsilon(G)$ and $T_\epsilon(G')$ may be empty.)*

Proof : Suppose, to the contrary, that $x \in T_\epsilon(G)$ and $x' \in T_\epsilon(G')$ with $d(x,x') \leq \delta$. It is easily seen that $\Gamma_\epsilon(x)$ and $\Gamma_\epsilon(x')$ are both subgroup of $\Gamma_{\epsilon+2\delta}(x)$, which by the Margulis lemma is virtually nilpotent and thus elementary. It follows that $\Gamma_{\epsilon+2\delta}(x)$ is contained in a maximal elementary subgroup G'' of Γ. We have $\Gamma_\epsilon(x) \subseteq G \cap G''$ and $\Gamma_\epsilon(x') \subseteq G' \cap G''$. Applying Corollary 3.2.3, we see that $G = G'' = G'$. ♢

In particular, we have shown:

Proposition 3.5.5 : *Suppose $\epsilon < \epsilon(n,\kappa)$, and $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete. Then $T_\epsilon(\Gamma)$ is a disjoint union of $T_\epsilon(G)$, as G ranges over all maximal infinite elementary subgroups of Γ.*

Corollary 3.5.6 : *Suppose $\epsilon < \epsilon(n,\kappa)$ and T_0 is an unbounded connected component of $T_\epsilon(\Gamma)$, then $T_0 = T_\epsilon(G)$ where $G = \mathrm{stab}_\Gamma T_0$.*

Proof : By Proposition 3.5.5, we know that T_0 is a component of $T_\epsilon(G)$ where G is a maximal infinite elementary subgroup of Γ. Since $T_\epsilon(G)$ contains an unbounded connected set, namely T_0, we see from the possible forms of $T_\epsilon(G)$ described above, that $T_\epsilon(G)$ must be connected. Thus $T_0 = T_\epsilon(G)$.

If $\gamma \in \mathrm{stab}_\Gamma T_0$, then $\gamma T_\epsilon(G) = T_\epsilon(\gamma G \gamma^{-1}) = T_\epsilon(G)$. By Proposition 3.5.4, we have that $\gamma G \gamma^{-1} = G$, i.e. $\gamma \in N_\Gamma(G)$. By Lemma 3.2.4, we have $\gamma \in G$. Thus $G = \mathrm{stab}_\Gamma T_0$. ♢

Corollary 3.5.7 : *Suppose $\epsilon < \epsilon(n,\kappa)$ and T_0 is a bounded connected component of $T_\epsilon(\Gamma)$, then T_0 is a connected component of $T_\epsilon(G)$, where G is a maximal loxodromic subgroup of Γ, of the second type.*

Proof : By Proposition 3.5.5, T_0 is a component of $T_\epsilon(G)$, where $G \subseteq \Gamma$ is maximal elementary. Since T_0 is bounded, the only possibility is for G to be loxodromic of the second type. ♢

Given $\epsilon < \epsilon(n,\kappa)$, and a discrete group Γ, set

$$\mathrm{thin}_\epsilon(M) = T_\epsilon(\Gamma)/\Gamma.$$

Thus $\mathrm{thin}_\epsilon(M)$ is a closed subset of the quotient orbifold $M = X/\Gamma$. We call $\mathrm{thin}_\epsilon(M)$ the *thin part* of M. By Propositions 3.5.3, 3.5.4 and 3.5.5, we see that $\mathrm{thin}_\epsilon(M)$ is (topologically) a disjoint union of its connected components, and that each such component has the form $T_\epsilon(G)/G$ where G is an maximal infinite elementary subgroup of Γ. If G is parabolic, we call $T_\epsilon(G)/G$ a *Margulis cusp*. If G is loxodromic, we call $T_\epsilon(G)/G$ a *Margulis tube*.

We write $\mathrm{thick}_\epsilon(M)$ for the closure of $M \setminus \mathrm{thin}_\epsilon(M)$ in M. We call $\mathrm{thick}_\epsilon(M)$ the *thick part* of M. We write $\mathrm{cusp}_\epsilon(M)$ for the union of all the Margulis cusps, and $\mathrm{noncusp}_\epsilon(M)$ for the closure of $M \setminus \mathrm{cusp}_\epsilon(M)$ in M. We call these respectively the *cuspidal* and *non-cuspidal* parts of M. Obviously, $\mathrm{cusp}_\epsilon(M) \subseteq \mathrm{thin}_\epsilon(M)$ and $\mathrm{thick}_\epsilon(M) \subseteq \mathrm{noncusp}_\epsilon(M)$.

Note that if M is a manifold (i.e. if Γ is torsion-free), then

$$\mathrm{thin}_\epsilon(M) = \{x \in M \mid \mathrm{inj}(x,M) \leq \epsilon/2\},$$

where $\mathrm{inj}(x, M)$ is the injectivity radius of M at x. In this case, Margulis tubes are either closed geodesics of length ϵ, or tubular neighbourhoods of closed geodesics of length less than ϵ.

4. Parabolic Groups.

In this chapter (and for the rest of this paper) we assume that all the sectional curvatures of X lie between $-\kappa^2$ and -1.

Suppose $p \in X_I$. In Section 1.1, we introduced the notation $x + t$ for $x \in X$ and $t \in [-\infty, \infty]$. Thus, suppose x lies in the bi-infinite geodesic $[y, p]$ where $y \in X_I$. If $t \geq 0$, then $x + t$ is the point of $[x, p]$ with $d(x, x + t) = t$. If $t \leq 0$, then $x + t$ is the point of $[y, x]$ with $d(x, x + t) = -t$. We set $x + \infty = p$ and $x - \infty = y$.

We defined a parabolic group $G \subseteq \mathrm{Isom}\, X$ as one for which $\mathrm{fix}\, G$ consists of a single point $p \in X_I$, and which preserves setwise each horosphere about p. It follows that G is infinite.

We begin by stating the following result proved in [Bo2], though we shall not need it for the rest of this chapter.

Proposition 4.1 : *A discrete parabolic group is finitely generated and virtually nilpotent.* ♢

The point here is that G is finitely generated. Given this, it is a simple consequence of the Margulis Lemma (Proposition 3.5.1), and convergence of geodesic rays (Proposition 1.1.11(2)) that G is virtually nilpotent.

Proposition 4.2 : *A discrete parabolic group contains a parabolic element.*

Proof : We give a proof without reference to Proposition 4.1. Suppose, for contradiction, that G is a discrete parabolic group with no parabolic element. Then G is torsion (every element has finite order). By the Margulis Lemma, and convergence of geodesic rays, we see that every finitely generated subgroup of G is virtually nilpotent, and hence finite. We may thus take an exhaustion of G by finite subgroups, $G = \bigcup_n G_n$, with $G_n \subseteq G_{n+1}$. Each set $\mathrm{fix}\, G_n$ is a non-empty subspace of X_C, and $\mathrm{fix}\, G_{n+1} \subseteq \mathrm{fix}\, G_n$. Clearly, the dimensions of the $\mathrm{fix}\, G_n$ must stablise, and so $\mathrm{fix}\, G = \bigcap_n \mathrm{fix}\, G_n$ meets X. Thus G is finite. This contradicts the supposition that G is parabolic. ♢

Note that the limit set $\Lambda(G)$ of a discrete parabolic group G consists of the fixed point p. Thus,

$$M_C(G) = (X_C \setminus \{p\})/G.$$

The main purpose of this chapter is to describe some relationships between certain naturally occurring closed G-invariant subsets of $X_C \setminus \{p\}$. The main results being aimed at are Propositions 4.12 and 4.14. Each of the G invariant subsets, S, we consider, has the property that $S \cup \{p\}$ is starlike about p. We make the following observation.

Lemma 4.3 : *Suppose $S \cup \{p\} \subseteq X_C$ is closed and starlike about p. Then, for any $r \geq 0$, the uniform neighbourhood $N_r(S \cup \{p\})$ is starlike about p.*

Proof : Using the monotonic convergence of geodesic rays, Proposition 1.1.11(1). $\diamondsuit$

If Q is a closed subset of $X_C \setminus \{p\}$, we write

$$N_r(Q) = N_r(Q \cup \{p\}) \setminus \{p\}.$$

Figure 4a.

The main types of sets that concern us are summarised below. Figure 4a gives a schematic representation of these sets, which is meant to evoke the upper half-space model for hyperbolic space with the fixed point p at ∞. We imagine G to be acting in a direction orthogonal to the paper.

One obvious type of G-invariant set is a horoball B about p (Figure 4a(1)).

If Q is a closed G-invariant subset of $X_I \setminus \{p\}$, we set $W = W(Q) = \bigcup\{[x,p] \mid x \in Q\} \setminus \{p\}$. We are principally interested in the case where Q/G is compact. In Figure 4a(2), a uniform neighbourhood $N_r(W)$ is also represented. Note that, by Lemma 4.3, $N_r(W) \cup \{p\}$ is starlike about p.

Of particular interest is the case where Q is the orbit, Gy, of a single point $y \in X_I \setminus \{p\}$. Figure 4a(3) shows $L = L(y) = W(Gy) = \bigcup_{\gamma \in G} \gamma[y,p] \setminus \{p\}$, and its uniform neighbourhood $N_r(L)$.

Again, if $Q \subseteq X_I \setminus \{p\}$ is closed in $X_I \setminus \{p\}$ and G-invariant, we we can consider the convex hull, $H = H(Q) = \mathrm{hull}(Q \cup \{p\}) \setminus \{p\}$ (Figure 4a(4)).

Given any $\epsilon \in (0,\infty)$, we defined the subset $T = T_\epsilon(G)$ in Section 3.5 (Figure 4a(5)). Of course, we are primarily interested in $T_\epsilon(G)$ when ϵ is less than the Margulis constant $\epsilon(n,\kappa)$, though we shall have no need to assume this in this chapter.

Let θ_0 be the constant of Proposition 2.5.1. Suppose $x \in X$. Then $\mathrm{hull}\,\mathrm{cone}(\overrightarrow{xp},\theta_0) \subseteq \mathrm{cone}(\overrightarrow{xp},\pi/2)$. Set $C = C(x) = \bigcap_{\gamma \in G} \gamma \mathrm{hull}\,\mathrm{cone}(\overrightarrow{xp},\theta_0) \setminus \{p\}$ (Figure 4a(6)). Clearly, C is a convex subset of $X_C \setminus \{p\}$. Thus, C/G is a convex suborbifold of $M_C(G) = (X_C \setminus \{p\})/G$ (Section 3.4). The complement of C/G in $M_C(G)$ is relatively compact. Also $\bigcap_{t \in [0,\infty)} C(x+t) = \emptyset$. We have shown:

Proposition 4.4 : *If G is discrete parabolic, then $M_C(G)$ has precisely one topological end. Moreover, we can find a system of neighbourhoods for the end consisting of convex suborbifolds of $M_C(G)$.*

We now begin a sequence of lemmas relating the various sets we have described.

Lemma 4.5 : *Suppose that B is a horoball about p and that $y \in X_I \setminus \{p\}$. Let $L = L(y)$. Let ρ be the restriction of proj_L to $X_C \setminus \{p\}$. Suppose $x \in L \cap B$. Let $C = C(x)$. Then*

$$C \subseteq \rho^{-1}(L \cap B)$$

where $\rho^{-1}(L \cap B) = \{z \in X_C \setminus \{p\} \mid \rho(z) \cap (L \cap B) \neq \emptyset\}$. *(Figure 4b.)*

Figure 4b.

Proof : Suppose $z \in C$. Let $w \in \rho(z)$. Without loss of generality, we can assume that $w, x \in [y, p]$. Now $C \subseteq \mathrm{cone}(\overrightarrow{xp}, \pi/2)$ and so by Lemma 2.4.1, $w = \mathrm{proj}_{[y,p]} z \in [x, p] \subseteq L \cap B$. Thus $\rho(z) \subseteq L \cap B$, and so certainly $\rho(z)$ meets $L \cap B$. Thus $z \in \rho^{-1}(L \cap B)$. $\diamondsuit$

Lemma 4.6 : *Suppose $S \subseteq X_C \setminus \{p\}$ is closed, and that $S \cup \{p\}$ is starlike about p. If B is any horoball about p, and $r \geq 0$, then*

$$N_r(S) \cap B \subseteq N_r(S \cap B).$$

(Figure 4c.)

Figure 4c.

Proof : If $x \in N_r(S) \cap B$, let $y = \mathrm{proj}_{S \cup \{p\}} x$. Then $x\hat{y}p \geq \pi/2$. It follows easily that $y \in B$, so $d(x, S \cap B) \leq d(x, y) \leq r$. $\diamondsuit$

Lemma 4.7 : *Suppose B is a horoball about p, and $Q \subseteq X_I \setminus \{p\}$ is closed and G-invariant with Q/G compact. Suppose $y \in X_I \setminus \{p\}$. Let $L = L(y)$. Then, there is some $r \geq 0$ such that*

$$W \cap B \subseteq N_r(L).$$

(Figure 4d.)

Figure 4d.

Proof : The map $[x \mapsto x - \infty]$ gives a homeomorphism of $(W \cap \partial B)/G$ onto Q/G. Thus $(W \cap \partial B)/G$ is compact, and so $W \cap \partial B \subseteq N_r(Gx)$, for some $r \geq 0$, where x is the point of intersection of $[y, p]$ and ∂B. It follows from the monotonic convergence of geodesic rays (Proposition 1.1.11(1)) that

$$\begin{aligned} W \cap B &= \bigcup \{[z, p] \mid z \in W \cap \partial B\} \setminus \{p\} \\ &\subseteq N_r(\bigcup G[x, p]) \setminus \{p\} \\ &\subseteq N_r(L). \end{aligned}$$

$\diamondsuit$

Lemma 4.8 : *There is some $r_0 > 0$ such that if $Q \subseteq X_I$ is closed, then*

$$H \subseteq N_{r_0}(W),$$

where $H = H(Q) = \mathrm{hull}(Q \cup \{p\}) \setminus \{p\}$, and $W = W(Q)$. (Figure 4e.)

Figure 4e.

Proof : The set $W \cup \{p\}$ is starlike about and hence λ_0-quasiconvex (Corollary 1.1.6). Proposition 2.5.4 gives us a constant r_0 such that

$$\mathrm{hull}(Q \cup \{p\}) = \mathrm{hull}(W \cup \{p\}) \subseteq N_{r_0}(W) \cup \{p\}.$$

$\diamondsuit$

Lemma 4.9 : *Suppose $\epsilon > 0$. Let $T = T_\epsilon(G)$. Suppose $Q \subseteq X_I \setminus \{p\}$ is closed and G-invariant with Q/G compact. Let $W = W(Q)$. Then, for any $r \geq 0$, there is a horoball B about p with*

$$T \cap N_r(W) \subseteq B.$$

(Figure 4f.)

Figure 4f.

Proof : Proposition 1.1.11(2) gives us a constant $R > 0$ such that if x and y lie in the same horosphere about x, and $d(x, y) \leq \epsilon$ then $d(x + t, y + t) \leq Re^{-t}$.

Choose any horoball B_0 about p. Now $(W \cap \partial B_0)/G$ is homeomorphic to Q/G and hence compact. Thus, there is a compact set $K \subseteq \partial B_0$ with $N_r(W) \cap \partial B_0 \subseteq \bigcup GK = \bigcup_{\gamma \in G} \gamma G$. Let $\eta = \frac{1}{2} \min\{d(x, \gamma x) \mid x \in K, \gamma \in G\} > 0$. Let $h = \max(0, \log_e(R/\eta))$. Let B be the horoball $N_h(B_0)$. We claim that $T \cap N_r(W) \subseteq B$.

Suppose, for contradiction, that there is some $x \in T \cap N_r(W) \setminus B$. We have $y = x + t \in \partial B_0$ for some $t \geq h$. Now, $N_r(W) \cup \{p\}$ is starlike about p (Lemma 4.3), and so $y \in N_r(W)$. Thus, $y \in \bigcup GK$, and so, without loss of generality, we can assume that $y \in K$. Since $x \in T$, there is some $\gamma \in G$ with $d(x, \gamma x) \leq \epsilon$. Thus $d(y, \gamma y) \leq Re^{-t} \leq Re^{-h} \leq \eta$ which contradicts the definition of η. $\diamondsuit$

Lemma 4.10 : *Given $\epsilon > 0$, let $T = T_\epsilon(G)$. Suppose $Q \subseteq X_I \setminus \{p\}$ is closed and G-invariant with Q/G compact. Let $H = H(Q) = \mathrm{hull}(Q \cup \{p\}) \setminus \{p\}$. Suppose $y \in X_I \setminus \{p\}$. Let $L = L(y)$. Then, there is some $r > 0$, and a horoball B about p such that*

$$H \cap T \subseteq H \cap B \subseteq N_r(L) \cap B.$$

(Figure 4g.)

Figure 4g.

Proof : Let $W = W(Q)$, so that by Lemma 4.8, $H \subseteq N_{r_0}(W)$. By Lemma 4.9, there is a horoball B about p so that $T \cap N_{r_0}(W) \subseteq B$. Thus $T \cap H \subseteq B$. By Lemma 4.7, there is some $r_1 \geq 0$ such that $W \cap B \subseteq N_{r_1}(L)$. Thus $H \cap B \subseteq N_{r_0}(W) \cap B \subseteq N_r(L)$, where $r = r_0 + r_1$, and so $H \cap T \subseteq H \cap B \subseteq N_r(L) \cap B$. $\diamondsuit$

Lemma 4.11 : *Given $\epsilon > 0$, let $T = T_\epsilon(G)$. Suppose $r > 0$, and $y \in X_I \setminus \{p\}$. Let $L = L(y)$. Then, there is some horoball B about p such that*

$$N_r(L) \cap B \subseteq T.$$

(Figure 4h.)

Figure 4h.

Proof : From the bounds on the volumes of balls, Propositions 1.1.12 and 1.2.4, we see that an R-ball in X can contain at most $M(r, \epsilon)$ disjoint $(\epsilon/2)$-balls, where $M(r, \epsilon)$ is the integer part of $V(\kappa r, n)/\kappa^n V(\epsilon/2, n)$. Set $R = r + 1 + \epsilon/2$, and $M = M(R, \epsilon)$.

By Proposition 4.2, G contains a parabolic element γ. By the convergence of geodesic rays, there is some $x \in [y, p]$ with $d(x, \gamma x) \leq 1/M$. Let B be the horoball about p with $x \in \partial B$. We claim that $N_r(L) \cap B \subseteq T$.

Suppose $z \in N_r(L) \cap B$. Let w be the nearest point to z in L. Translating everything by an element of G, we may as well suppose that $w \in [y, p]$. Since $p\hat{w}z = \pi/2$ and $z \in B$, we see that $w \in B$. Hence (Proposition 1.1.11(1)), $d(w, \gamma w) \leq d(x, \gamma x) \leq 1/M$. For any integer $i \in \{0, 1, \dots, M\}$, we have $d(\gamma^i z, w) = d(z, \gamma^{-i} w) \leq r + i/M \leq r + 1$. Thus, the $(\epsilon/2)$-balls about each $\gamma^i z$ are all contained in the $(R+1)$-ball about w. Thus, for some $i \neq j \in \{0, 1, \dots, M\}$, we have $N_{\epsilon/2}(\gamma^i z) \cap N_{\epsilon/2}(\gamma^j z) \neq \emptyset$. Thus $d(z, \gamma^{i-j} z) \leq \epsilon$ and so $\gamma^{i-j} \in \Gamma_\epsilon(z)$. It follows that $\Gamma_\epsilon(z)$ is infinite and so $z \in T_\epsilon(G) = T$. $\diamondsuit$

Proposition 4.12 : *Suppose $Q \subseteq X_I \setminus \{p\}$ is closed, G-invariant and non-empty, with Q/G compact. Let $H = H(Q) = \mathrm{hull}(Q \cup \{p\}) \setminus \{p\}$. Let $\rho : X_C \setminus \{p\} \longrightarrow H$ be the restriction of $\mathrm{proj}_{(H \cup \{p\})}$ to $X_C \setminus \{p\}$. Suppose $\epsilon > 0$. Let $T = T_\epsilon(G)$. Given any $y \in X_I \setminus \{p\}$, then there is some $x \in [y, p] \cap X$ such that*

$$C(x) \subseteq \rho^{-1}(H \cap T).$$

(Figure 4i.)

Figure 4i.

Proof : Let $L = L(y)$. Now, there is some $r > 0$, and a horoball B about p such that

$$H \cap B \subseteq N_r(L) \cap B.$$

By Lemma 4.11, there is another horoball B' with

$$N_r(L) \cap B' \subseteq T.$$

This follows either from Lemma 4.10, or (since we don't need the first inclusion) more directly from Lemmas 4.7 and 4.8. Without loss of generality, we can suppose that B' is strictly included in B. Thus $H \cap B' \subseteq H \cap B \subseteq N_r(L)$ and so

$$H \cap B' \subseteq T.$$

Now $H \cap B \subseteq N_r(L) \cap B \subseteq N_r(L \cap B)$ (Lemma 4.6). Also, since $Q \neq \emptyset$, it is clear that $L \cap B$ lies inside some uniform neighbourhood of $H \cap B$. In other words, the Hausdorff distance between $H \cap B$ and $L \cap B$ is finite. Now, $H \cap B$ is convex, and $(L \cap B) \cup \{p\}$ is starlike about p, and hence λ_0-quasiconvex (Corollary 1.1.6). Let $\rho_1 = \mathrm{proj}_{(H \cap B) \cup \{p\}}$ and $\rho_2 = \mathrm{proj}_{(L \cap B) \cup \{p\}}$. Proposition 2.2.2 gives us a constant $k \geq 0$ such that if $z \in X_C \setminus \{p\}$ then $\mathrm{diam}(\rho_1(z) \cup \rho_2(z)) \leq k$.

Let $x \in [y,p] \cap B'$ be the point distant k from the intersection of $[y,p]$ with $\partial B'$. We claim that $C(x) \subseteq \rho^{-1}(H \cap T)$.

Suppose $z \in C(x)$. Lemma 4.5 tells us that $\mathrm{proj}_{L \cup \{p\}}(z)$ meets B'', where B'' is the horoball about p with $x \in \partial B''$ (so that $B' = N_k(B'')$). Choose some $w \in \mathrm{proj}_{L \cup \{p\}}(z) \cap B''$. Since $B'' \subseteq B$, clearly $w \in \mathrm{proj}_{(L \cap B) \cup \{p\}}(z) = \rho_2(z)$. Since $H \cap B$ is convex, $\rho_1(z)$ consists of a single point $u \in H \cap B'$. We assumed that B' is strictly included in B, and so u lies in the interior of B. Thus, the point u locally minimises in H the distance to z (or locally maximises a Busemann function about z if $z \in X_I$). By Lemma 2.2.4, $u = \mathrm{proj}_{H \cup \{p\}}(z) = \rho(z)$. But $u \in H \cap B' \subseteq T$. Thus $\rho(z) \in H \cap T$. We have shown that $C(x) \subseteq \rho^{-1}(H \cap T)$. $\diamondsuit$

Lemma 4.13 : *Suppose $y \in X_I \setminus \{p\}$. Let $L = L(y)$. Suppose B is a horoball about B, and $r \geq 0$. Then $(N_r(L) \cap B)/G$ has finite volume. (Figure 4j.)*

Figure 4j.

Proof : We first prove the case where G is infinite cyclic, generated by a parabolic $\gamma \in G$.

Let x be the point of intersection of $\beta = [y,p]$ and ∂B. For $i \in \mathbf{N}$, let $x_i = x + i \log_e 2$. Thus $x_0 = x$. By the convergence of geodesic rays (Proposition 1.1.11(2)), we have $d(x + t, \gamma(x+t)) \leq Re^{-t}$ for some constant $R \geq 0$. Thus $d(x_i, \gamma x_i) \leq R/2^i$.

Let B_i be the horoball about p with $x_i \in \partial B_i$. (thus $B_i = N_{\log_e 2}(B_{i-1})$.)

Suppose $z \in N_{r+R}(\beta) \cap (B_0 \setminus B_1)$. Let w be the nearest point to z on β. Thus $p\hat{w}z = \pi/2$ and so $w \in B_0$. We have $d(w,z) \leq r + R$, and it is easily seen that $d(x_0, w) \leq d(z,w) + \log_e 2 \leq r + R + \log_e 2$. Thus $d(x_0, z) \leq 2r + 2R + \log_e 2$. This shows that $N_r(\beta) \cap (B_0 \setminus B_1) \subseteq N = N_{2r+2R+\log_e 2}(x_0)$. Since $L = \bigcup G\beta = \bigcup_{g \in G} g\beta$, we have $N_{r+R}(L) \cap (B_0 \setminus B_1) \subseteq \bigcup GN$. By Proposition 1.2.4, N has volume at most $V = V(2r + 2R + \log_e 2, n)$. It follows that

$$\mathrm{vol}\big((N_{r+R}(L) \cap (B_0 \setminus B_1))/G\big) \leq V.$$

So, certainly

$$\mathrm{vol}\big((N_r(L) \cap (B_0 \setminus B_1))/G\big) \leq V.$$

Now, given any $i \in \mathbf{N}$, let G_i be the subgroup of G generated by γ^{2^i}. Let $L_i = \bigcup G_i\beta$. We claim that $L \cap B_i \subseteq N_R(L_i) \cap B_i$. To see this, suppose that $z \in L \cap B_i$. Then $z \in g[x_i, p]$ for some $g \in G$. Now $g = h\gamma^{-j}$ where $h \in G_i$ and $j \in \{0, 1, \ldots, 2^i - 1\}$. Thus $\gamma^j z \in h[x_i, p] \subseteq L_i \cap B_i$. By Proposition 1.1.11(2), we have $d(z, \gamma^j z) \leq d(x_i, \gamma^j x_i) \leq j(R/2^i) \leq R$. Thus $z \in N_R(L_i)$ as claimed.

By Lemma 4.3, $N_R(L_i) \cup \{p\}$ is starlike about p. Applying Lemma 4.6,

$$N_r(L) \cap B_i \subseteq N_r(L \cap B_i) \subseteq N_r(N_R(L_i) \cap B_i) \subseteq N_{r+R}(L_i).$$

Thus $N_r(L) \cap (B_i \setminus B_{i+1}) \subseteq N_{r+R}(L_i) \cap (B_i \setminus B_{i+1})$. Exactly as with the first part of the argument (the case $i = 0$), we see that

$$\mathrm{vol}\big((N_{r+R}(L_i) \cap (B_i \setminus B_{i+1}))/G_i\big) \leq V$$

and so

$$\mathrm{vol}\big((N_r(L) \cap (B_i \setminus B_{i+1}))/G_i\big) \leq V.$$

Since G_i has index 2^i in G, we have

$$\mathrm{vol}\big((N_r(L) \cap (B_i \setminus B_{i+1}))/G\big) \leq V/2^i.$$

Since $B = \bigcup_{i=0}^{\infty}(B_i \setminus B_{i+1})$, we have

$$\mathrm{vol}\big((N_r(L) \cap B)/G\big) \leq V \sum_{i=0}^{\infty} 2^{-i} = 2V.$$

Now, if G is any discrete parabolic group, Proposition 4.2 tells us that G contains a parabolic element γ. Let G' be the subgroup of G generated by γ. Let $L' = \bigcup G'\beta$, where $\beta = [y, p]$. We have

$$\mathrm{vol}\big((N_r(L') \cap B)/G'\big) < \infty.$$

But $(N_r(L) \cap B)/G$ is a quotient of $(N_r(L') \cap B)/G'$ by an equivalence relation, and so also has finite volume. $\diamondsuit$

Proposition 4.14 : *Given $\epsilon > 0$, let $T = T_\epsilon(G)$. Suppose $Q \subseteq X_I \setminus \{p\}$ is closed and G-invariant with Q/G compact. Let $H = H(Q) = \mathrm{hull}(Q \cup \{p\}) \setminus \{p\}$. Then, for any $r \geq 0$, $N_r(H \cap T)/G$ has finite volume. (Figure 4k.)*

Figure 4k.

Proof : Let y be any point of $X_I \setminus \{p\}$, and set $L = L(y)$. By Lemma 4.10, there is some horoball B about p, and $r' \geq 0$ so that

$$H \cap T \subseteq N_{r'}(L) \cap B.$$

Thus,

$$N_r(H \cap T) \subseteq N_r(N_{r'}(L) \cap B) \subseteq N_{r+r'}(L) \cap N_r(B).$$

Note that $N_r(B)$ is a horoball about p. The result now follows from Lemma 4.13. $\diamondsuit$

5. Definitions.

We assume that all the sectional curvatures of X lie between $-\kappa^2$ and -1.

In this chapter, we give the four basic definitions of geometrical finiteness, F1, F2, F4 and F5. From property F1, we deduce that a geometrically finite group is finitely generated, and contains only finitely many conjugacy classes of finite subgroups.

5.1. Definition F1.

Suppose $\Gamma \subseteq \operatorname{Isom} X$ is discrete. Let $M_C(\Gamma) = (X \cup \Omega)/\Gamma$. As a topological space, $M_C(\Gamma)$ has associated to it a compact totally-disconnected space of ends. This space can be used to compactify $M_C(\Gamma)$. Suppose $E \subseteq M_C(\Gamma)$ is a closed subset with compact (topological) boundary. Then, the space of ends of E can be naturally identified as an open and closed subset of the space of ends of $M_C(\Gamma)$.

Write $\pi : X \cup \Omega \longrightarrow M_C(\Gamma)$ for the natural projection. Suppose $y \in \Lambda$, and $y_n \to y$, with each $y_n \in X$. Clearly, the sequence πy_n leaves every compact subset of $M_C(\Gamma)$. If it happens that πy_n tends to an end e of $M_C(\Gamma)$, we shall say that y is *associated* to e. In general, of course, a limit point has no ends associated to it, or indeed, it may have more that one, depending on the sequence (y_n). (Consider, for example, a double limit of quasifuchsian groups in $\mathbf{H}^3$.) We shall be interested only in very special cases.

Suppose that $E \subseteq M_C(\Gamma)$ is closed, connected, non-compact, and has compact topological boundary, $\partial_C E$. Let Y_0 be a connected component of the lift of $\pi^{-1}E$ of E to $X \cup \Omega$. Let $G = \operatorname{stab}_\Gamma Y_0$. Thus, G is determined by E up to conjugacy in Γ. Suppose that G is a parabolic group, with fixed point p. We may identify E as a subset, Y_0/G of $M_C(G) = (X_C \setminus \{p\})/G$. Thus, $\partial_C E$ is identified with $\partial_C Y_0/G$ where $\partial_C Y_0$ is the topological boundary of Y_0 in $X \cup \Omega$. We claim that E is closed in $M_C(G)$. This amounts to saying that $Y_0 \cup \{p\}$ is closed in X_C. Suppose, for contradiction, that $y \in \Lambda \setminus \{p\}$ lies in the closure of Y_0. Then there is a sequence of points $y_n \in Y_0$ tending to y. Choose any orbit $\Gamma z \subseteq X$ disjoint from Y_0. Since $y \in \Lambda$, there is a sequence $z_n \in \Gamma z$ with $z_n \to y$. Each geodesic $[y_n, z_n]$ meets $\partial_C Y_0$. Choosing $w_n \in [y_n, z_n] \cap \partial_C Y_0$, we see that $w_n \to y$. Since $\partial_C Y_0/G$ is compact, there is a compact set $K \subseteq X \cup \Omega$ with GK covering $\partial_C Y_0$. We see that the sets GK accumulate at y, contradicting the fact that G acts properly discontinuously on $X_C \setminus \{p\}$ (Proposition 3.2.6). This proves the claim. It follows easily that $\partial_C E = \partial_C Y_0/G$ is the topological boundary of E in $M_C(G)$.

We know from Proposition 4.4, that $M_C(G)$ has precisely one topological end. Thus E is a neighbourhood of that end. It follows that E itself has precisely one end. Thinking of E again as a subset of $M_C(\Gamma)$, we see that E is a neighbourhood of an end of $M_C(\Gamma)$. This end, e, is determined by E, and is isolated in the space of ends of $M_C(\Gamma)$.

By Proposition 4.4, we can assume that E is an orbifold with convex boundary. Thus, Y_0 has the form $Y \cup \Omega^Y(G)$ as described in Section 3.3, where $Y \subseteq X$ is closed and convex. Note that Y contains a horoball about p, and so for all $\gamma \in \operatorname{stab}_\Gamma p$, we have $\gamma Y \cap Y \neq \emptyset$, thus $\gamma Y = Y$. We see that $G = \operatorname{stab}_\Gamma Y = \operatorname{stab}_\Gamma p$. Thus G is a maximal parabolic subgroup of Γ.

Let $y_n \in X$ be any sequence of points tending to p. Let $\pi_G : X_C \setminus \{p\} \longrightarrow M_C(G)$

be the natural projection. Now, $\pi_G y_n$ leaves every compact set in $M_C(G)$. Since $M_C(G)$ has only one end, we have $\pi_G y_n \in (Y \cup \Omega^Y(G))/G \subseteq M_C(G)$ for all sufficiently large n. Thus, $\pi y_n \in E \subseteq M_C(\Gamma)$ for all sufficiently large n. It follows that the end e of $M_C(\Gamma)$ is associated to p, in the sense defined above, and that it is the unique end of $M_C(\Gamma)$ associated to p. Moreover, we see easily that any other limit point associated to e lies in the orbit Γp of p.

Definition : In the situation described above, we shall call e a *parabolic end* of $M_C(\Gamma)$, and E a *standard cusp region* (which we assume to be a convex suborbifold).

Suppose $p \in \Lambda = \Lambda(\Gamma)$ is associated to a parabolic end of $M_C(\Gamma)$. Then p is a parabolic fixed point, so that $G = \mathrm{stab}_\Gamma\, p$ is a maximal parabolic subgroup of Γ. Now $\Lambda \setminus \{p\} \subseteq \Omega(G) = X_C \setminus \{p\}$ and so $(\Lambda \setminus \{p\})/G \subseteq M_C(G)$. Let $E \subseteq M_C(\Gamma)$ be a standard cusp region. We may identify E with $(Y \cup \Omega^Y(G))/G \subseteq M_C(G)$, where $Y \subseteq X$ is closed and convex. Now $Y \cup \Omega^Y(G)$ does not meet $\Lambda \setminus \{p\}$, and so, in $M_C(G)$, E does not meet $(\Lambda \setminus \{p\})/G$. Since $(\Lambda \setminus \{p\})/G$ is closed in $M_C(G)$, and since E is a neighbourhood of the end, it follows that $(\Lambda \setminus \{p\})/G$ is compact.

Definition : A parabolic fixed point $p \in \Lambda$ is *bounded* if $(\Lambda \setminus \{p\})/\,\mathrm{stab}_\Gamma\, p$ is compact.

We have shown:

Lemma 5.1.1 : *If $p \in \Lambda$ is associated to a parabolic end, then p is a bounded parabolic fixed point.*

We shall see that the converse of Lemma 5.1.1 is also true (Corollary 6.3).

We can give an intrinsic characterisation of standard cusp regions as follows. We say that an orbifold E with boundary is an *intrinsic standard cusp region* if:

(1) E has the form $M_C^Y(G)$ (as described in Section 3.3), where Y is a metrically complete, simply connected manifold, with all sectional curvatures between $-\kappa^2$ and -1, with convex boundary ∂Y, and where G acts properly discontinuously on Y,

(2) $\partial Y/G$ is relatively compact in $M_C^Y(G)$, and

(3) G is infinite, and has a unique fixed point in Y_I.

These properties (1)–(3) characterise standard cusp regions in the following sense.

Proposition 5.1.2 : *Suppose $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete, and $E \subseteq M_C(\Gamma)$ is a convex suborbifold which is an intrinsic standard cusp region. Then, E is a standard cusp region in $M_C(\Gamma)$.*

Proof : From the definitions of convex suborbifold, we know that E has the form $M_C^{Y'}(G')$ where $Y' \subseteq X$ and $G' \subseteq \Gamma$. We can identify Y' with Y, and G' with G, in the definition of intrinsic standard cusp region. Since Y is G-invariant, and G has a unique fixed point in Y_I, we see that G is parabolic. The statement that E is a convex suborbifold tells us that the topological boundary of E in $M_C(\Gamma)$ is the closure of $\partial Y/G$ in E, which, by hypothesis, is compact. Also E is connected and non-compact. From the discussion above, it follows that E is a standard cusp region. $\diamondsuit$

We can now give the first definition of geometrical finiteness.

Definition : The discrete group $\Gamma \subseteq \mathrm{Isom}\, X$ is "F1" if $M_C(\Gamma)$ has finitely many ends, each a parabolic end.

Another way to say this that $M_C = M_C(\Gamma)$ is the union of a compact set, K, and a finite number of standard cusp regions, E_i for $1 \leq i \leq k$. We can suppose that K is the closure of $M_C \setminus \bigcup_{i=1}^k E_i$ in M_C, and thus a suborbifold with boundary. Note that for any standard cusp regions $E_i' \subseteq E_i$, we will have that $M_C \setminus \bigcup_{i=1}^k E_i$ is relatively compact. In this way, we can always arrange that $d(E_i, E_j)$ is arbitrarily large for $i \neq j$.

5.2. Definition F2.

The second definition gives a description of geometrical finiteness intrinsic to the action of Γ on Λ.

We shall need the notion of "conical limit point" which is based on the following observation.

Proposition 5.2.1 : *Suppose $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete and not loxodromic. Suppose (γ_n) is a sequence of distinct elements of Γ, and that $y \in \Lambda$. Then the following are equivalent.*

1a(1b): For some (each) $x \in X$ and some (each) geodesic ray β tending to y, we have $\gamma_n x \to y$ and $d(\gamma_n, \beta)$ bounded.

2a(2b): For some (each) geodesic ray β tending to y, and for every subsequence (γ_{n_i}) of (γ_n), the sets $\gamma_{n_i}^{-1}\beta$ accumulate somewhere in X.

3a: For each $z \in \Lambda \setminus \{y\}$, the sequence of ordered pairs $(\gamma_n^{-1} y, \gamma_n^{-1} z)$ remains in a compact subset of $(\Lambda \times \Lambda) \setminus \Delta(\Lambda)$, where $\Delta(\Lambda) = \{(x,x) \mid x \in \Lambda\}$.

Proof : The only implication that requires comment is that (3) implies (1). If Γ is finite or parabolic, this is vacuous. Since Γ is not loxodromic, we can suppose that there are distinct points z and z' in $\Lambda \setminus \{y\}$. Let α be the bi-infinite geodesic $[y, z]$, and let x be any point of X. Saying that $(\gamma_n^{-1} y, \gamma_n^{-1} z)$ remains in a compact subset $(\Lambda \times \Lambda) \setminus \Delta(\Lambda)$ is the same as saying that $d(\gamma_n^{-1}\alpha, x)$ is bounded. Thus $d(\alpha, \gamma_n x)$ is bounded. It follows that the set of accumulation points of $\{\gamma_n x\}$ is a subset of $\{y, z\}$. Similarly, this set of accumulation points is also a subset of $\{y, z'\}$, and thus equal to $\{y\}$. In other words, $\gamma_n x \to y$. $\diamondsuit$

Definition : Suppose $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete. Then $y \in \Gamma$ is a *conical limit point* if there is a sequence (γ_n) of distinct elements of Γ so that, for each $z \in \Gamma \setminus \{y\}$, the sequence $(\gamma_n y, \gamma_n z)$ lies in a compact subset of $(\Lambda \times \Lambda) \setminus \Delta(\Gamma)$.

Thus, y is a conical limit point if and only if for some (or each) $x \in X$, and some (or each) geodesic ray β tending to y, then for some $r \geq 0$, the set $\Gamma x \cap N_r(\beta)$ accumulates at y.

Alternatively, y is a conical limit point if and only if for some (or each) ray β tending to y, $\pi(\beta \cap X)$ accumulates in $M = X/\Gamma$, where $\pi : X \longrightarrow M$ is the natural projection. (This means that there is a sequence of points, $z_n \in \beta \cap X$, tending to y, with πz_n convergent in M.)

Note that, in the above two statements, we need make no special qualifications for loxodromic groups.

Saying that $\pi(\beta \cap X)$ does not accumulate in M is the same as saying that the orbit $\Gamma\beta$ of β is locally finite in X. From Proposition 3.3.1, it follows that $\Gamma\beta$ is locally finite on $X \cup \Omega$, in other words, $\pi(\beta \cap X)$ does not accumulate in M_C. Thus:

Lemma 5.2.2 : *Suppose $y \in \Lambda$, and β is a geodesic ray tending to y. Let π be the projection from $X \cup \Omega$ to M_C. If $\pi(\beta \cap X)$ accumulates in M_C, then y is a conical limit point.* $\diamondsuit$

Note that a parabolic fixed point, $p \in \Lambda$, may be recognised from the action of Γ on Λ. Thus, p is a parabolic fixed point if and only if it is the unique fixed point in Λ of the infinite group $\mathrm{stab}_\Gamma\, p$. The definition of bounded parabolic fixed point given in Section 5.1 is already intrinsic to Λ.

Definition : The discrete group $\Gamma \subseteq \mathrm{Isom}\, X$ is "F2" if the limit set Λ consists entirely of conical limit points and bounded parabolic fixed points.

It is easily seen that a limit point cannot be both a conical limit point and a bounded parabolic fixed point. We shall see (Lemma 6.4) that if Γ is geometrically finite, then every parabolic fixed point is bounded.

5.3. Definition F4.

Suppose $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete. The (closed) convex hull of the limit set, hull(Λ), is Γ-invariant. Thus we may define

$$\mathrm{core}(M) = (\mathrm{hull}(\Lambda))/\Gamma \subseteq M = X/\Gamma.$$

We call core(M) the *convex core* of M. If core(M) has non-empty interior, then it has the structure of an orbifold with boundary, though in general, it may not. (However, for any $\eta > 0$, the η-neighbourhood, $N_\eta \mathrm{core}(M)$, is a convex suborbifold of M.)

Definition : The discrete group $\Gamma \subseteq \mathrm{Isom}\, X$ is "F4" if, for some $\epsilon \in (0, \epsilon(n,\kappa))$, we have that $\mathrm{core}(M) \cap \mathrm{thick}_\epsilon(M)$ is compact.

Here, $\epsilon(n,\kappa)$ is the Margulis constant (Section 3.5). Note that the thick part of the convex core, $\mathrm{core}(M) \cap \mathrm{thick}_\epsilon(M)$, is defined intrinsically to core(M).

There are several variations on this definition one could give. For example, we could replace $\mathrm{core}(M)$ by $(\mathrm{join}(\Lambda))/\Gamma$. Instead of $\mathrm{thick}_\epsilon(M)$ we could take $\mathrm{noncusp}_\epsilon(M)$. Instead of saying "for some $\epsilon \in (0, \epsilon(n, \kappa))$", we could say "for all $\epsilon \in (0, \epsilon(n, \kappa))$". That all combinations arising in this way give rise to the same notion of geometrical finiteness should be apparent from the proofs of equivalence given in the next chapter.

5.4. Definition F5.

Definition : Suppose $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete. Γ is "F5" if there is a bound on the orders of every finite subgroup of Γ, and if, for some $\eta > 0$, $N_\eta \mathrm{core}(M)$ has finite volume.

Again, we could replace $\mathrm{core}(M)$ by $(\mathrm{join}(\Lambda))/\Gamma$, or say that every η-neighbourhood has finite volume.

We have already remarked that $N_\eta \mathrm{core}(M)$ is a convex suborbifold. The group Γ is the orbifold fundamental group of $N_\eta \mathrm{core}(M)$. Thus, the definition F5 is intrinsic to $N_\eta \mathrm{core}(M)$. This statement becomes more transparent given:

Proposition 5.4.1 : *If $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete, and neither finite nor parabolic, then every finite subgroup, G, of Γ has a fixed point in* $\mathrm{hull}(\Lambda)$.

Proof : We know that G has a fixed point in X. Since the projection to $\mathrm{hull}(\Lambda)$ is Γ-equivariant, we see that the projection of this point to $\mathrm{hull}(\Lambda)$ is also fixed by G. ♢

Thus, the bound on the order of finite subgroups of Γ translates to a bound on the orders of the subgroups of the orthogonal group defining the orbifold structure of $N_\eta \mathrm{core}(M)$.

In fact, I suspect that this bound is superfluous, i.e. it should be implied by the statement that $N_\eta \mathrm{core}(M)$ has finite volume. This is certainly the case if M itself has finite volume.

Proposition 5.4.2 : *Suppose $\Gamma \subseteq \mathrm{Isom}\, X$ is discrete, and that $M = X/\Gamma$ has finite volume. Then, there is a bound on the orders of finite subgroups of Γ.*

Proof : Given the lower bound on the volumes of uniform balls in X, (Proposition 1.1.12), the proof is essentially the same as that in the constant curvature case given in [Bo1]. We shall not reproduce the argument here. ♢

In fact, we shall see that if M has finite volume, then it is topologically finite as an orbifold (Proposition 6.6).

5.5. Basic group-theoretic properties.

Proposition 5.5.1 : *If Γ is F1, then Γ is finitely generated.*

Proof : Write $M_C = K \cup \bigcup_{i-1}^{k} E_i$, where each E_i is a standard cusp region, and K is a compact set which we can take to be a suborbifold with boundary. The orbifold fundamental group of each E_i is isomorphic to the corresponding maximal parabolic subgroup. Such a subgroup is finitely generated by Proposition 4.1. The result now follows by the orbifold van-Kampen theorem. $\diamondsuit$

Proposition 5.5.2 : *If Γ is F1, then Γ has finitely many conjugacy classes of finite subgroups.*

Proof : We have already observed (Section 3.2), that every finite subgroup of Γ lies inside at least one maximal finite subgroup. It thus suffices to show that there are only finitely many conjugacy classes of maximal finite subgroups.

At the end of Section 3.2, we defined the locally finite collection, $\hat{\mathcal{G}}$ of disjoint subsets of $M_C(\Gamma)$, which are in bijective correspondence with the conjugacy classes of maximal finite subgroups of Γ. We see that only finitely many elements of $\hat{\mathcal{G}}$ can meet the compact set K. On the other hand, if an element of $\hat{\mathcal{G}}$ meets a standard cusp region E_i, we see that (up to conjugacy in Γ) the corresponding maximal finite subgroup lies inside the maximal parabolic subgroup corresponding to E_i. Now, Proposition 4.1 tells us that a parabolic subgroup of Γ is finitely generated and virtually nilpotent. The proposition thus reduces to the following group-theoretic statement (Lemma 5.5.3). $\diamondsuit$

Lemma 5.5.3 : *A finitely generated virtually nilpotent group has finitely many conjugacy classes of finite subgroups.*

Proof : Suppose P is finitely generated, and contains a nilpotent subgroup N of finite index. Then N is also finitely generated, and we can suppose that N is normal in P. Let Z be the centre of N. It is well known that Z is finitely generated. (Alternatively, we could take Z to be the first group of the lower central series, which is clearly finitely generated.) Let T be the torsion subgroup of Z. Thus T is finite. Since T and Z are characteristic in N, they are normal in P. By induction on the height of N, we can suppose that P/Z has only finitely many conjugacy classes of finite subgroup.

Suppose F_1 and F_2 are finite subgroups of P. We can assume that F_1Z and F_2Z are conjugate in P. Thus, we can take $F_1Z = F_2Z = K$ say. We claim that K/T contains only finitely many transversal subgroups to Z/T up to conjugacy in K/T. Given this, we can assume that $F_1T = F_2T$. But this group is finite, and so contains only finitely many subgroups. This, then, completes the proof.

To prove the claim, let $H = K/T$, and $A = Z/T$. Thus, A is free abelian, and normal and of finite index in H. The rest of the argument is standard group cohomology. Let G be a transversal subgroup to A in H, i.e. $GA = H$, and $G \cap A = \{0\}$. If G' is another transversal subgroup, we have a unique monomorphism $\theta : G \longrightarrow H$ such that the image $\theta(G)$ equals G', and $\theta(g)g^{-1} \in A$ for all $g \in G$.

Suppose that the transversal subgroups G_1 and G_2 have corresponding monomorphisms θ_1 and $\theta_2 : G \longrightarrow H$. If $a \in A$ and $g \in G$, then

$$a\theta_1(g)a^{-1}g^{-1} = a(\theta_1(g)g^{-1})ga^{-1}g^{-1} = (\theta_1(g)g^{-1})aga^{-1}g^{-1}.$$

Thus, $\theta_2(g) = a\theta_1(g)a^{-1}$ if and only if $\theta_2(g)\theta_1(g)^{-1} = aga^{-1}g^{-1}$. We say that G_1 and G_2 are *A-conjugate* if for some $a \in A$, we have $\theta_2(g)\theta_1(g)^{-1} = aga^{-i}g^{-1}$ for all $g \in G$. We see that A-conjugacy is an equivalence relation on transversal subgroups (defined independently of the choice of G). Clearly, A-conjugate transversals are conjugate as subgroups of H.

Now, the set of all maps from G into A form a free abelian group under multiplication in A, of rank equal to $|G|\operatorname{rank} A$. Those maps of the form $[g \mapsto \theta(g)g^{-1}]$, for a monomorphism θ, form a free abelian subgroup C. Those of the form $[g \mapsto aga^{-1}g^{-1}]$, for $a \in A$, form a subgroup B of C. Thus, A-conjugacy classes of transversal subgroups are in bijective correspondence with the elements of C/B. (C/B is the first cohomology group $H^1(G, A)$.) We claim that C/B is finite. Since C is finitely generated free abelian, it suffices to see that C/B is a torsion group.

Given a map $[g \mapsto \theta(g)g^{-1}] \in C$, let $b = \prod_{h\in G}(\theta(h)h^{-1}) \in A$. Then, for any $g \in G$,

$$\begin{aligned} b &= \prod_{h\in G} \theta(gh)(gh)^{-1} = \prod_{h\in G} (\theta(g)g^{-1})g(\theta(h)h^{-1})g^{-1} \\ &= (\theta(g)g^{-1})^n gbg^{-1} \end{aligned}$$

where $n = |G|$. Thus,

$$(\theta(g)g^{-1})^n = bgb^{-1}g^{-1},$$

and so

$$[g \mapsto \theta(g)g^{-1}]^n = [g \mapsto bgb^{-1}g^{-1}] \in B.$$

$\diamondsuit$

6. Proofs of equivalence.

We assume that X has pinched negative curvature. The main aim of this chapter is to show the equivalence of the main definitions of geometrical finiteness from Chapter 5.

Theorem 6.1 : *The properties F1, F2, F4 and F5, of a discrete subgroup of* Isom X*, are all equivalent.*

This will be largely a matter of tying up loose ends—most of the work has already been done. We shall give proofs of the following implications:

We include F1 $\Rightarrow$ F2 since it admits a direct proof much simpler than following the cycle. The proof of F1 $\Rightarrow$ F5 uses F1 $\Rightarrow$ F4.

Proof of F1 $\Rightarrow$ F2 : Suppose Γ is F1. We write $M_C(\Gamma) = K \cup \bigcup_{i=1}^{k} E_i$ where each E_i is a standard cusp region, and K is compact. Let $\pi : X \cup \Omega \longrightarrow M_C(\Gamma)$ be the projection. Each E_i corresponds to an orbit, Π_i, of a parabolic fixed point. Let $\Pi = \bigcup_{i=1}^{k} \Pi_i \subseteq \Lambda$. By Lemma 5.1.1, each element of Π is a bounded parabolic fixed point.

Suppose $y \in \Lambda \setminus \Pi$. Let β be any geodesic ray tending to y. Now, each component of $\bigcup_{i=1}^{k} \pi^{-1} E_i$ is a convex set whose closure meets Λ in a single point of Π. It follows easily that $\beta \cap \pi^{-1} K$ is unbounded. We see that $\pi(\beta \cap X)$ must accumulate somewhere in $K \subseteq M_C(\Gamma)$. By Lemma 5.2.2, y is a conical limit point. $\diamondsuit$

Next, we aim to prove F2 $\Rightarrow$ F1.

Lemma 6.2 : *Suppose $\Gamma \subseteq \operatorname{Isom} X$ is discrete. Let $\Pi \subseteq \Lambda$ be the set of all bounded parabolic fixed points. Write Π as a disjoint union $\Pi = \bigsqcup_{i \in I} \Pi_i$ of orbits under Γ, where I is a finite or countable indexing set. Then, each orbit Π_i is associated to a standard cusp region $E_i \subseteq M_C(\Gamma)$. Moreover, the E_i are all disjoint in $M_C(\Gamma)$. In fact, given any $r > 0$, we can arrange that $d(E_i, E_j) \geq r$ if $i \neq j$.*

Proof : Let $H = \operatorname{hull}(\Lambda)$. Let $\rho = \operatorname{proj}_H : X_C \longrightarrow H$. Choose any $\epsilon \in (0, \epsilon(n, \kappa))$, and let $T = T_\epsilon(\Gamma) \subseteq X$ as in Section 3.5.

Suppose $p_i \in \Pi_i$. Let $G_i = \operatorname{stab}_\Gamma p_i$. Thus G_i is maximal parabolic, and $T_i = T_\epsilon(G_i)$ is a connected component of T. Since p_i is a bounded parabolic fixed point, $(\Lambda \setminus \{p_i\})/G_i$ is compact. Thus, by Proposition 4.12, we can find a convex set $C_i \subseteq \rho^{-1}(H \cap T_i) \subseteq X_C \setminus \{p_i\}$ so that C_i/G_i is a standard cusp region in $M_C(G_i)$. Clearly, $\rho^{-1}(H \cap T_i)$ cannot meet Λ, and so $C_i \subseteq X \cup \Omega$. We see that C_i/G_i descends to a standard cusp region $E_i = (\bigcup \Gamma C_i)/\Gamma$ in $M_C(\Gamma)$. Note that $\bigcup \Gamma C_i = \rho^{-1}(H \cap (\bigcup \Gamma T_i))$.

We perform this construction for each $i \in I$. By Proposition 3.5.4, we have that, for some $\delta > 0$, if $i \neq j$, then $d(\bigcup \Gamma T_i, \bigcup \Gamma T_j) \geq \delta$. It follows that the E_i are disjoint. In fact, by Proposition 4.4, we can arrange that $d(E_i, E_j)$ is arbitrarily large for $i \neq j$. $\diamondsuit$

Corollary 6.3 : *A limit point is associated to a parabolic end of $M_C(\Gamma)$ if and only if it is a bounded parabolic fixed point.*

Proof : By Lemmas 5.1.1 and 6.2. $\diamondsuit$

Proof of F2 ⇒ F1 : Suppose that Γ is F2. Let $\Pi \subseteq \Lambda$ be the set of all bounded parabolic fixed points. Write Π as a disjoint union $\Pi = \bigsqcup_{i \in I} \Pi_i$, where I is a finite or countable indexing set. Lemma 6.2 gives us a corresponding collection of standard cusp regions, $E_i \subseteq M_C(\Gamma)$. We may suppose that $d(E_i, E_j) \geq 1$ if $i \neq j$. Let E_i° be the topological interior of E_i in M_C. We claim that $M_C \setminus \bigcup_{i \in I} E_i^\circ$ is compact. It then follows that I is finite, and so Γ is F1.

Let $\pi : X \cup \Omega \longrightarrow M_C$ be the quotient map. Let $Z = \bigcup_{i \in I} \pi^{-1} E_i^\circ = (X \cup \Omega) \setminus \pi^{-1} K$. To each point $p \in \Pi$ is associated a component $Y(p)$ of Z, so that $Y(p)/G(p)$ is a neighbourhood of the end of $M_C(G(p))$, where $G(p)$ is the maximal parabolic group $\mathrm{stab}_\Gamma\, p$. Distinct components of Z are at least a distance 1 apart, thus $Y(p)$ is open in $X \cup \Omega$, and hence in $X_C \setminus \{p\}$. We see that $(X_C \setminus (Y(p) \cup \{p\}))/G(p)$ is compact.

Let $D \subseteq X_C$ be any Dirichlet domain for Γ (Section 3.3). Thus D is closed in X_C and quasiconvex. Write ΓD for the collection of images of D under Γ. By Corollary 3.3.2, ΓD is locally finite on $X \cup \Omega$, and so $X \cup \Omega \subseteq \bigcup \Gamma D$. We see that $\pi^{-1} K = (X \cup \Omega) \setminus Z \subseteq \bigcup \Gamma (D \setminus (\Lambda \cup Z))$ and so $K = \pi(D \setminus (\Lambda \cup Z))$. To prove the claim, it thus suffices to see that $D \setminus (\Lambda \cup Z)$ is compact.

Since D is quasiconvex, and ΓD is locally finite on X, it follows easily that D cannot contain any conical limit point of Γ. Since Γ is F2, we have that $D \cap \Lambda \subseteq \Pi$, and so $D \setminus (\Lambda \cup Z) = D \setminus \bigcup_{p \in \Pi} (Y(p) \cup \{p\})$. Since D is compact Hausdorff, it thus suffices to see that $D \cap (Y(p) \cup \{p\})$ is open in D for all $p \in \Pi$.

Fix $p \in \Pi$, and let $Y = Y(p)$ and $G = G(p)$. By Corollary 3.3.2, we know that GD is locally finite on $\Omega(G) = X_C \setminus \{p\}$. Now, certainly $D \setminus (Y \cup \{p\})$ is closed in $X_C \setminus \{p\}$, and $(X_C \setminus (Y \cup \{p\}))/G$ is compact. We conclude that $D \setminus (Y \cup \{p\})$ is compact, and hence closed in D. Thus $D \cap (Y \cup \{p\})$ is open in D. ♢

Proof of F1 ⇒ F4 : Suppose Γ is F1. Let $e_1, \ldots, e_k$ be the parabolic ends of M_C. Suppose $p_i \in \Lambda$ is associated to e_i. By Lemma 5.1.1, p_i is a bounded parabolic fixed point. Let $G_i = \mathrm{stab}_\Gamma\, p_i$, so that $(\Lambda \setminus \{p_i\})/G_i$ is compact. Given any $\epsilon \in (0, \epsilon(n, \kappa))$, let $T_i = T_\epsilon(G_i)$ be as defined in Section 3.5. Let $H = \mathrm{hull}(\Lambda)$ and $\rho = \mathrm{proj}_H : X_C \longrightarrow H$. Proposition 4.12 gives us a convex set $C_i \subseteq X_C \setminus \{p_i\}$, with $C_i \subseteq \rho^{-1}(H \cap T_i)$, and such that C_i/G_i is a closed neighbourhood of the end of $M_C(G_i)$. It follows that $\bigcup \Gamma C_i$ projects to a standard cusp region $E_i \subseteq M_C(\Gamma)$, which is a neighbourhood of the end e_i. Since $C_i \subseteq \rho^{-1}(H \cap T_i)$, we certainly have $H \cap C_i \subseteq T_i$.

We perform this construction for each $i \in \{1, 2, \ldots, k\}$. Thus $H \cap (\bigcup_{i=1}^k \bigcup \Gamma C_i) \subseteq \bigcup_{i=1}^k \bigcup \Gamma T_i$.

Projecting to M_C, we have

$$\mathrm{core}(M) \cap (\bigcup_{i=1}^{k} E_i) \subseteq \mathrm{cusp}_\epsilon(M),$$

and so

$$\mathrm{core}(M) \setminus \mathrm{cusp}_\epsilon(M) \subseteq M_C \setminus \bigcup_{i=1}^{k} E_i.$$

Since Γ is F1, it follows that the closure, $\mathrm{core}(M) \cap \mathrm{noncusp}_\epsilon(M)$, of $\mathrm{core}(M) \setminus \mathrm{cusp}_\epsilon(M)$ is compact. Thus $\mathrm{core}(M) \cap \mathrm{thick}_\epsilon(M)$ is compact. $\diamondsuit$

Lemma 6.4 : *If Γ is F4, then every parabolic fixed point is bounded.*

Proof : Let $H = \mathrm{hull}(\Lambda)$. Suppose $p \in \Lambda$ is a parabolic fixed point. Let $G = \mathrm{stab}_\Gamma p$. Given $\epsilon \in (0, \epsilon(n, \kappa))$ let $T = T_\epsilon(G)$. Let ∂T be the topological boundary of T in X. Let $v : X_C \setminus \{p\} \longrightarrow X_I \setminus \{p\}$ be the map $[x \mapsto x - \infty]$. Thus v is a G-equivariant continuous retraction of $X_C \setminus \{p\}$ onto $X_I \setminus \{p\}$. From the form of T described by Proposition 3.5.2, it is clear that $v(\partial T) = X_I \setminus \{p\}$. (In fact, $v|\partial T$ is a homeomorphism.) From Section 3.5, we know that T/G may be identified as a component of $\mathrm{thin}_\epsilon(M)$. Thus, $\partial T/G$ may be identified as a boundary component, S, of $\mathrm{thick}_\epsilon(M)$. Under this identification, $(H \cap \partial T)/G$ is identified with $\mathrm{core}(M) \cap S$, which is a closed subset of $\mathrm{core}(M) \cap \mathrm{thick}_\epsilon(M)$ and hence compact. Now $v(H \cap \partial T) \supseteq \Lambda \setminus \{p\}$, and so $(\Lambda \setminus \{p\})/G$ is a closed subset of $v(H \cap \partial T)/G$ and hence compact. Thus, p is a bounded parabolic fixed point. $\diamondsuit$

Corollary 6.5 : *If Γ is F1, then Γ has finitely many conjugacy classes of maximal parabolic subgroups.*

Proof : We know that Γ is also F4, and so by Lemma 6.4, every parabolic fixed point is bounded. Now, maximal parabolic subgroups of Γ are in bijective correspondence with orbits of parabolic fixed points. These in turn (applying Corollary 6.3) are in bijective correspondence with parabolic ends of M_C. Since Γ is F1, there are only finitely many such ends. $\diamondsuit$

Proof of F4 $\Rightarrow$ F2 : Suppose Γ is F4. Thus $\mathrm{core}(M) \cap \mathrm{thick}_\epsilon(M)$ is compact for some $\epsilon \in (0, \epsilon(n, \kappa))$. Since Margulis tubes do not accumulate in M (Proposition 3.5.4), it follows that $\mathrm{core}(M) \cap \mathrm{thick}_\epsilon(M)$ meets only finitely many Margulis tubes. Since each such tube is compact, it follows that $\mathrm{core}(M) \cap \mathrm{noncusp}_\epsilon(M)$ is compact.

Let $\Pi \subseteq \Lambda$ be the set of all parabolic fixed points. By Lemma 6.4, each such fixed point is bounded. Suppose $y \in \Lambda \setminus \Pi$. Choose any X in $X \cap \mathrm{hull}(\Lambda)$, and let β be the ray $[x, y]$. Thus $\beta \subseteq \mathrm{hull}(\Lambda)$. From the form of Margulis cusps described by Proposition 3.5.2, it is clear that $\beta \cap \pi^{-1}\mathrm{noncusp}_\epsilon(M)$ is unbounded, where $\pi : X \cup \Omega \rightarrow M_C$ is the natural projection. It follows that $\pi(\beta \cap X)$ must accumulate somewhere in $\mathrm{core}(M) \cap \mathrm{noncusp}_\epsilon(M) \subseteq M$. Thus y is a conical limit point.

Proof of F1 $\Rightarrow$ F5 : Suppose Γ is F1. Proposition 5.5.2 tells us that there is a bound on the orders of finite subgroups of Γ. Suppose $\epsilon \in (0, \epsilon(n, \kappa))$ and $\eta > 0$.

Since Γ is F4, we know that $\mathrm{core}(M) \cap \mathrm{noncusp}_\epsilon(M)$ is compact. Thus $N_\eta(\mathrm{core}(M) \cap \mathrm{noncusp}_\epsilon(M))$ is compact. We thus need to show that $N_\eta(\mathrm{core}(M) \cap \mathrm{cusp}_\epsilon(M))$ has finite volume.

By Corollary 6.5, we know that $\mathrm{cusp}_\epsilon(M)$ consists of finitely many Margulis cusps. Each Margulis cusp has the form $T_\epsilon(G)/G$, where $G = \mathrm{stab}_\Gamma p$ is a maximal parabolic subgroup of Γ, with fixed point p. By Lemma 6.4, $(\Lambda \setminus \{p\})/G$ is compact, and so by Proposition 4.14, $N_\eta(H \cap T_\epsilon(G))/G$ has finite volume, where $H = \text{hull}(\Lambda)$. Summing over the set of Margulis cusps, we conclude that $N_\eta(\text{core}(M) \cap \text{cusp}_\epsilon(M))$ has finite volume. $\diamondsuit$

Proof of F5 $\Rightarrow$ F4 : Suppose that Γ is F5. Thus, for some $\eta > 0$, $N_\eta(\text{core}(M))$ has finite volume, V_0 say. Also, there is a bound, m, on the orders of finite subgroups of Γ. Given $\epsilon \in (0, \epsilon(n,\kappa))$, we want to show that $\text{core}(M) \cap \text{thick}_\epsilon(M)$ is compact. Let $\delta = \min(\eta, \epsilon/2)$.

Let $\pi : X \longrightarrow M$ be the projection. Suppose $a \in \text{thick}_\epsilon(M)$. Choose $x \in X$ with $\pi x = a$. Thus $x \in T_\epsilon(\Gamma)$, and so $\Gamma_\epsilon(x)$ is finite, of order at most m. Since $\delta \leq \epsilon/2$, it follows that $N_\delta(x)$ meets at most m images of itself under Γ. Applying Proposition 1.1.12, we see that $\pi N_\delta(x) \subseteq M$ has volume at least $V(\delta, n)/m$. Now, $\pi N_\delta(x)$ is the uniform δ-ball, $N_\delta(a)$ about a in M. We thus have a lower bound on the volumes of δ-balls in M centred on $\text{thick}_\epsilon(M)$.

Now choose a maximal subset $A \subseteq \text{core}(M) \cap \text{thick}_\epsilon(M)$, such that the balls $\{N_\delta(a) | a \in A\}$ are disjoint in M. Since $\delta \leq \eta$ and $N_\eta(\text{core}(M))$ has finite volume, the set A must be finite. (It has at most $mV_0/V(\delta, m)$ elements.) Since A is maximal, we have $\text{core}(M) \cap \text{thick}_\epsilon(M) \subseteq N_{2\delta}(A)$. Thus $\text{core}(M) \cap \text{thick}_\epsilon(M)$ is compact. $\diamondsuit$

This concludes the proofs of equivalence (Theorem 6.1).

To finish off, we give the following result, which is a generalisation of a well-known result in case of manifolds. We say that an orbifold is *topologically finite* if it is orbifold-homeomorphic to the interior of a compact orbifold with boundary (Section 3.4).

Proposition 6.6 : *Suppose* $\Gamma \subseteq \text{Isom}\, X$ *is discrete. If* $M = X/\Gamma$ *has finite volume, then it is topologically finite as an orbifold.*

Proof : By Proposition 5.4.2, there is a bound on the orders of finite subgroups of Γ. Thus Γ is F5, and so F1. We can thus write $M_C(\Gamma) = K \cup \bigcup_{i=1}^k E_i$, where each E_i is standard cusp region, and K is compact subset which we can take to be an orbifold with boundary.

Now, each E_i has the form C_i/G_i, where $G_i \subseteq \Gamma$ is a maximal parabolic subgroup, with fixed point p_i, and C_i is a closed convex G_i-invariant subset of $X_C \setminus \{p_i\}$. Let v be the retraction $[x \mapsto x - \infty]$ of $X_C \setminus \{p_i\}$ to $X_I \setminus \{p_i\}$. Since E_i has finite volume, it is clear that $v(\partial C_i) = X_I \setminus \{p_i\}$, where ∂C_i is the boundary of C_i in $X_C \setminus \{p_i\}$. Since E_i is a neighbourhood of the end of $M_C(G_i)$, we have that $\partial C_i/G_i$ is compact. Since v is G_i-equivariant, we have that $(X_I \setminus \{p_i\})/G_i = v(\partial C_i)/G_i$ is compact. Now, E_i is topologically (as an orbifold) a product of $(X_I \setminus \{p_i\})/G_i$ and a half-open interval, and thus topologically finite. $\diamondsuit$

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