Group actions on trees and dendrons

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0. Introduction.

The main objective of this paper will be to describe how certain results concerning isometric actions on **R**-trees can be generalised to a wider class of treelike structures. This enables us to analyse convergence actions on dendrons, and hence on more general continua, via the results of [Bo1]. The main applications we have in mind are to boundaries of hyperbolic and relatively hyperbolic groups. For example, in the case of a one-ended hyperbolic group, we shall see that the existence of a global cut point gives a splitting of the group over a two ended subgroup (Corollary 5). Pursuing these ideas further, one can prove the non-existence of global cut points for strongly accessible hyperbolic groups [Bo5], and indeed for hyperbolic groups in general [Sw]. Reset in a general dynamical context, one can use these methods to show, for example, that every global cut point in the limit set of a geometrically finite group is a parabolic fixed point [Bo6]. These results have implications for the algebraic structure of such groups. Some of these are discussed in [Bo3].

The methods of this paper are essentially elementary and self-contained (except of course for the references to what have now become standard results in the theory of **R**-tree actions). A different approach has been described by Levitt [L2], which gives a generalisation of the central result (Theorem 0.1) to non-nesting actions on **R**-trees. Levitt's work makes use of ideas from the theory of codimension-1 foliations (in particular the result of Sacksteder [St]). This seems a more natural result, though our version of Theorem 0.1 suffices for the applications we have in mind at present. There are many further questions concerning the relationship between actions on **R**-trees and dendrons, which seem worthy of further investigation. Some of these are described in [Bo2].

The notion of an **R**-tree was formulated in [MoS]. It can be given a number of equivalent definitions. For example, it can be defined simply as a path-metric space which contains no embedded circle. For more discussion of **R**-trees, see, for example, [Sh,P2]. There is a powerful machinery for studying isometric group actions on **R**-trees, due to Rips, and generalised by Bestvina and Feighn [BeF] and Gaboriau, Levitt and Paulin [GaLP1]. In most applications, such actions arise from some kind of degeneration of a hyperbolic metric, so that the **R**-trees obtained come already equipped with a natural metric. However, there are potential applications, as we shall describe, where one obtains, a-priori, only an action by homeomorphism with certain dynamical properties.

To deal with this situation, one might attempt either to generalise the Rips machinery to cover such cases, or one might attempt to construct a genuine **R**-tree starting with a more general action. It is the latter course that is followed in this paper (and that of Levitt

[L2]).

Let's begin by giving more precise definitions of the objects we are working with:

Definition : A *real tree*, T, is a hausdorff topological space which is uniquely arc connected, and locally arc connected.

More precisely, if $x, y \in T$, then there is a unique interval, $[x, y]$ connecting x to y. (Here "interval" means a subset homeomorphic to a closed real interval.) Moreover, given any neighbourhood, U, of x, there is another neighbourhood, V, of x, such that if $y \in V$, then $[x, y] \subseteq U$. We shall define a *dendron* to be a compact real tree.

We shall say that a metric, d, on T, is *monotone* if given $x, y, z \in T$ with $z \in [x, y]$, then $d(x, z) \leq d(x, y)$. We shall say that d is *convex*, if for all such x, y, z, we have $d(x, y) = d(x, z) + d(z, y)$. Clearly, any convex metric is monotone.

An equivalent definition of an $\mathbf{R}$-tree is thus a real tree, together with a continuous convex metric. (We specify continuous, with respect to the real tree topology, since we shall later be obliged to consider discontinuous metrics.) We also have the more general notion:

Definition : A *monotone tree* consists of a real tree, together with a continuous monotone metric.

A result of Mayer and Oversteegen [MaO] tells us that that any metrisable real tree admits the structure of an $\mathbf{R}$-tree (i.e. a continuous convex metric). However the $\mathbf{R}$-tree metric arising from this construction is not in any way canonical, and it's not clear how this procedure could be generalised to give $\mathbf{R}$ tree metrics which are invariant under some group action — even starting from an invariant metric. Here we shall use a different construction, which gives a somewhat weaker result, but which is equivariant.

Suppose Γ acts by homeomorphism on the real tree T. Given $x \in T$, write $\Gamma(x) = \Gamma_T(x) = \{g \in \Gamma \mid gx = x\}$. We say the action is *parabolic* if there is a point $x \in T$ with $\Gamma(x) = \Gamma$. (This is frequently termed "trivial" in the literature. We use the term "parabolic" for consistency with the terminology of convergence actions.) If $x, y \in T$ are distinct, we write $\Gamma(A)$ (or $\Gamma_T(A)$) for the pointwise stabiliser of the interval $A = [x, y]$. Thus, $\Gamma(A) = \Gamma(x) \cap \Gamma(y)$. A subgroup of the form $\Gamma(A)$ is referred to as an *edge stabiliser*, or if we need to be more specific, a *T-edge stabiliser*. A sequence of subgroups $(G_i)_{i \in \mathbf{N}}$ is referred to as a *chain of T-edge stabilisers* if there is a sequence of non-trivial intervals, $(A_i)_{i \in \mathbf{N}}$, such that $G_i = \Gamma(A_i)$ and $A_{i+1} \subseteq A_i$ for all i, and such that $\bigcap_{i \in \mathbf{N}} A_i$ consists of a single point. Thus, $(G_i)_{i \in \mathbf{N}}$ is an ascending chain of subgroups. The action of Γ in T is said to be *stable* if every chain of edge stabilisers is eventually constant. The results concerning actions on $\mathbf{R}$-trees already alluded to, refer to non-parabolic stable isometric actions, usually with some conditions imposed on the types of groups that can arise as edge stabilisers.

Here we shall show that much of this theory can be generalised, as least as far as isometric actions on monotone trees. Specifically, we show:

Theorem 0.1 : *Suppose Γ is a finitely presented group, which admits a non-parabolic isometric action on a monotone tree, T. Then Γ also admits a non-parabolic isometric action on an $\mathbf{R}$-tree, Σ, such that each Σ-edge stabiliser is contained in a T-edge stabiliser. Moreover, if $(G_i)_{i\in\mathbf{N}}$ is a chain of Σ-edge stabilisers, then there is a chain of T-edge stabilisers, $(H_i)_{i\in\mathbf{N}}$, such that $G_i \leq H_i$ for all $i \in \mathbf{N}$.*

In many cases of interest, the stability of the action on T will imply the stability of the action on Σ. Suppose, for example, that each subgroup of Γ which fixes an interval of T (i.e. any subgroup of any T-edge stabiliser) is finitely generated. Then, if the action of Γ on T is stable, then so is the action on Σ. This additional assumption is likely to hold for most plausible applications, where the T-edge stabilisers are constrained to be reasonably nice subgroups. Of particular interest here is the case where all T-edge stabilisers are assumed to finite.

Now, a consequence of the results of [BeF], is that a finitely presented group which acts isometrically, stably and non-parabolically on an $\mathbf{R}$-tree with finite edge stabilises must be virtually abelian or split over a finite or two-ended subgroup. (Note that "two-ended" is the same as "virtually cyclic".) Applying Theorem 0.1, we thus obtain the same result for monotone trees:

Corollary 0.2 : *Suppose Γ is a finitely presented group, which acts isometrically, stably, non-parabolically, and with finite edge-stabilisers, on a monotone tree. Then, either Γ is virtually abelian, or it splits over a finite or two-ended subgroup.*

In the case where Γ is virtually abelian, it must fix some subtree of T, homeomorphic to the real line. This case is of no particular interest to us here.

A principal application of this result concerns discrete convergence actions on non-trivial dendrons. (Recall that a "dendron" is a compact real tree. The term "non-trivial" simply means that it is not a point. The notion of a "discrete convergence" action was defined in [GeM]. For further discussion, see [T] or [Bo4]. Putting results of [Bo2] together with Corollary 0.2, we obtain:

Theorem 0.3 : *Suppose Γ is a finitely presented infinite group, such that any ascending chain of finite subgroups eventually stabilises. Suppose that Γ admits a discrete convergence action on a dendron. Then, Γ splits over a finite or two-ended subgroup.*

(It's not clear that the assumption on chains of finite subgroups is necessary. It is also probable that the result holds for finitely generated groups.)

This result in turn has applications to the boundaries of certain hyperbolic groups. Suppose that Γ is a one-ended word hyperbolic group (in the sense of Gromov [Gr]). Then, Γ is finitely presented, has a bound on the orders of finite subgroups, and does not split over any finite subgroup. Moreover the boundary, $\partial\Gamma$, is connected and hence a continuum — a connected, compact, hausdorff topological space. (See [GhH]). It was conjectured in [BeM] that such a case, $\partial\Gamma$ must be locally connected. They showed that if $\partial\Gamma$ is not locally connected, then it must contain a global cut point. (A converse was obtained in

[Bo3].) Moreover, it was shown in [Bo2] that if $\partial\Gamma$ contains a global cut, then it admits a Γ-invariant quotient which is a non-trivial (separable and hence metrisable) dendron. Now, Γ acts as a convergence group on this dendron, and so we obtain:

Corollary 0.4 : *If Γ is a one-ended hyperbolic group with a global cut point in its boundary, then Γ splits over a two-ended subgroup.*

In fact, Swarup [Sw] showed how one can adapt these ideas to prove the cut point conjecture in general. This result can be generalised to a dynamical context [Bo6], which also has applications to limit sets of geometrically finite kleinian groups (and groups acting on pinched hadamard manifolds). One can obtain most of the results of the present paper without explicit use of these refinements.

Now, if we assume that $\partial\Gamma$ has no global cut point, then it's locally connected (by [BeM]) so we can bring the results of [Bo3] into play. In particular, we obtain:

Theorem 0.5 : *A one-ended non-fuchsian hyperbolic group splits over a two-ended subgroup if and only if its boundary contains a cut point.*

Here "cut point" should be interpreted as either local or global. A *local cut point* may be defined as a point $x \in \partial\Gamma$ such that $\partial\Gamma \setminus \{x\}$ has more than one end. A *fuchsian group* is one which contains a finite index subgroup which is the fundamental group of a closed surface. (The case of fuchsian groups can be explicitly described, see for example [Bo3].)

The importance of splitting over over two-ended subgroups is well-known, see, for example [P1,Se]. The set of such splittings determines the structure of the outer automorphism group. In particular, if a one-ended hyperbolic group has infinite outer automorphism group, then it splits over a two-ended subgroup. We say that a one-ended hyperbolic group is *strongly rigid* if there is no such splitting. We see that, modulo fuchsian groups, strong rigidity of a one-ended hyperbolic group can be recognised from the topology of the boundary. In particular, we see:

Corollary 0.6 : *For non-fuchsian one-ended hyperbolic groups, strong rigidity is a geometric property.*

Here "geometric" means "quasiisometry invariant". We refer to [Bo3] for more details.

We can give a refinement of Theorem 0.1 as described in Section 7. Thus, if $H_1, \ldots, H_n$ are finitely presented subgroups of Γ which act parabolically on T, then they can be assumed to act parabolically also on Σ. This leads to refinements of Corollary 0.2 and Theorem 0.3 which find application in [Bo6].

We make a few observations about Theorem 0.1, and generalisations. First we need another definition:

Definition : Suppose that g is a homeomorphism of a real tree, T. We say that g is *non-nesting* if, given any compact interval, $A \subseteq T$ such that either $A \subseteq gA$ or $gA \subseteq A$, then $gA = A$.

We say that a group of homeomorphisms is *non-nesting* if every element is.

Non-nesting homeomorphisms have many of the features of isometries of **R**-trees, for example, they can be classified into types, as described later. (Note that any isometry of a monotone tree is necessarily non-nesting.) Levitt [L2] has generalised Theorem 0.1 to the case of non-nesting actions on a real tree. This generalisation is remarkable in that the dynamics of the original action might not resemble that of an isometric action. For example, the tree might contain wandering intervals, in which case action would certainly not be topologically conjugate to an isometric action. Note that, in proving Theorem 0.3, this result would enable us to bypass the construction of the monotone metric (see Section 6).

We remark that Levitt does not obtain explicitly our result about chains of edge stabilisers, and hence the preservation of stability in the case of finitely generated edge stabilisers. It's possible that a careful analysis of Levitt's construction would yield this. In any case, for the applications which interest us here (where edge stabilisers are finite, and there is no infinite torsion subgroup) stability is automatic.

Levitt's construction proceeds by translating the problem into the language of one-dimensional pseudogroups, and appealing to some of the results developed in the theory of codimension-one foliations in this context. In particular a theorem of Sacksteder [Sa] gives methods for finding invariant Borel measures on pseudogroups. One then constructs an **R**-tree from such a measure. Here, we shall phrase everything in terms of pseudometrics rather than measures.

We suspect that Theorem 0.1 remains true if "finitely presented" is replaced by "finitely generated". Perhaps the argument presented here can be made to work in that generality, using "normal covers", as described, for example in [LP], though we have not worked out the details.

One might also ask when an action of a (finitely presented) group on a real tree is conjugate to an action on an **R**-tree. Here "conjugate" might mean "topologically conjugate", or probably more naturally "conjugate under a pretree isomorphism" (see below). One certainly needs stronger constraints on the dynamics than just the non-nesting hypothesis. Perhaps the existence of an invariant monotone metric would be sufficient. In particular, one might ask if this is true of the monotone tree arising from a convergence action on a metrisable dendron, as constructed in [Bo2]. On the other hand, going in the opposite direction, from an **R**-tree action to an action on a dendron, can be described fairly easily in terms of a compactification process. Of course one needs a constraint on the **R**-tree action for it to give rise to a convergence action. This is discussed in [Bo2]. In general, the relationship between **R**-trees and dendrons seems subtle, and worthy of further investigation.

We remark that the topology on the real tree is not directly relevant to anything we do. All we really need is the relation of "betweenness", which we can regard as determining a preferred class of subsets of the set T, namely the closed intervals. One can write down explicit axioms for such a structure, which in [Bo2] is referred to as a "real pretree" (see also [W] and [AN]). This structure is weaker than that of a topology, in that different real tree topologies might give rise to the same pretree structure. It would probably be more natural to phrase everything in terms of isomorphisms of real pretrees, rather than homeomorphisms of real trees. However the extra generality is spurious, since it can be shown that every real pretree admits a certain canonical topology as a real tree (one which admits a hausdorff compactification).

The principal tools we shall use are foliations on 2-complexes. We shall use the term “track complex” for such an object, since our formulation is analogous to the tracks of Dunwoody [D]. However, they are essentially the same as foliated 2-complexes discussed in [LP]. A track complex (or foliated 2-complex as in [LP]) essentially consists of a locally finite 2-dimensional simplicial complex, together with a partition into leaves, in such a way that the set of leaves meets any given simplex in one of specific number of patterns. They are closely related to the band complexes of [BeF], and everything we do could be rephrased in these terms, or indeed in terms of pseudogroups. However, since we shall not be applying the Rips machinery directly, we prefer to use this intuitively simpler picture.

The idea of the proof of Theorem 0.1 is roughly as follows. Suppose Γ is a finitely presented group which acts on a real tree T. By taking a “resolution” of such an action (cf. [BeF]), we construct a track complex based on a finite 2-dimensional simplicial complex, K, with $\Gamma \equiv \pi_1(K)$. An invariant continuous monotone metric on K induces a kind of “transverse” metric on K. By a compactness argument, we use this to construct a transverse convex pseudometric. This gives rise to a genuine path-pseudometric on the universal cover, $\tilde{K}$, which we proceed to show is “0-hyperbolic”. The $\mathbf{R}$-tree, Σ, thus arises as the hausdorffification of $\tilde{K}$. We finally have to verify the statements about edge stabilisers and non-parabolicity of the action.

I am indebted to Mladen Bestvina and Gilbert Levitt who suggested to me that results of this nature should be possible. It was Gilbert Levitt, who put me on to the article by Sacksteder, which partly inspired the arguments of this paper, even if I don’t apply these results directly. Most of this paper was prepared while visiting the Universität Zürich, at the invitation of Viktor Schroeder. It was revised somewhat while I was visiting the University of Melbourne at the invitation of Walter Neumann and Craig Hodgson. I would like to thank both institutions for their hospitality.

1. Pseudometrics.

In this section, we make some elementary observations about pseudometrics.

Let M be a hausdorff topological space. A *pseudometric* on M is a function $d : M \times M \longrightarrow [0, \infty)$ satisfying $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$. It is a *metric* if $d(x, y) = 0$ implies $x = y$. Note that we do not assume that d is continuous, unless explicitly stated.

Suppose d is a continuous pseudometric on M. We may define the rectifiable length, $\mathrm{length}_d\,\beta \in [0, \infty]$ of any continuous path, β, in M, in the usual way. We say that d is a *path-pseudometric* if given any $x, y \in M$, and any $\epsilon > 0$, there is a path β in M, joining x to y with $\mathrm{length}_d\,\beta \leq d(x, y) + \epsilon$. Note that path-pseudometrics are assumed to be continuous. By a *geodesic* in M joining x to y, we mean a path, β, with $\mathrm{length}_d\,\beta = d(x, y)$.

Given a continuous pseudometric space, (M,d), we define an equivalence relation, $\sim$, on M by $x \sim y$ if $d(x,y)=0$. Thus, d descends to a continuous metric on the quotient, $M/{\sim}$, which we shall also denote by d. We call $M/{\sim}$ the *hausdorffification* of M. Note that if d is a path-pseudometric on M, then the metric on the quotient is a path-metric.

We shall say that a path-pseudometric is *0-hyperbolic* if, given any $x,y,z,w \in M$, the largest two of the three quantities, $d(x,y)+d(z,w)$, $d(x,z)+d(y,w)$ and $d(x,w)+d(y,z)$ are equal. Note that an $\mathbf{R}$-tree can be defined as a 0-hyperbolic path-metric space. (The equivalence of this with the usual definition is shown, for example, in [Bo1], where the formulation of hyperbolicity we have used is referred to as "H1". In fact, as noted in [Sho], any 0-hyperbolic connected metric space is an $\mathbf{R}$-tree.) In particular, we see that in 0-hyperbolic path-metric space, every pair of points can be joined by a geodesic. This need not be true in a 0-hyperbolic path-pseudometric space. Note that the hausdorffification of a 0-hyperbolic path-pseudometric space is an $\mathbf{R}$-tree.

We shall be particularly interested in pseudometrics on intervals, or disjoint unions of intervals. Let $I=[0,1]$ be the unit interval in $\mathbf{R}$. We say that a pseudometric, d, on I is *monotone*, in whenever $x<y<z$, we have $d(x,y) \le d(x,z)$ and $d(y,z) \le d(x,y)$. We say that d is *convex* if whenever $x<y<z$, we have $d(x,z)=d(x,y)+d(y,z)$. Clearly a convex pseudometric is monotone. In any case, let $l(I,d)=d(0,1)$. Clearly, a monotone pseudometric, d, is identically 0 if and only if $l(I,d)=0$.

Let $V(I)=(I\times\{-,+\})\setminus\{(0,-),(1,+)\}$. We think of $V(I)$ as the set of unit tangent vectors to I. Thus, $(x,+)$ is the vector based at x pointing towards 1. Suppose that d is a monotone pseudometric on I. For $x\in[0,1)$ we define $\mu_d(x,+)=\inf\{d(x,y) \mid y>x\}$. We similarly define $\mu_d(x,-)$ for $x\in(0,1]$. This gives a map $\mu_d : V(I) \longrightarrow [0,\infty)$. We refer to $v\in V(I)$ as an *atom* if $\mu_d(v)>0$. Thus, a monotone pseudometric is continuous if and only if there are no atoms.

Suppose that d is convex pseudometric. If $x<y\in I$, then we see easily that $\mu_d(x,+)+\mu_d(y,-)\le d(x,y)$. Thus, if $0=x_0<x_1<\cdots<x_m=1$, we see that $\sum_{i=0}^m(\mu_d(x_i,-)+\mu_d(x_i,+)) \le \sum_{i=1}^m d(x_i,x_{i-1}) = d(0,1) = l(I,d)$ (with the convention that $\mu_d(0,-)=\mu_d(1,+)=0$). Thus, $\sum_{v\in V(I)}\mu_d(v) \le l(I,d) < \infty$. (In particular, there are at most countably many atoms.)

We may apply exactly the same discussion to a finite set of disjoint intervals, $J = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n$. Here the pseudometric is assumed to be defined on $P(J)=\bigsqcup_{i=1}^n I_i^2$. We say that d is monotone (convex) if $d|I_i^2$ is monotone (convex) on I_i for each i. Let $l(J,d)=\sum_{i=1}^n l(I_i,d|I_i^2)$. Note that d is identically zero of and only if $l(J,d)=0$. We say that d is *normalised* if $l(J,d)=1$.

Let $\mathcal{M}(J)$ be the set of normalised monotone pseudometrics on J. We can embed $\mathcal{M}(J)$ in the Tychonoff cube $[0,1]^{P(J)}$, by taking the (x,y)-coordinate of d to be $d(x,y)$. Now each of the relations defining a monotone pseudometric is closed, and so $\mathcal{M}(J)$ is a closed subset of $[0,1]^{P(J)}$. Thus, with the subspace topology, $\mathcal{M}(J)$ is compact.

We define $V(J)=\bigsqcup_{i=1}^n V(I_i)$. Given a monotone pseudometric d, we may define $\mu_d : V(J) \longrightarrow [0,\infty)$ as before. As in the case of a single interval, we note

Lemma 1.1 : *If d is a monotone pseudometric on J, then $\sum_{v\in V(J)}\mu_d(v)<\infty$.* $\diamondsuit$

We finally make a few remarks about convex pseudometrics. Suppose d is a non-zero continuous convex pseudometric on a closed interval I. Let $q : I \longrightarrow I'$ be the quotient map to the hausdorffification I'. Thus, I' is also an interval, and d is a continuous convex metric on I'. Thus, (I', d) is isometric to a closed real interval. The pullback of Lebesgue measure on I' gives us an atomless regular Borel measure on I. The measure of any subinterval of I is thus the same as its d-length. We write $\mathrm{supp}\, d$ for the support of this measure. Now, q collapses each of an (at most) countable set of disjoint closed intervals to a point, and is injective on the complement of the union of these intervals. These intervals are precisely the closures of the components of the complement of $\mathrm{supp}\, d$.

2. Foliations on 2-complexes.

In this section, we discuss foliations on 2-complexes. Variations of these ideas have been studied by several authors in relation to **R**-trees. The theory has been recently developed formally by Levitt and Paulin in [LP]. Here we shall use the term "track complex" for a foliated 2-complex. Our formulation differs slightly from that given in [LP], though for all practical purposes, it amounts to the same thing.

We shall be interested in certain "transverse" structures to these foliations, which we phrase in terms of pseudometrics. Thus, an atomless transverse regular Borel measure, as described in [LP] translates to a continuous convex pseudometric. We shall not want to assume that the measure has full support, as has typically been done elsewhere. However, the relevant arguments would seem to generalise without problems.

One of the main aims of this section is to describe how transverse pseudometrics give rise to **R**-trees. In the context of pseudogroups (or "systems of isometries"), this has been examined by several authors, see, in particular, [L1,GaLP2]. It is also described for foliated 2-complexes in [LP], refering to this earlier work. For completeness, we include an alternative, direct argument here.

We begin with some definitions.

Let M be a connected locally finite simplicial 2-complex. (In this paper, the term "complex" is always taken to mean "simplicial complex".) We refer to the 1-simplices of M as *edges* of M, and 0-simplices as *vertices*. Let $\mathcal{F}$ be a partition of M is a disjoint connected subsets. Given a simplex, σ, of M, a *stratum* of σ is defined to be a connected component of the intersection of σ with an element of $\mathcal{F}$.

We say that $\mathcal{F}$ is a *track foliation* of M if the following hold:

(1) If σ is an edge of M, then the set of strata in σ either consists of just one element, namely σ itself, or else consists of the set of all points of σ.

(2) If σ is a 2-simplex of M, then each stratum of σ is topologically a point, a closed interval, or a closed disc. Moreover the set of strata forms, up to homeomorphism relative to the set of vertices, one of the four patterns, (A)–(D), given by Figure 1 (see below).

(3) Each stratum of a 2-simplex intersects each face of that simplex in a stratum of that face.

(If we imagine a 2-simplex, σ, as an equilateral triangle, then up to homeomorphisms, the patterns can be described as follows. In picture (A), there is a single stratum, namely σ itself. In picture (B), the strata consist of all lines parallel to one edge, together with the vertex opposite that edge. In picture (C), the strata consist of all lines perpendicular to one edge, together with the two endpoints of that edge. In picture (D), one statum is the triangular convex hull of the midpoints of the three edges. The remaining strata are lines parallel to one of the edges of this hull, together with all three vertices of the simplex.)

We refer to an edge of M as *essential* if each stratum is a point. If σ is a 2-simplex, a *singular* stratum, is either one which contains a vertex, or one which intersects all three 1-dimensional faces. (Thus all strata which are topologically discs are singular.) A point on an essential edge is *singular* if it is a vertex, or if it lies in a singular stratum of an incident face. Thus, the set of singular points in each edge is finite.

Without loss of generality, we can assume that each stratum of a 2-simplex is a point, a euclidean arc, or bounded by euclidean arcs.

Definition : A *track complex*, $(M, \mathcal{F})$, consists of a locally finite simplicial complex, M, together with a track foliation, $\mathcal{F}$, on M. An element of $\mathcal{F}$ is referred to as a *leaf* of the foliation. A leaf is *singular* if it contains a singular stratum.

Note that a leaf itself has the structure of a locally finite 2-complex. A non-singular leaf is a locally finite 1-complex.

Note that we can recover a track foliation from the set of strata in each leaf as follows. Define a relation on M, by saying that two points are related if they lie in the same stratum of some simplex. Take the equivalence relation on M generated by this relation. The equivalence classes are precisely the the leaves of the foliation. Note that for this to work, we need only that the set of strata satisfies axioms (1)–(3) above. We can thus define a track foliation by describing the set of strata, and so it can be thought of as really a "local" structure.

Now any subcomplex of M inherits a structure as a track complex. More generally, if L is a connected locally finite 2-complex, and $f : L \longrightarrow M$ is a simplicial map, we can pull back the track foliation to L. (Note that if f collapses a 2-simplex, σ, to an edge, then the pattern induced on σ will be of type (B) or type (A), depending in whether of not the edge is essential.) Clearly, f maps each leaf in L into a leaf in M.

Another point to note is that we can find arbitrarily fine simplicial subdivisions of $\mathcal{M}$ which admit track foliations with precisely the same set of leaves. We refer to such a subdivision as a *foliated subdivision.* The Simplicial Approximation Theorem now tells us that, up to small homotopy, every continuous map of simplicial 2-complex into M can be assumed to be simplicial.

The definition of a foliation given in [LP] is essenially the same, except that they only allow for pictures (B) and (C). Since our complex is locally finite, we can clearly eliminate picture (D) after subdivision. (This may be more natural for our purposes, though one might imagine contexts in which it might be useful to allow it, for example to study the space of foliations on a fixed complex.) One could also collapse down the union of all the simplices of type (A) without doing much damage, though it is technically convenient to allow the possibility here.

Here are some more definitions:

Definition : A track complex, M, is *simple* if each leaf intersects each edge in a single stratum.

Definition : A path β in M is *piecewise straight* if it is the finite union of subpaths, each of which maps injectively either into an edge or a stratum of a 2-simplex.
A piecewise straight path, β, is *taut* if the preimage of every leaf in the domain of β is connected. (In other words, β never leaves and re-enters the same leaf.)

It's easy to see that every path can be homotoped, relative to its endpoints, to a piecewise straight path (though not necessarily to one which is taut — even for simple track complexes). Note that every piecewise straight path can be assumed to lie in the 1-skeleton of some foliated subdivision of M. Finally, note that any edge of a simple track complex can be viewed as a taut path.

We shall be interested in certain kinds of "transverse" pseudometrics to a track foliation. Let $(M, \mathcal{F})$ be a track complex. Let $J(M)$ be the (abstract) disjoint union of all the essential edges of M.

Definition : An *edge pseudometric*, d, on $(M, \mathcal{F})$ is a pseudometric on $J(M)$ satisfying the following. Suppose σ is a 2-simplex of M, and e, e' are essential 1-dimensional faces of σ. Suppose $x, y \in e$ and $x', y' \in e'$ such that x and x' lie in the same stratum of σ, and y and y' lie in the same stratum of σ. Then, $d(x, y) = d(x', y')$.

Definition : A *transverse pseudometric*, d, on M, is a pseudometric on M such that if $x \in M$, and if $y, y' \in M$ lie in the same leaf, then $d(x, y) = d(x, y')$.

Note that (for a transverse pseudometric) it follows, in addition, that if x, x' lie in the same leaf, then $d(x, y) = d(x', y')$ and $d(x, x') = 0$.

Clearly, a transverse pseudometric gives rise immediately to an edge pseudometric. Note that we can refer to an edge pseudometric, and hence a transverse pseudometric, as being "monotone" or "convex", as described in Section 1. Also, a transverse pseudometric is continuous (on M) if and only if the corresponding edge pseudometric is continuous (on $J(M)$). By a *transverse path-pseudometric*, we simply mean a transverse pseudometric which also a path-pseudometric (and hence continuous).

Now, it is easily seen that any rectifiable path can be homotoped to a piecewise smooth path, while, at worse, increasing its length by and arbitrarily small amount. We deduce:

Lemma 2.1 : *Suppose ρ is a transverse path-pseudometric on M. Then, given any $x, y \in M$ and $\epsilon > 0$, we can join x to y by a piecewise straight path of ρ-rectifiable length at most $\rho(x, y) + \epsilon$.* ◇

We omit a detailed proof, since in the cases of interest, this property follows directly from the construction.

Suppose now that d is a continuous convex edge pseudometric on M. Given a piecewise straight path β in M, we define its "d-length", $\text{length}_d\,\beta$, to be the sum of the lenghs of each component of β which lie in an essential edge. (Since d is convex, the length of such a component is just the distance between its endpoints. Note that there is no clash of notation if d happens to be the restriction of a transverse pseudometric, since the remaining parts of β is automatically have zero length.) Given $x, y \in M$, we define $\rho(x, y)$ to be $\inf\{\text{length}_d \beta\}$, as β ranges over all piecewise straight paths joining x to y.

Lemma 2.2 : *ρ is a transverse path-pseudometric on M.*

Proof : The fact that ρ is a transverse pseudometric is essentially trivial. Also, since d is convex, it's clear that $\rho \leq d$ on each edge. Since d is assumed to be continuous, it follows that ρ is continuous on each edge, and hence on M.

Suppose β is a piecewise smooth path. Now the ρ-length of each component of β in a stratum of a 2-simplex is 0 (since ρ is identically zero on each leaf). Moreover, since $\rho \leq d$, we see that the ρ-rectifiable length is at most the d-length on each essential edge. Thus, $\text{length}_\rho \beta \leq \text{length}_d \beta$. But now, by definition, if $x, y \in M$ and $\epsilon > 0$, we can join x to y by a piecewise straight path β, with $\text{length}_d \beta \leq \epsilon$. This shows that ρ is a path-pseudometric. ◊

Note that we have, in fact, verified the conclusion of Lemma 2.1, for a path-pseudometric arising in this way. We refer to ρ as the "induced transverse path-pseudometric".

Given any track complex, $(M, \mathcal{F})$, we can view the leaf space $\mathcal{F}$ as a topological space with the quotient topology from M. Viewed in this way, we shall write it as $F(M)$. Note that the transverse pseudometric gives rise to a pseudometric on $F(M)$, which we shall also denote by ρ. In general, $F(M)$ will not be hausdorff. However, in certain cases, we get a nice space. Of special interest is the following situation.

Suppose that M is finite, and $(M, \mathcal{F})$ is simple. In this case, each leaf is compact, and there are only finitely many singular leaves. If the remove all the singular leaves, each component of the complement is topologically of the form $G \times \mathbf{R}$, where G is a finite 1-complex. Moreover, the leaves have the form $G \times \{x\}$ for $x \in \mathbf{R}$. We conclude that, in this case, $F(M)$ is a finite 1-complex. We shall write $p : M \longrightarrow F(M)$ for the quotient map. Of particular interest is the case where M is topologically a disc:

Lemma 2.3 : *Suppose $(D, \mathcal{F})$ is a simple track complex on the topological disc, D. Then $F(D)$ is a finite tree.*

Proof : Note that any non-singular leaf in D is an arc connecting two points of the boundary, ∂D. Suppose e is an edge of $F(D)$ with endpoints x and y, and let z be any interior point. Now, $p^{-1}z$ is an arc in D which must locally, and hence globally, separate any point in $p^{-1}x$ from any point in $p^{-1}y$. Thus, z separates x from y in $F(M)$. Thus, e cannot lie in any embedded circle in $F(M)$. ◊

Corollary 2.4 : *Suppose that ρ is a transverse path-pseudometric on D. Then ρ is 0-hyperbolic.*

Proof : Now ρ induces a pseudometric on $F(D)$, which is also a path-pseudometric. It's easily verified that any path-pseudometric on a (finite) tree must be 0-hyperbolic. It follows that ρ is also 0-hyperbolic on D. $\diamondsuit$

We now move on to consider a more general situation. We aim to show:

Proposition 2.5 : *Suppose that $(M, \mathcal{F})$ is a simply connected simple track complex, and that d is a continuous convex edge pseudometric on M. Let ρ be the induced transverse path-pseudometric. Then ρ is 0-hyperbolic, and induces the original metric, d, on each edge of M.*

¿From this we see immediately that:

Corollary 2.6 : *The hausdorffification of (M, ρ) is an* $\mathbf{R}$*-tree.* $\diamondsuit$

A similar result, in the context of pseudogroups, is given in [GiS]. This is also discussed in [L1]. A proof along similar lines to the one we give here can be found in [P2].

We begin by showing:

Lemma 2.7 : *Suppose β is a taut piecewise straight path in M connecting points x and y. Then* $\mathrm{length}_d\,\beta = \rho(x, y)$.

Proof : By the definition of ρ, we have $\rho(x, y) \leq \mathrm{length}_d\,\beta$. Suppose that $\rho(x, y) < \mathrm{length}_d\,\beta$. Let $\epsilon = \mathrm{length}_d\,\beta - \rho(x, y)$. By the definition of ρ, there is a piecewise straight path, β' joining y to x with $\mathrm{length}_d\,\beta' < \rho(x, y) + \epsilon = \mathrm{length}_d\,\beta$. Now, $\gamma = \beta \cup \beta'$ is a loop in M. After subdivision, we can suppose that γ lies in the 1-skeleton of M. (Note that simplicity is preserved under subdivision.) Since M is simply connected, we can find a triangulation of the disc D, and a simplicial map $f : D \longrightarrow M$ such that $f|\partial D = \gamma$. We can now pull back the track foliation and edge pseudometric to D. Clearly D is also simple. Let λ be the transverse path-pseudometric on D induced by this edge metric. Note that any piecewise smooth path in D maps under f to a piecewise smooth path in M. We see that f is distance non-increasing from (D, λ) to (M, ρ).

Now, let α and α' be, respectively, the pullbacks of the paths β and β'. Thus $\partial D = \alpha \cup \alpha'$. Now, $\mathrm{length}_d\,\alpha = \mathrm{length}_d\,\beta$ and $\mathrm{length}_d\,\alpha' = \mathrm{length}_d\,\beta'$, and so $\mathrm{length}_d\,\alpha' < \mathrm{length}_d\,\alpha$. Also, α is an taut path in D.

Let $p : D \longrightarrow F(D)$ be the projection to leaf space, $F(D)$. Thus, λ also gives a path-pseudometric on $F(D)$. By Lemma 2.3, $F(D)$ is a finite tree. Since α is taut, it maps to an interval in $F(D)$. Moreover, the projection of α to $F(D)$ is injective on the union of segments lying in essential edges of M. Since $F(D)$ is a tree, we see that the projection of the path α' to $F(D)$ must contain the projection of α. ¿From this, we arrive easily at the contradiction that $\mathrm{length}_d\,\alpha \leq \mathrm{length}_d\,\alpha'$. $\diamondsuit$

Now, as observed earlier, we must have $\rho \leq d$ on each edge of M. Thus, we see that $\rho(x, y) \leq \mathrm{length}_\rho\,\beta \leq \mathrm{length}_d\,\beta = \rho(x, y)$, and so these quantities are all equal. Thus, β is a ρ-geodesic.

In the particular case where β is a segment of an edge of M, we have (by convexity of d, that $\text{length}_d \beta = d(x, y)$. Since all edges are taut (by simplicity), we deduce that $\rho(x, y) = d(x, y)$. This shows that ρ induces the original edge metric, d.

To prove Proposition 2.5, it remains to show:

Lemma 2.8 : *ρ is 0-hyperbolic.*

Proof : If not, then there are points, $y_1, y_2, y_3, y_4 \in M$ such that the largest two of the three quantities, $\rho(y_i, y_j) + \rho(y_k, y_l)$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ are distinct. Thus, without loss of generality, we can assume that

$$\max\{\rho(y_1, y_2) + \rho(y_3, y_4), \rho(y_2, y_3) + \rho(y_4, y_1)\} < \rho(y_1, y_3) + \rho(y_2, y_4) - 2\epsilon$$

for some $\epsilon > 0$.

Let β_i piecewise straight path joining y_i to y_{i+1} in M with $\text{length}_d \beta_i \leq \rho(y_i, y_{i+1}) + \epsilon$. Let γ be the loop $\beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4$. After subdivision, we can suppose that γ lies in the 1-skeleton of M. Let $f : D \longrightarrow M$ and λ be as in the proof of Lemma 2.7. Thus f is distance non-increasing from (D, λ) to (M, ρ). Let x_i be the preimage of y_i and α_i be the pullback of β_i to D. Thus, $\partial D = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$. Note that $\rho(y_i, y_j) \leq \lambda(x_i, x_j)$. In particular,

$$\rho(y_1, y_3) + \rho(y_2, y_4) \leq \lambda(x_1, x_3) + \lambda(x_2, x_4).$$

Also $\lambda(x_i, x_{i+1}) \leq \text{length}_d \alpha_i = \text{length}_d \beta_i \leq \rho(x_i, x_{i+1})$. Putting this back in the earlier inequality for ρ, we find that

$$\max\{\lambda(x_1, x_2) + \lambda(x_3, x_4), \lambda(x_2, x_3) + \lambda(x_4, x_1)\} < \lambda(x_1, x_3) + \lambda(x_2, x_4).$$

This contradicts the fact that (D, λ) is 0-hyperbolic (Corollary 2.4). $\diamondsuit$

This proves Proposition 2.5.

3. From convex pseudometrics to R-trees.

A the end of Section 2, we described how a continuous convex edge pseudometric in a simply connected simple track complex gives rise to an **R**-tree. In this section we consider the case where there is a group action which respects this construction. We explain how to recognise edge stabilisers in the complex, and give a criterion for the action to be minimal.

Suppose that K is a finite track complex.

Definition : A subset, Q, of K is *elementary* it is a union of leaves, and intersects every edge of K in a closed connected set (possible empty).

It follows that Q is itself closed. Note that ∂Q meets every edge in a finite set (at most two points). We see that ∂Q is intrinsically a 1-complex and lies inside a finite union of leaves. Thus each component of ∂Q lies inside a single leaf.

Definition : We say that K is *efficient* if it does not contain a proper elementary subset which carries all the fundamental group.

Suppose now that K is a finite track complex. Let $\Gamma = \pi_1(K)$, and let $\tilde{K}$ be the universal cover of K. Let $\pi : \tilde{K} \longrightarrow K$ be the quotient map. We can pull back the foliation on K to give a Γ-invariant track foliation on $\tilde{K}$. We shall assume that $\tilde{K}$ is simple.

Suppose that K carries a continuous convex edge-pseudometric, d. We can pull this back to one on $\tilde{K}$, which we also denote by d. Let ρ be the induced transverse path-pseudometric on $\tilde{K}$, so that ρ agrees with d on every edge (Proposition 2.5). Let Σ be the hausdorffification of $(\tilde{K}, \rho)$. We also denote the induced metric on Σ by ρ. Thus, by Corollary 2.6, (Σ, ρ) is an $\mathbf{R}$-tree. We write $\psi : \tilde{K} \longrightarrow \Sigma$ for the projection map. Since the construction is Γ-invariant, we see that Σ admits an isometric action by Γ.

Note that this is precisely the contruction given in by Levitt and Paulin [LP], except that they assume that d is an edge-metric. They describe actions arising in this way as "geometric", and also characterise geometric actions as ones which are not strong limits. We note that the proof of Theorem 2.5 of [LP] should work under our weaker hypotheses to show that the actions we obtain are also geometric.

Proposition 3.1 : *If K is efficient, then Σ contains no proper closed Γ-invariant subtree.*

Proof : Suppose that $S \subseteq \Sigma$ is a closed proper Γ-invariant subtree. Let $\tilde{Q} = \psi^{-1}S \subseteq \tilde{K}$. Thus, $\tilde{Q}$ is a closed Γ-invariant set. Let $Q = \pi(\tilde{Q}) \subseteq K$. Thus, Q is closed, and a union of leaves.

Suppose e is an edge of $\tilde{K}$. By Proposition 2.5, we know that the path-pseudometric ρ restricted to e agrees with the original convex metric d defining ρ. Suppose that $x, y \in e \cap \tilde{Q}$, and $z \in [x, y] \subseteq e$. By convexity, we have $\rho(x, y) = \rho(x, z) + \rho(z, y)$ and so, projecting to Σ, we have $\rho(\psi x, \psi y) = \rho(\psi x, \psi z) + \rho(\psi z, \psi y)$. Thus, $\psi z \in [\psi x, \psi y] \subseteq S$, and so $z \in Q$. This shows that $e \cap \tilde{Q}$ is convex. Projecting to K, we see that Q also meets every edge in a convex set, and so Q is elementary.

Now, the set of singular points, $R(e)$ on each edge e of $\tilde{K}$ is finite. Suppose that $e \cap \tilde{Q}$ is a proper non-empty subinterval of e. Let $\epsilon(e) = \rho(e \cap \tilde{Q}, R(e) \setminus \tilde{Q})$. (This is defined, since every vertex is a singular point, and so $R(e) \setminus \tilde{Q}$ is non-empty.) We must have $\epsilon(e) > 0$. (For if $x \in e \cap \tilde{Q}$ and $y \in e \setminus \tilde{Q}$, we have $\rho(x, y) > 0$, otherwise $\psi x = \psi y$, contradicting the definition of $\tilde{Q}$ as $\psi^{-1}S$.) Let $\epsilon = \min\{\epsilon(e)\}$ as e ranges over all edges of $\tilde{Q}$. Thus $\epsilon > 0$ (since $K = \tilde{K}/\Gamma$ is a finite complex). Now, it's easy to see that any piecewise straight path connecting two distinct components, $\tilde{Q}_1$ and $\tilde{Q}_2$, of $\tilde{Q}$ must have d-length at least ϵ. Thus $\rho(\tilde{Q}_1, \tilde{Q}_2) \geq \epsilon$. If $Q_1 = \psi(\tilde{Q}_1)$ and $Q_2 = \psi(\tilde{Q}_2)$, then $\rho(Q_1, Q_2) \geq \epsilon$. But now S is a disjoint union of sets of this form. Since S is connected, we see that there can only be one such set. In other words, $\tilde{Q}$ is connected. It follows that Q carries all of the fundamental group if K, contradicting the hypothesis that K is efficient. $\diamondsuit$

Note that it was shown in [LP] that the minimal subtree of a geometric action is necessarily closed. Thus, given the observation preceeding Proposition 3.1, it follows that the action on Σ is, in fact, minimal.

We next want to consider edge stabilisers in Σ.

Suppose that e is an edge of $\tilde{K}$, and that $x, y \in e$ are distinct points. Let β be the interval $[x, y]$, and write $\operatorname{int}\beta$ for its *interior*, i.e. the open interval (x, y). Suppose that β is *length-minimal* in the sense that there does not exist a proper subinterval $\beta' \subseteq \beta$ with $\operatorname{length}_d \beta' = \operatorname{length}_d \beta$. It follows that $x, y \in \operatorname{supp} d$, where $\operatorname{supp} d$ is the support of d as described at the end of Section 1. It also follows that $\rho(x, y) = d(x, y) > 0$, and so $\psi x \neq \psi y$. Let $\Gamma(\psi\beta) = \Gamma(\psi x) \cap \Gamma(\psi y)$ be the stabiliser of the interval $\psi\beta$ in Σ.

Given a point $z \in \tilde{K}$, write $L(z)$ for the leaf containing z.

Lemma 3.2 : *Suppose that β is a length-minimal subinterval of an edge of $\tilde{K}$. If $z \in \operatorname{int}\beta \cap \operatorname{supp} d$, then $\Gamma(\psi\beta)$ preserves setwise the leaf $L(z)$.*

In fact, if K as the "non-nesting" property (see Section 4), then the conclusion is true for all $z \in \operatorname{int}\beta$.

Proof : Let $\epsilon = \min\{d(x, z), d(y, z)\}$. ¿From the length-minimality assumption on β it follows that $\epsilon > 0$. Suppose $g \in \Gamma(\psi\beta)$. Since $g \in \Gamma(\psi x)$ we have $g(\psi x) = \psi x$ and so $\rho(x, \psi x) = 0$. Thus, we can join x to gx by a piecewise straight path, α, with $\operatorname{length}_d \alpha < \epsilon$. Similarly, we can join y to gy by a piecewise straight path α' with $\operatorname{length}_d \alpha' < \epsilon$. Write $x' = gx$, $y' = gy$, $z' = gz$ and $\beta' = g\beta$. Thus, the paths $\beta, \alpha, \beta', \alpha'$ form a piecewise smooth loop γ, which, after subdivision of $\tilde{K}$, can be assumed to lie in the 1-skeleton. (It may no longer be the case that β lies inside a single edge, but that is unimportant in what follows.)

Let $f : D \longrightarrow \tilde{K}$, and λ be as in the proof of Lemma 2.7. Thus, $f : (D, \lambda) \longrightarrow (\tilde{K}, \rho)$ is distance non-increasing. Let, $p : D \longrightarrow F(D)$ be the projection to the leaf space of D, which, by Lemma 2.3, is a finite tree. Note that λ descends to a pseudometric, λ on $F(D)$, also denoted by λ. Now, the map $\psi \circ f : (D, \lambda) \longrightarrow (\Sigma, \rho)$ collapses each leaf of D to a point, and so gives rise to a continuous, distance non-increasing map $q : (F(D), \lambda) \longrightarrow (\Sigma, \rho)$.

Now, the loop γ can be lifted to ∂D and then projected back down to $F(D)$. We write h for the map thus defined. (Strictly speaking h is defined on the domain of γ.) Now, $h|\beta$ and $h|\beta'$ are homeomorphisms onto the arcs $h\beta$ and $h\beta'$ in $F(D)$. Moreover, $\lambda(hx, hz) = \rho(x, z) \geq \epsilon$. Similarly, $\lambda(hy, hz) \geq \epsilon$, $\lambda(hx', hz') \geq \epsilon$ and $\lambda(hy', hz') \geq \epsilon$. Also, we have $\operatorname{length}_\lambda h\alpha = \operatorname{length}_d \alpha < \epsilon$. Similarly, $\operatorname{length}_\lambda h\alpha' < \epsilon$. Now, the paths $h\beta$, $h\alpha$, $h\beta'$ and $h\alpha'$ form a loop cyclically connecting the points hx, hy, hy', hx', hx. Now, since $F(D)$ is a tree, it follows easily that $hz, hz' \in h\beta \cap h\beta'$.

Now, $q(hz') = \psi(z') = \psi(gz) = g(\psi z) = \psi z = q(hz)$. Also $q|h\beta$ and $q|h\beta'$ are precisely the hausdorffification maps to $\psi\beta$ and $\psi\beta'$. It follows that $\lambda(hz, hz') = 0$. If z does not lies in any of the closed intervals of $h\beta$ which get collapsed to a point under the hausdorffification, then if follows that $hz = hz' = h(gz)$. On the other hand, since $z \in \operatorname{supp} d$, we can always find some $w \in \operatorname{int} b$ arbitrarily close to z which does not lie in any such interval. (Note that $\operatorname{supp} d$ does not contain any isolated points.) In this case we deduce, by the same argument, $hw = h(gw)$. It follows, by continuity, that we again have $hz = h(gz)$.

But now, this means that the preimages of z and gz in D lie in the same leaf of D, and so z and gz lie in the same leaf of $\tilde{K}$. This shows that $L(z) = L(gz) = gL(z)$ as required. $\diamondsuit$

To relate this to edge stabilisers more generally, we note:

Lemma 3.3 : *Suppose that $a, b \in \Sigma$ are distinct points. Then, there is an edge e of $\tilde{K}$, and points $x, y \in e$, such that $\psi x = a$, $\psi y \in (a, b]$, and $[x, y]$ is length-minimal.*

Proof : Choose any $z \in \psi^{-1}a$ and $w \in \psi^{-1}b$. Join z to w by a piecewise straight path α. Now, since Σ is a tree, $[a, b] \subseteq \psi\alpha$, and so there must be a segment, α', of α lying in some essential edge e of $\tilde{K}$, with $[a, c] \subseteq \psi\alpha'$ for some $c \in (a, b]$. Now, let $[x, y] \subseteq \alpha'$ be the minimal interval such that $\psi x = a$ and $\psi y = c$. $\diamondsuit$

It follows that for any edge stabliser, $G \leq \Gamma$, we can find a set of G-invariant leaves in the manner described by Lemma 3.2.

4. From monotone metrics to convex pseudometrics.

In this section, we show how to obtain convex edge pseudometrics on a finite track complex, starting with a monotone edge metric. We start with a few definitions.

Suppose $(K, \mathcal{F})$ is a finite track complex. Let $J = J(K)$ be the (abstract) disjoint union of all the essential edges of K. If L is a leaf of K, we can view the intersection of L with the union of the essential edges of K, as an equivalence class in J, where the equivalence relation may be defined as follows. If e, e' are essential edges in a 2-simplex, σ of K, and $x \in e$ and $x' \in e'$, then we write $x \approx x'$ if x and x' lie in the same stratum of σ. The equivalence relation in question is that generated by all the relations $\approx$ for all 2-simplexes of K. We refer to this equivalence relation as the *pushing relation*. Thus we say that one point can be "pushed to" another if they are equivalent under this relation.

The point of making this observation, is that we can generalise the idea to subintervals of J. Thus, again if e, e', σ are as above, and α, α' are respectively subintervals of e and e', then we write $\alpha \approx \alpha'$ if there is a homeomorphism of α onto α' such that each point of α gets mapped to a point of α' in the same stratum of σ. Again, the "pushing relation" is the equivalence relation on the set of all subintervals of J generated by the relations $\approx$ for all simplices of K.

Finally, we can also define a pushing relation on the set, $V(J)$, of unit tangent vectors to J. Thus, if $v, v' \in V(J)$, then v can be pushed to v' if and only if there are (non-trivial) closed intervals, $\alpha, \alpha' \subseteq J$, such that α can be pushed onto α', and v and v' are respectively the initial tangent vectors of α and α'. (Thus the basepoint of v is pushed onto the basepoint of v'.)

In all cases, we refer to equivalence classes under the pushing relation as *orbits*.

Definition : We say that $(K, \mathcal{F})$ is non-nesting if there is no subinterval of J which can be pushed onto a proper subinterval of itself.

Note that if K admits a monotone edge metric, then it is necessarily non-nesting.

The theorem of Levitt [L2] can be reinterpreted as asserting that if $(K, \mathcal{F})$ is non-nesting, then it admits a non-zero continuous convex edge pseudometric. Here, we shall prove the weaker result:

Proposition 4.1 : *If $(K, \mathcal{F})$ admits a non-zero continuous monotone edge metric, then it also admits a non-zero continuous convex edge pseudometric.*

In fact, we could replace "metric" by "pseudometric" in the hypothesis, provided we assume, in addition, that $(K, \mathcal{F})$ is non-nesting. This is all we essentially use in the proof. Note also that the term "non-zero" in the hypothesis is redundant, unless the whole of K consists of a single leaf.

Recall, from Section 1, that $\mathcal{M}(J)$ is the compact space of normalised monotone pseudometrics on J. Let $\mathcal{E}(K)$ be the subset of edge pseudometrics on K. Since the property of being an edge metric is given by a set of closed relations, we see that $\mathcal{E}(K)$ is a closed subset of $\mathcal{M}(J)$. Thus, $\mathcal{E}(K)$ is compact. The hypotheses of the proposition tell us that $\mathcal{E}(K)$ is non-empty.

Proof : Choose any $d \in \mathcal{E}(K)$. Given any $\epsilon > 0$, we define another metric on J as follows. Suppose that e is a component of J, with linear order $<$. Suppose $x, y \in e$ with $x < y$. By an "ϵ-sequence" from x to y, we mean a finite sequence, $t = (t_i)_{i=0}^m$, of points of e such that $x = t_0 < t_1 < \cdots < t_m = y$ and $d(t_{i-1}, t_i) \leq \epsilon$ for all $i \in \{1, \ldots, m\}$. Let $\lambda(t) = \sum_{i=1}^m d(t_{i-1}, t_i)$, and let $d_\epsilon(x, y) = \inf\{\lambda(t)\}$ as t ranges over all ϵ-sequences from x to y. We perform this construction on each component of J.

Now, it's simple a exercise to verify that d_ϵ is a monotone metric on J. Moreover, if x, y, z lie in some component, e, with $x < y < z$, then $d_\epsilon(x, y) + d_\epsilon(y, z) \leq d_\epsilon(x, z) + \epsilon$. Also, $d_\epsilon \geq d$, and so $l(J, d_\epsilon) \geq l(J, d) = 1$. Let $\rho_\epsilon = d_\epsilon / l(J, d_\epsilon)$. Since the construction was natural, we see that ρ_ϵ is an edge metric, and so $\rho_\epsilon \in \mathcal{E}(K)$.

Let $\rho \in \mathcal{E}(K)$ be an accumulation point of the set $\{\rho_{1/n} \mid n \in \mathbf{N}\}$. We claim that ρ is convex. To see this, let e be any component of J, and suppose that $x, y, z \in e$ with $x < y < z$. Choose any $\eta > 0$. ¿From the definition of the topology, we can find a neighbourhood, U, of ρ in $\mathcal{E}(K)$, such that for all $\rho' \in U$, we have $\max\{|\rho(x) - \rho'(x)|, |\rho(y) - \rho'(y)|, |\rho(z) - \rho'(z)|\} < \eta$. Now, for some $n > 1/\eta$, we have $\rho_{1/n} \in U$. Now, $\rho_{1/n}(x, y) + \rho_{1/n}(y, z) - \rho_{1/n}(x, z) \leq 1/n < \eta$, and so $\rho(x, y) + \rho(y, z) \leq \rho(x, z) + 4\eta$. The claim now follows by letting η tend to 0, and noting that x, y, z were arbitrary.

Now, if ρ is continuous, we are done. We are thus left to consider the case where there is some atom, i.e., some vector $v_0 \in V(J)$, with $\mu_\rho(v_0) > 0$. (Recall the definitions from Section 1.) Let $W \subseteq V(J)$ be the orbit of v_0 (under the pushing relation). Now, for each $v \in W$, we have $\mu_\rho(v) = \mu_\rho(v_0)$ and so it follows from Lemma 1.1 that W is finite.

Given $v \in V(J)$, let $x(v)$ be the basepoint of v. Let $F \subseteq J$ be the union of $\{x(v) \mid v \in W\}$ and the set of all singular points of J. Thus, F is finite. Given $v \in W$, let $h(v) = \min\{d(x(v), y)\}$ as y ranges over the set of all points of $F \setminus \{x(v)\}$ which lie in the same edge as x, and are such that v points towards y. Let $h = \min\{h(v) \mid v \in W\}$. Thus $h > 0$. Given any $v \in W$, let $a(v)$ and $b(v)$ be the points in the same edge as $x(v)$ such that v point towards both $a(v)$ and $b(v)$, and such that $d(x(v), a(v)) = h/5$ and

$d(x(v), b(v)) = 2h/5$. (Such points exist, since vertices are considered as singular points.) Let $\beta(v)$ be the interval $[a(v), b(v)]$. Thus, $\beta(v) \cap F = \emptyset$, and $\beta(v) \cap \beta(w) = \emptyset$ if $v \neq w$. Also, since there are no singular points in any of the half-open intervals $(x(v), b(v)]$, we see easily that $\{\beta(v) \mid v \in W\}$ is precisely the orbit of a single interval. (Note that we only really need the non-nesting property to construct the intervals $\beta(v)$ from the orbit of vectors, W. However, the construction is more succinctly expressed in terms of an edge metric.)

Finally, we obtain a non-zero continuous convex edge pseudometric by choosing any such pseudometric on one of the intervals $\beta(v)$ and transporting to all the other such intervals using the pushing operation. $\diamondsuit$

In fact, for applications, the last paragraph of the proof is superfluous. The finite orbit of intervals immediately gives us a splitting of the group; so it's a bit pointless to contruct a pseudometric, only to recover another splitting subsequently. However, the observation saves us having to deal with this as a special case.

5. From monotone trees to monotone metrics.

In fact we shall deal generally with non-nesting actions on real trees. We show how these give rise to finite track complexes in the case of finitely presented groups. We can always arrange that such a complex be efficient. If the group acts by isometry on a monotone tree, then we can pull back the metric to obtain a continuous edge metric on the complex. First we make a few general observations.

Let T be a real tree. Recall from the introduction that a homeomorphism, g, of T is "non-nesting" if neither g nor g^{-1} maps any closed interval of T into a proper subinterval of itself. (Here we use the term "closed interval" to mean a set of the form $[x, y]$ for $x, y \in T$.) We say that a non-nesting homeomorphism, g, is *loxodromic* if there is a g-invariant closed subset, $\beta \subseteq T$, which is homeomorphic to the real line, and on which g has no fixed point. (Thus $\beta/\langle g \rangle$ is a circle.) It's easily seen that in this case g has no fixed points in T, and that the *axis* β is unique. Moreover, all powers of g are also loxodromic. Conversely, we have:

Lemma 5.1 : *If g is a non-nesting homeomorphism of T with no fixed point, then g is loxodromic.*

Proof : (Sketch) Choose any $x \in T$, and consider the combinatorial possibilities for the finite tree, τ, spanned by the points $\{x, gx, g^2x, g^3x\}$, i.e. $\tau = \bigcup_{i,j \in \{0,1,2,3\}} [g^i x, g^j x]$. (We do not know a-priori that these four points are distinct.) If all the intervals, $[g^i x, g^j x]$ intersect pairwise, then they must all meet at a point, and this point would have to be fixed by g. Now, it's fairly easy to rule out the possibility that $[x, g^3x] \cap [gx, g^2x] = \emptyset$. If $[x, g^2x] \cap [gx, g^3x] = \emptyset$, then the interval $[x, gx] \cap [g^2x, g^3x]$ would have to contain a fixed point of g. Finally, if $[x, gx] \cap [g^2x, g^3x] = \emptyset$, let $\alpha = [x, g^2x] \cap [gx, g^3x]$. In this case, $\bigcup_{i \in \mathbf{Z}} g^i \alpha$ is a loxodromic axis. $\diamondsuit$

On the other hand, if g is non-nesting and has a fixed point, then the fixed point set, $\operatorname{fix} g$ must be a closed subtree of T.

We say that a group, Γ, of homeomorphisms is *non-nesting* if every element is. It is *non-parabolic* if there is no point fixed by the whole of Γ.

Lemma 5.2 : *Suppose that Γ is a finitely generated group with a non-parabolic non-nesting action on a real tree, T. Suppose Γ_0 is a finite symmetric system of generators for Γ (i.e. if $g \in \Gamma_0$, then $g^{-1} \in \Gamma_0$). Then there is a point $x \in T$, and elements $g, h \in \Gamma_0$, such that x lies in the open interval (gx, hx).*

Proof : If Γ_0 contains a loxodromic element, g, choose any point x in the axis of g. Thus, $x \in (gx, g^{-1}x)$. So, suppose that for all $g \in \Gamma_0$, $\operatorname{fix} g \neq \emptyset$. Now an analogue of Helly's theorem for trees tells us that any finite set of pairwise intersecting closed subtrees of a real tree must have non-empty intersection. Now, since the action of Γ is assumed to be non-parabolic, we have $\bigcap_{g \in \Gamma_0} \operatorname{fix} g = \emptyset$. Thus, there exist $g, h \in \Gamma_0$ with $\operatorname{fix} g \cap \operatorname{fix} h = \emptyset$. Now there is a (unique) closed interval in T, meeting both $\operatorname{fix} g$ and $\operatorname{fix} h$ in a single point. Choose any x in the interior of this interval. It is easily verified that $x \in (gx, hx)$ as required. $\diamondsuit$

We remark that it's easy to see that a non-parabolic non-nesting action must contain a loxodromic element. (Consider the product gh, where $\operatorname{fix} g \cap \operatorname{fix} h = \emptyset$.) Thus, if we were to allow ourselves to change the generating set, we could always take $h = g^{-1}$ in the above result.

We now want to obtain track complexes from real trees. This will be done using "resolutions" as in [BeF].

Definition : Suppose T is a real tree, and $(M, \mathcal{F})$ is a track complex. A *resolving map* is a continuous map $\phi : M \longrightarrow T$ such that every leaf of M gets mapped to a point of T, and such that ϕ is injective on each essential edge of M.

Note that it follows that M is necessarily simple.

We can construct resolving maps as follows. Let P be the set of vertices of M. Suppose we are given any map $\phi : P \longrightarrow T$. Suppose e is an edge of M, with endpoints $x, y \in P$. We extend ϕ to e by mapping e to $[\phi x, \phi y]$, either homeomorphically (if $\phi x \neq \phi y$) or by collapsing it to a point (if $\phi x = \phi y$). This gives a map of the 1-skeleton of M into T. Suppose now that σ is a 2-simplex of M. We have ϕ already defined on $\partial\sigma$, and $\phi(\partial\sigma)$ must be a point, an interval, or a tripod. Now, it's easily seen that we can extend ϕ over σ, in such a way that the set of preimages of points in σ conforms to one of the patterns (A)–(D) in the definition of a track complex (Section 2). Calling each preimage a "stratum" of σ, we obtain a simple track foliation on M, so that the map $\phi : M \longrightarrow T$ is a resolving map.

Suppose now that T admits non-nesting action of a finitely presented group Γ. Let K be a finite 2-complex with $\pi_1(K) = \Gamma$. Let $\tilde{K}$ be the universal cover of K.

Definition : A *resolution* of the action of Γ on T consists of a track foliation on K together with a Γ-equivariant resolving map $\phi : \tilde{K} \longrightarrow T$.
A resolution is *efficient* if the track foliation on K is efficient (as defined in Section 3).

Now it's easy to see that we can construct a resolution for any Γ-action, starting with any 2-complex K with $\pi_1(K) = \Gamma$. First, choose any Γ-equivariant map of the vertices of $\tilde{K}$ into T, and then extend over $\tilde{K}$, in the manner described above, taking care to do so equivariantly. The track complex thus defined on $\tilde{K}$ is necessarily simple, and descends to one on K.

The following gives a condition under which the resolution is efficient:

Lemma 5.3 : *Suppose that T admits a non-nesting action by the group Γ. Suppose that $\phi : \tilde{K} \longrightarrow T$ is a resolution of this action. Suppose that each vertex, a, of $\tilde{K}$ lies in the interior of a path, β, consisting of a sequence, $e_0, \ldots, e_n$, of edges of $\tilde{K}$ such that $e_1, \ldots, e_{n-1}$ are non-essential, and such that $\phi a \in (\phi b, \phi c)$, where b and c are the endpoints of β. Then, ϕ is efficient.*

Proof : Suppose, to the contrary, that Q is a proper elementary subset of K which carries all of the fundamental group Γ. Since Q meets every edge of K in a connected set, it cannot contain all vertices of K. Let R be a component of $K \setminus Q$, and let $V(R)$ be the set of vertices of K lying in R. We can suppose that $V(R) \neq \emptyset$. Note that R cannot lie inside a leaf of K (since Q is a union of leaves) and so must contain an open interval of some essential edge. Let G be the subgroup of Γ carried by R.

Now ∂R is a finite 1-complex in K, which lies inside ∂Q. Some neighbourhood of ∂R in K retracts onto ∂R. Thus, by the Van Kampen Theorem, we see that ∂R is connected and carries all of G. Let $\tilde{R}$ be a lift of R to $\tilde{K}$. We see that its boundary, $\partial \tilde{R}$, is connected. It follows that $\partial \tilde{R}$ lies in a leaf of $\tilde{K}$, and so $\phi(\partial \tilde{R})$ is a single point, say y, in T. Since $\tilde{R}$ is G-invariant, we see that y is fixed by G. Let $S = \phi(\tilde{R})$. Thus, S is a G-invariant subtree of T containing the point y. Since R contains an open interval of some essential edge, we see that $S \neq \{y\}$.

Let $V(\tilde{R})$ be the set of vertices of $\tilde{K}$ which lie in $\tilde{R}$. We claim that there is some $a \in V(\tilde{R})$ such that ϕa is terminal in S and not equal to y. For if not, given that $V(\tilde{R})/G \equiv V(R)$ is finite, we could find $p \in V(\tilde{R})$ and $g \in G$ such that $\phi p \in (y, \phi(gp))$. But now, $[y, \phi p]$ is a proper subset of $[y, \phi(gp)] = g[y, \phi p]$, contrary to the non-nesting hypothesis. This proves the claim.

Now, given a, let β, $e_0, \ldots, e_n$ and $b, c \in \tilde{K}$ be as given by the hypotheses. (Thus b and c are endpoints, respectively of the edges e_0 and e_n.) Now, $e_1, \ldots, e_{n-1}$ all lie in a leaf of $\tilde{K}$, and so all project under ϕ to the point $\phi a \in S \setminus \{y\}$. By hypothesis, $\phi a \in (\phi b, \phi c)$, and so without loss of generality, we have $\phi a \in (y, \phi b)$. It follows that $y \notin [\phi a, \phi b] = \phi(e_0)$. Thus, $\partial \tilde{R} \cap e_0 = \emptyset$ and so $e_0 \subseteq \tilde{R}$. Thus, $b \in V(\tilde{R})$ and so $\phi b \in S$. This contradicts the fact that ϕa is terminal in S. $\diamondsuit$

Proposition 5.4 : *Suppose T admits a non-nesting action of a finitely presented group Γ. Then the action admits an efficient resolution.*

Proof : Let K by any finite 2-complex, with $\pi_1(K) = \Gamma$. If Γ acts parabolically, then (for what it's worth) we get a an efficient resolution by mapping all of $\tilde{K}$ to any Γ-invariant point of T. So, we can assume that the action is non-parabolic.

Let τ be a maximal simplicial subtree of the 1-skeleton of K. Relative to any basepoint in τ, any directed edge K gives rise to a (possibly trivial) element of $\pi_1(K) = \Gamma$. Thus, any undirected edge gives rise to a pair of the form $\{g, g^{-1}\} \subseteq \Gamma$. Let Γ_0 be the set of elements of Γ arising in this way. Thus, Γ_0 is a finite symmetric generating set for Γ. By Lemma 5.2, we can find a point $x \in T$, such that $x \in (gx, hx)$ for some $g, h \in \Gamma_0$.

Let $\tilde{\tau}$ be a lift of τ to $\tilde{K}$. Let P be the set of all vertices of $\tilde{K}$. We define a map $\phi : P \longrightarrow T$ by sending every point of $P \cap \tilde{\tau}$ to x, and mapping the rest of P Γ-equivariantly. As discussed above, this gives rise to a resolution $\phi : \tilde{K} \longrightarrow T$. Note that $\tilde{\tau}$ lies inside a leaf of $\tilde{K}$, and so τ lies in a leaf of K. We claim that this resolution is efficient.

To see this, we verify the hypotheses of Lemma 5.3. Let a be any vertex of $\tilde{K}$, which we can assume to lie in $\tilde{\tau}$. Let $g, h \in \Gamma_0$ be the elements given by Lemma 5.2. By construction, there are edges e and e' of $\tilde{K}$ which connect $\tilde{\tau}$ respectively to $g\tilde{\tau}$ and $h\tilde{\tau}$. We connect e and e' by a path in $\tilde{\tau}$ to give a path β connecting a point $b \in g\tilde{\tau}$ to a point $c \in h\tilde{\tau}$. Now, $\phi b = gx$ and $\phi c = hx$, and so $\phi a \in (\phi b, \phi c)$ as required. ♢

We noted in the introduction, that any isometric action in a monotone tree is non-nesting. We can pull back the metric on T to get a Γ-invariant continuous monotone edge metric on $\tilde{K}$. This descends to an edge metric on K. The only way for an edge metric to be identically zero is for it to be defined on an empty set. In other words, all edges are inessential, and so K conists of a single leaf. In this case the action of Γ would have to be parabolic. In summary, for a non-parabolic action on a monotone tree, we get a non-zero continuous monotone edge metric on K. Moreover, we can assume K to be efficient, and $\tilde{K}$ to be simple.

6. Proofs and applications.

In this section, we complete the proof of Theorem 0.1, and describe in more detail some of the applications outlined in the introduction.

Proof of Theorem 0.1 : Suppose, then, that Γ is a finitely presented group, with a non-trivial isometric action on a monotone tree, (T, d). Let K be a finite 2-complex with $\pi_1(K) = \Gamma$. By Proposition 5.4, there exists an efficient resolution, $\phi : \tilde{K} \longrightarrow T$. As described at the end of Section 5, we can pull back the metric to obtain a non-zero continuous monotone edge metric on K. By Proposition 4.1, K also admits a non-zero continuous convex edge pseudometric. We lift this pseudometric to $\tilde{K}$, and let ρ be the induced transverse path-pseudometric on $\tilde{K}$. Since $\tilde{K}$ is simple and simply connected, Corollary 2.6 tells us that the hausdorffification, (Σ, ρ) of $(\tilde{K}, \rho)$ is an $\mathbf{R}$-tree. Since ρ is non-zero, Σ in non-trivial, i.e. not a point. Since the whole construction has been Γ-invariant, we get an isometric action of Γ on Σ. Since K is efficient, Proposition 3.1 tells us that Σ has no proper closed invariant subtree. In particular, the action on Σ is non-trivial. Let ψ be the quotient map to Σ. In summary, we have two Γ-equivariant maps of $\tilde{K}$ to trees, namely $\phi : \tilde{K} \longrightarrow T$ and $\psi : \tilde{K} \longrightarrow \Sigma$.

It remains to verify the statements about edge stabilisers. Suppose, then, that $G \subseteq \Gamma$ is a Σ-edge stabiliser. In other words, there are distinct points, $a, b \in \Sigma$, such that $G = \Gamma_\Sigma([a, b])$. Now, by Lemma 3.3, there is an edge e of $\tilde{K}$, and points, $x, y \in e$, with $\psi x = a$, $\psi y \in (a, b]$, and such that $[x, y]$ is length-minimal in e with respect to the metric ρ. Note that $G \subseteq \Gamma_\Sigma(\psi[x, y])$. Now, since $\psi x \neq \psi y$, e is essential, and so, since ϕ is a resolving map, $\phi|e$ is injective. In particular, $\phi x \neq \phi y$, and ϕ maps $[x, y]$ homeomorphically onto $[\phi x, \phi y] = \phi[x, y]$. Let H be the T-edge stabiliser $\Gamma_T(\phi[x, y])$.

Suppose $g \in \Gamma(\psi[x, y])$, and $w \in (\phi x, \phi y)$. Now, $w = \phi z$ for some $z \in (x, y)$. Since K is non-nesting, using Lemma 3.2, and the subsequent observation, we see that g preserves setwise the leaf $L(z)$ passing through z. In other words, $L(gz) = L(z)$, and so $\phi(gz) = \phi(z)$. It follows that $gw = w$. This shows that g fixes pointwise the open interval $(\phi x, \phi y)$, and so, by continuity, also the closed interval $[\phi x, \phi y]$. It follows that $G \subseteq \Gamma_\Sigma(\psi[x, y]) \subseteq \Gamma_T(\phi[x, y]) = H$.

We have shown that every Σ-edge stabiliser in Γ is contained in a T-edge stabiliser. It remains to worry about chains of edge stabilisers.

Suppose, then, that $(G_i)_{i \in \mathbf{N}}$ is a chain of Σ-edge stabilisers. In other words, there is a decreasing sequence of closed intervals $(A_i)_{i \in \mathbf{N}}$ of Σ, with $\bigcap_{i \in \mathbf{N}} A_i$ a singleton, $\{a\}$, and $G_i = \Gamma_\Sigma(A_i)$ for all i. We can find some point, $b \in A_0$, such that $(a, b] \cap A_i \neq \emptyset$ for all i. By Lemma 3.3, there is some edge, e of $\tilde{K}$, and points $x, y_0 \in e$ with $\psi x = a$, $\psi y_0 \in (a, b]$ and such that $[x, y_0]$ is length-minimal with respect to the metric ρ. Let $b_0 = \psi y_0$. We now, choose points, b_i inductively, so that $b_i \in A_i$ and $b_{i+1} \in (a, b_i]$. Thus, $[a, b_i] \subseteq A_i$ and so $G_i \subseteq \Gamma_\Sigma([a, b_i])$. For each $i > 0$, let y_i be the point of $(x, y_0] \subseteq e$ closest to x (i.e. with $[x, y_i]$ minimal) subject to $\psi y_i = b_i$. Thus, it's easily seen that $[x, y_i]$ is length-minimal with respect to ρ. Moreover, $y_{i+1} \in [x, y_i]$ for all i. Now, e is essential, and so $\phi|e$ is injective. Thus, we get a decreasing chain, $\phi[x, y_i]$, of non-trivial intervals of T. Let H_i be the T-edge stabiliser $\Gamma_T(\phi[x, y_i])$. Now, as the previous argument, we obtain, $G_i \subseteq \Gamma_\Sigma(\psi[x, y_i]) \subseteq \Gamma_T(\phi[x, y_i]) = H_i$. Finally, note that $\rho(x, y_i)$ must tend to 0. Since each $[x, y_i]$ is edge-minimal, we see easily that $\bigcap_{i \in \mathbf{N}} [x, y_i] = \{x\}$. Thus the intersection of the intervals $\phi[x, y_i]$ is just a point, $\{\phi x\}$. We thus have that $(H_i)_{i \in \mathbf{N}}$ is a chain of T-edge stabilisers in the sense defined in the introduction. $\diamondsuit$

Suppose that every subgroup of Γ which fixes a non-trivial interval of T is finitely generated. Suppose also that the action on T is stable. Given a chain, $(G_i)_{i \in \mathbf{N}}$, of Σ-edge stabiliser, let $(H_i)_{i \in \mathbf{N}}$ be the chain of T-edge stabilisers given by the theorem. Since the action on T is stable, we see that $H = \bigcup_{i \in \mathbf{N}} H_i$ is a T-edge stabiliser. Now, $\bigcup_{i \in \mathbf{N}} G_i \subseteq H$ is subgroup of a T-edge stabiliser, and so, by hypothesis, is finitely generated. It follows that the sequence $(G_i)_{i \in \mathbf{N}}$ must stabilise. This shows that the action on Σ is also stable.

The observation of the last paragraph applies to most actions of interest, where the edge stabililisers are constrained to lie in some class of reasonably nice groups, for example finitely generated virtually abelian groups. In particular, if all the edge stabilisers are finite, then Theorem 9.5 of [BeF] tells us that Γ is either virtually abelian or splits over a finite or two-ended subgroup. This gives us Corollary 0.2.

Suppose Γ is a finitely generated infinite virtually abelian group acting by isometries on a real tree, with finite kernel (for example if the edge stabilisers are finite). Let Γ'

be a free abelian subgroup of finite index in Γ. Now every non-trivial element of Γ' is loxodromic, and all the loxodromic axes coincide. In fact, this axis will be Γ-invariant, so we get an action of Γ on the real line. Now if this action preserves a mononote metric (or any continuous metric), then it's not too hard to see that it will be topologically conjugate to a linear action. All we need to observe here, is that if Γ is not virtually cyclic, then the action on this line cannot be properly discontinuous. In particular, we can find an infinite sequence of distinct elements, $(g_i)_{i\in\mathbf{N}}$, and a non-trivial interval, A, such that g_iA converges on a non-trivial interval.

We are now ready to prove Theorem 0.3. We should first note that there is no loss of generality in assuming that the dendron is metrisable. (Since Γ is countable, the smallest subcontinuum containing any Γ-orbit will be a Γ-invariant separable dendron, and hence metrisable). The only likely applications are to metrisable dendrons anyway.

The first part of the argument (the construction of the monotone tree) was described in detail in [Bo2], so we only outline the procedure here.

Proof of Theorem 0.3 : Suppose that Γ is a finitely presented infinite group acting on the metrisable dendron P, with the hypotheses of Theorem 0.3. A theorem of Bing/Moise tells us that P admits a continuous convex metric, δ. (This also follows from [MaO].) Let T be the connected subset of P consisting of the set of non-terminal points. Given $x, y \in T$, let $d(x, y) = \max\{\delta(gx, gy) \mid g \in \Gamma\}$. ¿From the convergence property of the action, this maximum is necessarily attained. (In fact, for each $r > 0$, the set of $\gamma \in \Gamma$ such that $\delta(\gamma x, \gamma y) \geq r$ is finite. For otherwise, we could find points $a, b \in P$, and a sequence $(\gamma_i)_i$ in Γ such that $\gamma_i|P \setminus \{a\}$ converges locally uniformly to b, and with $\delta(\gamma_i x, \gamma_i y)$ bounded away from 0. Since $[x, y]$ lies in the interior of a larger interval in P, it is easy to deduce that both sequences $(\gamma_i x)_i$ and $(\gamma_i y)_i$ must tend to b, giving us a contradiction.) It is also easily seen that d is a monotone metric on T, and so (T, d) is a monotone tree. Since the construction is natural, we see that the action of Γ on T is isometric. (For further details, see [Bo2]).

Now, again from the convergence property, we see easily that edge stabilisers have to be finite. The hypothesis about chains of finite subgroups now tells us that the action on T is stable. Corollary 0.2 now tells us that Γ is either virtually abelian of splits over a finite or two-ended subgroup.

Suppose Γ were virtually abelian but not virtually cyclic. Then the observation about convergence of intervals made immediately before the proof is easily seen to contradict the convergence property of the action.

Finally, note that any virtually cyclic group splits over a finite subgroup. $\diamondsuit$

More details of the applications of this result to hyperbolic groups, as outlined in the introduction, can be found in [Bo2] and [Bo3].

7. A refinement of the main theorem.

In this section, we briefly describe a relative version of Theorem 0.1, which finds application in [Bo6]. Specifically, we show:

Theorem 7.1 : *Suppose that Γ is a finitely presented group, and that $H_1, \ldots, H_n$ are finitely presented subgroups. Suppose that Γ admits a non-parabolic isometric action on a monotone tree, T, such that the actions of each of the groups H_i is parabolic. Then Γ also admits an isometric action on an $\mathbf{R}$-tree, Σ, with the properties described by Theorem 0.1, and such that the action of each of the groups H_i on Σ is parabolic.*

We note that, using [BeF], we get a refinement of Corollary 0.2, which tells us that, in this case, (assuming that Γ is not virtually abelian) the splitting can be chosen such that each of the groups H_i is conjugate into one of the vertex groups. This also carries over to convergence actions on dendrons, given a refinement of Theorem 0.3. More discussion of this is given in [Bo6]. (In this context, we should note that the fixed point of a parabolic element acting on a dendron cannot be terminal, and so the induced isometry on the monotone tree, as constructed in [Bo2], is also parabolic.)

Before giving the proof of Theorem 7.1, we make a few preliminary observations. Suppose that Γ is a group of non-nesting homeomorphisms of a real tree, T. Suppose $S \subseteq T$ is a Γ-invariant subtree. If $H \leq \Gamma$ is parabolic on T, then it's parabolic on S. (Note that each $x \in T$ determines a unique $y \in S$ with the property that $[x, y] \cap S = \{y\}$. If x is fixed by H, then so is y.) Thus, without loss of generality, we can assume that the Γ-action on T is minimal. It follows that every point of T lies in some loxodromic axis. (Note that the union of all the loxodromic axes is non-empty and connected, and hence an invariant subtree — this is a standard argument for $\mathbf{R}$-trees.)

Proof of Theorem 7.1 : We need a variation of the construction of Proposition 5.4.

As observed above, we can suppose that the action of Γ on T is minimal. Let x_i be a fixed point of H_i in T. Let $\gamma_i \in \Gamma$ be a loxodromic whose axis contains x_i. In particular, we have $x_i \in (\gamma_i^{-1} x_i, \gamma_i x_i)$. For each i, we construct a finite simplicial 2-complexes, $L_i \subseteq K_i$, with fundamental groups H_i and Γ respectively. In fact, we choose a basepoint, $p_i \in L_i$ such that the inclusion of pointed complexes (L_i, p_i) into (K_i, p_i) induces the inclusion of H_i into Γ. We construct a maximal tree in the 1-skeleton of L_i, and extend it to a maximal tree, τ_i, in the 1-skeleton of K_i. Thus, each directed edge of K_i not in τ_i gives us an element of Γ. Note that, if this edge lies in L_i, then the corresponding element of Γ lies in H_i. We can assume (by enlarging K_i if necessary) that at least one edge of K_i gives us the element $\gamma_i \in \Gamma$.

We now construct a finite simplicial 2-complex K, with base point p, such that $\pi_1(K, p) = \Gamma$. We can assume that K contains disjoint copies of each of the complexes K_i. We can also assume that there is a tree, τ, in the 1-skeleton of K, containing the point p, and such that $\tau \cap K_i = \{p_i\}$. We thus get homomorphisms from $\pi_1(K_i, p_i)$ to $\pi_1(K, a)$ which we can assume to be the identity on Γ. Now, $\tau \cup \bigcup_{i=1}^n \tau_i$ is a tree, which we extend to a maximal tree, σ, in the 1-skeleton of K. We can find disjoint subtrees, $\sigma_1, \ldots, \sigma_n$, of σ such that $\tau_i \subseteq \sigma_i$ and $\bigcup_{i=1}^n \sigma_i$ contains all the vertices of K.

Now let $\tilde{\sigma}$ be a lift of σ to $\tilde{K}$. Let $\tilde{\sigma}_i$ be the lift of σ_i which is contained in $\tilde{\sigma}$. We define a Γ-equivariant resolving map, $\phi : \tilde{K} \longrightarrow T$ by sending each $\tilde{\sigma}_i$ to the point x_i. Thus, if $\tilde{L}_i$ is the lift of L_i meeting $\tilde{\sigma}_i$, then $\phi(\tilde{L}_i) = \{x_i\}$. We see that L_i lies inside single leaf of K.

We next verify the hypothesis of Lemma 5.3. Suppose that a is any vertex of $\tilde{K}$, which we can suppose lies in some $\tilde{\sigma}_i$. ¿From the construction, we can find edges, e and e' of $\tilde{K}$, which connect $\tilde{\sigma}_i$ respectively to $\gamma_i^{-1}\tilde{\sigma}_i$ and $\gamma_i\tilde{\sigma}_i$. Let α be any path in $\tilde{\sigma}_i$ containing a, and connecting the points $e \cap \tilde{\sigma}_i$ and $e' \cap \tilde{\sigma}_i$. Let β be the path $e \cup \alpha \cup e'$. Now, $\phi(a) = x_i$, and the endpoints of β get mapped respectively to $\gamma_i^{-1}x_i$ and $\gamma_i x_i$. Now, $x_i \in (\gamma_i^{-1}x_i, \gamma_i x_i)$, so the hypotheses of Lemma 5.3 are satisfied. It follows that ϕ is efficient.

Now, let Σ be the **R**-tree constructed as in the proof of Theorem 0.1. This admits a natural isometric action of the group Γ. Since each L_i lies inside a leaf of K, we see that all the subgroups H_i are parabolic on Σ. ♢

As mentioned earlier, this gives us refinements of Corollary 0.2 and Theorem 0.3. For more details and applications, see [Bo6].

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